Unanimity and Local Incentive Compatibility

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Abstract

We study the relationship between unanimity and local incentive constraints of deterministic social choice functions (or voting mechanisms). We consider a standard Bayesian environment where agents have private and strict preference orderings on a finite set of alternatives. We show that with independent and generic priors, locally ordinal Bayesian incentive compatibility of a social choice function combined with unanimity implies the tops-only property. Also, assuming unanimity invokes the sufficiency of local incentive constraints for full incentive constraints. Furthermore, unanimity helps our results hold in a broad class of domains — a few of its constituents being the unrestricted domain, the single-peaked domain, the single-dipped domain, and some other connected domains.

Keywords: Unanimity, Incentive compatibility, Local incentive compatibility, Tops-only property, Connected domains

JEL Classification: C72, D01, D02, D72, D82.

1 Introduction

Incentive compatibility of a social choice function (or a voting mechanism) has been one of the foremost concerns in mechanism design. Gibbard (1973), Satterthwaite (1975) and Moulin (1980) are some seminal papers that pertain to this matter. However, when the number of alternatives or the set of admissible preferences is large,

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it becomes onerous and costly to verify that a social choice function (scf) satisfies every incentive constraint. Therefore, the sufficiency of *local* constraints — on small distortions in reporting preferences — for full incentive compatibility has attracted substantial attention from mechanism design literature: some recent papers include Carroll (2012), Sato (2013), and Mishra (2016).

Meanwhile, *unanimity* of an scf is a mild form of efficiency[1] — whenever every agent agrees on an alternative as the best, the scf should choose it. Hence, it is quite natural to require an scf to be unanimous. In this paper, we argue that requiring an scf to be unanimous significantly eases the problem of designing an incentive compatible scf in three aspects. First, on the range of scfs, local incentive compatibility of an scf combined with unanimity invokes a strong and useful property called *tops-onlyness*; it exclusively responds to changes in the tops of preference profiles. Therefore, it is sufficient to consider scfs having this property as candidates. Secondly, concerning incentive constraints, unanimity helps inducing local incentive constraints to be sufficient to imply full incentive compatibility. Lastly, regarding the domain of admissible preferences, such sufficiency holds in a broad class of domains under unanimity.

Our framework is built on a standard Bayesian environment where individuals have private and strict preference orderings on a finite set of alternatives. We consider profiles of independent and *generic* priors, introduced in Majumdar and Sen (2004) and studied in Mishra (2016)[2]. Also, we restrict our attention to deterministic ordinal scfs that only account for the ordinal preferences of individuals. We consider two main concepts of incentive compatibility for these scfs: dominant strategy incentive compatibility (DSIC) and ordinal Bayesian incentive compatibility (OBIC) introduced by d'Aspremont and Peleg (1988). An scf is OBIC if for any agent, his interim outcome probability vector from truth-telling first-order stochastic-dominates any vector obtained from lying. Similarly to Mishra (2016), we focus on OBIC with respect to generic priors (G-OBIC). Local incentive constraints are respectively weakened versions of each full incentive compatibility, those that merely pertain to local distortions: local dominant strategy incentive compatibility (LDSIC)[3] and generic-

[1]Holströmes and Myerson (1983) classify the concepts of Pareto efficiency depending on the stage of information regarding the types of agents, among which ex-post Pareto efficiency is the weakest. Azrieli and Kim (2014) explain that any Ex-post Pareto efficient scf is unanimous.

[2]To be specific, Majumdar and Sen (2004) prove that these priors are generic in a topological sense under the unrestricted domain. Mishra (2016) explains that the identical proof works in restricted domains.

[3]LDSIC corresponds to AM-proof in Sato (2013).

local ordinal Bayesian incentive compatibility (G-LOBIC).

For the domain of preferences, we assume that the set of admissible preferences is *connected*, following the notions in Sato (2013): from any preference ordering to another, there is a path consisting of *adjacent* orderings. That is, any *large* distortion in preferences can be decomposed into a sequence of *local (or small)* distortions. Sato (2013) also defines an important subclass of connected domains, *connected domains without restoration*, where a certain kind of decomposition is possible. For any pair of preferences, there exists a path in which any adjacent distortion is not reversed (or restored) later in the sequence. Many well-known and widely studied domains such as the unrestricted domain and the full single-peaked domain lie in this class of domains. However, we mainly consider an even larger class of domains, *weakly* connected domains without restoration[4]. Whereas the former class necessitates the existence of a single path where the ranking between *any* two alternatives is not restored, our main domains merely demand the existence of one path for *each* pair of alternatives where the ranking between them is not restored. Note that the gap between two classes of domains can be larger as the number of alternatives grows.

In our main results, we study the implications unanimity of scfs have on local incentive constraints. Our first main result shows that the search for G-LOBIC scfs can be restricted to tops-only ones. Tops-onlyness is desirable not only for a mechanism designer (or a social planner) since it saves costs in collecting and processing data from agents, but also for agents who reveal their preferences in terms of privacy. Our result is in line with the literature on tops-onlyness — such as Weymark (2008) and Chatterji and Sen (2011) — which shows that DSIC with unanimity implies tops-onlyness on several domains. We generalize this result by weakening the incentive constraint to G-LOBIC and extending domains.

Next, we study the sufficiency of local incentive constraints. Sato (2013) shows that weakly connected domains without restoration are necessary but not sufficient for the equivalence between LDSIC and DSIC. However, our second main result is that assuming unanimity not only restores the equivalence between LDSIC and DSIC but also invokes the equivalence of G-LOBIC and G-OBIC. While the former equivalence has been discussed in the literature, we are the first to our knowledge to study the

[4]To be clear, Sato (2013) introduces this domain as a necessary, yet not a sufficient domain for the equivalence between LDSIC and DSIC. Since this domain is sufficient for our main results, we name it.

equivalence of Bayesian incentive constraints. This result is especially relevant when a mechanism designer considers DSIC to be demanding. For example, DSIC and unanimous scfs are inevitably dictatorial on the unrestricted domain, leaving OBIC scfs as natural substitutes. We also show that connected domains without restoration are sufficient but not necessary for the equivalence between G-LOBIC and G-OBIC without unanimity, which highlights the importance of unanimity.

Finally, we illustrate one of the theoretical implications of our results. Mishra (2016) — work most related to ours— studies the conditions for the equivalence of G-LOBIC and DSIC in restricted domains. We strengthen his results by generalizing the sufficient domain and relaxing the restriction on scfs.

The rest of the paper is organized as follows. We present a detailed framework in Section 2. In Section 3, we demonstrate our main results. All of the proofs are in the appendix.

2 The Model

2.1 Framework

Consider a standard Bayesian environment with private types[5]. The set of agents is $N = \{1, 2, ..., n\}$ and the set of alternatives is A with $m \equiv |A| \geq 3$. Let $\mathcal{P}$ denote the set of all strict linear orders over A. Then $\mathcal{P}$ is the *unrestricted domain* and a proper set $\mathcal{D} \subset \mathcal{P}$ is a restricted domain. Each agent $i \in N$ has a private preference ordering (or a type) $P_i \in \mathcal{D}$. For any preference ordering $P \in \mathcal{D}$ and any pair of alternatives $\{a, b\} \in A$, $a\,P\,b$ if and only if a is strictly preferred to b by P.

A deterministic and ordinal scf is a mapping, $f : \mathcal{D}^n \to A$. We focus on scfs that choose an alternative whenever it is agreed by all agents as the best alternative; unanimous scfs. For any preference ordering $P \in \mathcal{D}$ and any integer $k \in K \equiv \{1, ..., m\}$, let $P(k)$ denote the kth-ranked alternative for P.

Definition 1. An scf f is **unanimous** if for any $\boldsymbol{P} \in \mathcal{D}^n$ and $a \in A$, $f(\boldsymbol{P}) = a$ whenever $a = P_i(1)$ for every agent $i \in N$.

We assume that each agent independently draws his preference using a probability distribution $\mu_i : \mathcal{D} \to [0, 1]$, which is common knowledge for every agent. For any $Q \subseteq \mathcal{D}^{n-1}$, agent i's belief of others having a preference profile in Q is $\mu(Q) =$

[5] We borrow several key concepts from Mishra (2016).

$\sum_{\boldsymbol{P}_{-i}\in Q} \times_{j\neq i} \mu_j(P_j)$. We mainly consider the following profile of priors.

Definition 2 (Majumdar and Sen 2004)**.** A profile of priors $\{\mu_i\}_{i\in N}$ is **generic** if for every $Q, R \subseteq \mathcal{D}^{n-1}$ we have $[\mu(Q) = \mu(R)] \Rightarrow [Q = R]$.

2.2 Incentive Compatibility and Connected Domains without Restoration

We now define several IC constraints: from the most stringent one, DSIC.

Definition 3. An scf is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, every $P_i \in \mathcal{D}$ and every $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$, there exists no $P'_i \in \mathcal{D}$ such that $f(P'_i, \boldsymbol{P}_{-i})\, P_i\, f(P_i, \boldsymbol{P}_{-i})$.

A weaker concept of incentive compatibility is LDSIC. We define some necessary concepts. For any two types $P, P' \in \mathcal{D}$, we say that $\boldsymbol{P'}$ **is an** $\boldsymbol{(a,b)}$**-swap of** $\boldsymbol{P}$ if for some $a, b \in A$ and $k \in K$, $P(k) = P'(k+1) = a$, $P(k+1) = P'(k) = b$ and $P'(j) = P(j)$ for all $j \in K \setminus \{k, k+1\}$. Also, a pair of types $P, P' \in \mathcal{D}$ is **adjacent** if P' is an (a,b)-swap of P for some $\{a,b\} \subset A$ and denote the adjacent alternatives by $A(P, P') = \{a, b\}$.

Definition 4 (Mishra 2016)**.** An scf is **locally dominant strategy incentive compatible (LDSIC)** if for every $i \in N$, every $P_i \in \mathcal{D}$ and every $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$, there exists no **adjacent** type $P'_i \in \mathcal{D}$ to P_i such that $f(P'_i, \boldsymbol{P}_{-i})\, P_i\, f(P_i, \boldsymbol{P}_{-i})$.

Denote the union of a and the set of alternatives preferred to alternative a to type P_i as $B(a, P_i) = \{a' \in A : a' = a \text{ or } a' P_i a\}$. Also, for each agent $i \in N$, let $\pi_i^f(a, P_i) \equiv \sum_{\boldsymbol{P}_{-i}\in\mathcal{D}^{n-1}: f(P_i, \boldsymbol{P}_{-i}) = a} \mu(\boldsymbol{P}_{-i})$. Now we define the ordinal notion of Bayesian incentive compatibility.

Definition 5 (d'Aspremont and Peleg 1988)**.** An scf f is **ordinally Bayesian incentive compatible (OBIC)** with respect to $\{\mu_i\}_{i\in N}$ if for all $i \in N$, $P_i, P'_i \in \mathcal{D}$ and all $a \in A$, we have

$$\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P'_i).^6$$

[6] With some abuse of notation, let $\pi_i^f(B(a, P_i), P'_i)$ denote the sum of probabilities of preference profiles such that $f(\boldsymbol{P}) \in B(a, P_i)$ when reporting P'_i. That is, $\pi_i^f(B(a, P_i), P'_i) \equiv \sum_{\boldsymbol{P}_{-i}\in\mathcal{D}^{n-1}: f(P'_i, \boldsymbol{P}_{-i}) \in B(a, P_i)} \times_{j\neq i} \mu_j(P_j)$

f is **G-OBIC** if it is OBIC with respect to a profile of generic priors μ.

As DSIC is weakened to LDSIC, OBIC can be weakened in the same spirit.

Definition 6 (Mishra 2016)**.** An scf f is **locally ordinally Bayesian incentive compatible (LOBIC)** with respect to $\{\mu_i\}_{i\in N}$ if for all $i \in N$, for all $a \in A$, and for all pair of adjacent types $P_i, P_i' \in D$ we have

$$\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i'). \tag{1}$$

f is **G-LOBIC** if it is LOBIC with respect to a profile of generic priors $\{\mu_i\}_{i\in N}$.

Next we define some concepts on the domain of preferences. Any pair of types $P, P' \in \mathcal{D}$ is **connected** if there exists a sequence of types $(P = P^0, P^1, ..., P^h, P^{h+1} = P')$ in $\mathcal{D}$ such that for every $l \in \{0, 1...., h\}$, P^l and P^{l+1} are adjacent. For each pair $\{a, b\} \subset A$, a sequence in $\mathcal{D}$ is **with $\{a,b\}$-restoration** if for some distinct $l, l' \in \{0, 1, ..., h\}$, $A(P^l, P^{l+1}) = A(P^{l'}, P^{l'+1}) = \{a, b\}$ and **without $\{a,b\}$-restoration** if there exist no such swaps. A sequence is **without restoration** if it is without $\{a, b\}$-restoration for any $\{a, b\} \subset A$.

We first define a domain in which every pair of preference orderings are connected without restoration.

Definition 7 (Sato 2013)**.** A domain $\mathcal{D} \subseteq \mathcal{P}$ is **connected without restoration** if any pair of types $P, P' \in \mathcal{D}$ is connected without restoration.

Mishra (2016) discusses several examples of connected domains without restoration such as the unrestricted domain, the single-peaked domain, the single-dipped domain, and some single-crossing domains. However, we consider an even broader class of domains that includes connected domains without restoration: weakly connected domains without restoration.

Definition 8. A domain $\mathcal{D} \subseteq \mathcal{P}$ is **weakly connected without restoration** if for each $\{a, b\} \subset A$, any pair of types $P, P' \in \mathcal{D}$ is connected without $\{a, b\}$-restoration.

3 Results

3.1 Tops-Onlyness

An scf is tops-only if it only takes into account the top alternative of each agent.

Definition 9. An scf f is **tops-only** if for any $\boldsymbol{P}, \boldsymbol{P'} \in \mathcal{D}^n$, $f(\boldsymbol{P}) = f(\boldsymbol{P'})$ whenever $P_i(1) = P'_i(1)$ for all $i \in N$.

Our first main result shows that tops-onlyness is necessary for G-LOBIC scfs under unanimity.

Theorem 1. Let $f : \mathcal{D}^n \to A$ be an scf where $\mathcal{D} \subseteq \mathcal{P}$ is a weakly connected domain without restoration. If f is unanimous and G-LOBIC, then it is tops-only.

3.2 Local Domains and Sufficiency of Local Incentive Constraints

Mishra (2016) calls a domain *local* if LDSIC is equivalent to DSIC in it. Sato (2013) shows that connected domains without restoration are local. We show in the following proposition that the equivalence of G-LOBIC and G-OBIC also holds in these domains.

Proposition 1. Let $f : \mathcal{D}^n \to A$ be an scf where $\mathcal{D}$ is a connected domain without restoration. Then f is G-LOBIC if and only if it is G-OBIC.

While G-LOBIC and LDSIC are sufficient respectively for G-OBIC and DSIC in connected domains without restoration, it can be shown that there exist scfs that are LDSIC but not DSIC nor OBIC in weakly connected domains without restoration[7]. However, the following theorem shows that under unanimity, the equivalences are restored.

Theorem 2. Let $f : \mathcal{D}^n \to A$ be a unanimous scf where $\mathcal{D}$ is a weakly connected domain without restoration. Then f is G-LOBIC (resp. LDSIC) if and only if it is G-OBIC (resp. DSIC).

3.3 Ordinal Bayesian Incentive Compatibility and Dominant Strategy Incentive Compatibility

Now, using the results from above, we study the relationship between G-LOBIC and DSIC. Mishra (2016) investigates two weak versions of Maskin monotonicity in the context of this relationship.

Definition 10 (Mishra 2016)**.** An scf f satisfies **elementary monotonicity** if for every $i \in N$, every $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$ and every $P_i, P'_i \in \mathcal{D}$ such that P'_i is an (a, b)-swap of P_i for some $a, b \in A$ and $f(P_i, \boldsymbol{P}_{-i}) = b$, we have $f(P'_i, \boldsymbol{P}_{-i}) = b$.

[7] Example 3.2 in Sato (2013) serves this purpose.

He first shows that G-LOBIC combined with elementary monotonicity implies DSIC in local domains. Then he further relaxes monotonicity to hold only in a restricted set of preference profiles. For a domain $\mathcal{D} \subset \mathcal{P}$, a profile of preferences $\boldsymbol{P} \in \mathcal{D}^n$ is a **top-2 profile** if for every $i, j \in N$, $P_i(k) = P_j(k)$ for all $k > 2$. Let $\mathcal{D}^2(2)$ be the set of all top-2 profiles in $\mathcal{D}$.

Definition 11 (Mishra 2016)**.** An scf $f : \mathcal{D}^n \to A$ satisfies **weak elementary monotonicity** if f restricted to $\mathcal{D}^n(2)$ satisfies elementary monotonicity.

Similarly to our spirit, Mishra (2016) proves that under unanimity, elementary monotonicity can be replaced by weak elementary monotonicity for the sufficiency of G-LOBIC for DSIC in the single-peaked domain. In connected domains without restoration, tops-onlyness is additionally necessary for the sufficiency. However, the following lemma shows that an intermediate step (for the sufficiency of G-LOBIC for LDSIC) works without tops-only property even in weakly connected domains without restoration.

Lemma 1. Let $f : \mathcal{D}^n \to A$ be a unanimous scf where $\mathcal{D}$ is a weakly connected domain without restoration. Then f is LDSIC if and only if it is G-LOBIC and satisfies weak elementary monotonicity.

Combining Theorem 2 and Lemma 1 leads to our last theorem.

Theorem 3. Let $f : \mathcal{D}^n \to A$ be a unanimous scf where $\mathcal{D}$ is a weakly connected domain without restoration. Then f is DSIC if and only if it is G-LOBIC and satisfies weak elementary monotonicity.

Theorem 3 is a twofold strengthening of Mishra (2016)'s results. First, whereas Mishra (2016) presents the sufficient conditions contingent on the types of domains — single-peaked or connected without restoration —, we present an inclusive result. That is, we show that tops-onlyness required in Mishra (2016)'s result is redundant. In fact, this redundancy follows from Theorem 1. Secondly, we extend the domain. Showing the equivalence of LDSIC with DSIC is a crucial step in the proof of Theorem 3. While Mishra (2016) relies on the observation that connected domains without restoration are local, we use Theorem 2 for this step. Recall that the class of weakly connected domains without restoration is lager than the class of local domains in Mishra (2016).

3.4 Necessity

Appendix

Proof of Theorem 1. Suppose for the sake of contradiction that f is unanimous and G-LOBIC but not tops-only. Then there exist an agent (say agent 1), $\boldsymbol{P}_{-1} \in \mathcal{D}^{n-1}$ and two types $P_1, \bar{P}_1 \in \mathcal{D}$ such that $a^* \equiv P_1(1) = \bar{P}_1(1)$ and $f(\bar{P}_1, \boldsymbol{P}_{-1}) \neq f(P_1, \boldsymbol{P}_{-1})$.[8]

The first step of our proof is to show that the outcome of f is invariant to adjacent manipulations in preferences when the top alternative is fixed (Lemma 4). In order to do so, we introduce a property called 'swap monotonicity (SM)' and a result — both from Mishra (2016) — that any G-LOBIC scf satisfies SM (Lemma 2)[9]. Then, we show that on weakly connected domains without restoration, an scf that satisfies SM has a special property (Lemma 3) and use Lemma 2 and Lemma 3 to prove Lemma 4. Finally, we generalize the invariance of the outcome of f for larger manipulations in preferences with common tops, which shows that f is tops-only.

Definition 12 (Mishra 2016)**.** An scf f satisfies **swap monotonicity (SM)** if for every $i \in N$ and $P_i, P'_i \in \mathcal{D}$ such that $A(P_i, P'_i) = \{a, b\} \subset A$, we have for every $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$ that $f(P'_i, \boldsymbol{P}_{-i}) = f(P_i, \boldsymbol{P}_{-i})$ if $f(P_i, \boldsymbol{P}_{-i}) \notin \{a, b\}$ and $f(P'_i, \boldsymbol{P}_{-i}) \in \{a, b\}$ if $f(P_i, \boldsymbol{P}_{-i}) \in \{a, b\}$.

Lemma 2 (Mishra 2016)**.** Let $f : \mathcal{D}^n \to A$ be a G-LOBIC scf where $\mathcal{D} \subseteq \mathcal{P}$. Then f satisfies swap monotonicity.

Lemma 3. Let $f : \mathcal{D}^n \to A$ be an scf where $\mathcal{D}$ is a weakly connected domain without restoration. If f satisfies swap monotonicity, then it has the following property. For any $i \in N$, $P_i \in \mathcal{D}$, $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$ and $\{a, b\} \subset A$, if $f(P_i, \boldsymbol{P}_{-i}) \neq f(P'_i, \boldsymbol{P}_{-i})$ where P'_i is an (a, b)-swap of P_i, then $f(P_i, \boldsymbol{P}'_{-i}) = f(P_i, \boldsymbol{P}_{-i})$ and $f(P'_i, \boldsymbol{P}'_{-i}) = f(P'_i, \boldsymbol{P}_{-i})$ for any $\boldsymbol{P}'_{-i} \in \mathcal{D}^{n-1}$ such that for every $j \in N \setminus \{i\}$, $a\, P'_j\, b$ if and only if $a\, P_j\, b$.

Proof. WLOG, assume that $f(P_1, \boldsymbol{P}_{-1}) \neq f(P'_1, \boldsymbol{P}_{-1})$ for some $P_1 \in \mathcal{D}$, $\boldsymbol{P}_{-1} \in \mathcal{D}^{n-1}$ and $\{a, b\} \subset A$ such that $A(P_1, P'_1) = \{a, b\}$. Then by SM, $\{f(P_1, \boldsymbol{P}_{-1}), f(P'_1, \boldsymbol{P}_{-1})\} = \{a, b\}$. Assume without loss of generality that $f(P_1, \boldsymbol{P}_{-1}) = a$ and $f(P'_1, \boldsymbol{P}_{-1}) = b$ and that $a\, P_2\, b$. Suppose $P'_2 \in \mathcal{D}$ is a disinct type of agent 2 such that $a\, P'_2\, b$. Then there exists a path from P_2 to P'_2 ($P_2 = P_2^0, P_2^1, ..., P_2^h, P_2^{h+1} = P'_2$) in $\mathcal{D}$ that is

[8] $\boldsymbol{P}_{-1} \equiv (P_2, ..., P_n)$, $\boldsymbol{P}_{-\{1,2\}} \equiv (P_3, ..., P_n)$.

[9] This result is conveniently used in the proofs of Mishra (2016)'s results and also has a crucial role in ours.

without $\{a,b\}$-restoration. For simplicity, for any $l \in \{0,...,h+1\}$, let $\bar{f}(P_2^l) \equiv f(P_1, P_2^l, \boldsymbol{P}_{-\{1,2\}})$ and $\bar{f}'(P_2^l) \equiv f(P_1', P_2^l, \boldsymbol{P}_{-\{1,2\}})$.

We argue that $\forall l \in \{0,...,h\}$, $[\bar{f}(P_2^l) = a$ and $\bar{f}'(P_2^l) = b] \Rightarrow [\bar{f}(P_2^{l+1}) = a$ and $\bar{f}'(P_2^{l+1}) = b]$. Since the path is without $\{a,b\}$-restoration, $A(P_2^l, P_2^{l+1}) \neq \{a,b\}$. This leaves two possible cases: $A(P_2^l, P_2^{l+1}) \cap \{a,b\} = \emptyset$ or $A(P_2^l, P_2^{l+1}) \cap \{a,b\} \neq \emptyset$. If it is the former case, then by SM, $\bar{f}(P_2^{l+1}) = a$ and $\bar{f}'(P_2^{l+1}) = b$. For the latter case, assume without loss of generality that $A(P_2^l, P_2^{l+1}) = \{a,c\}$ where $c \in A$ and $c \neq b$. Then, $\bar{f}(P_2^{l+1}) \in \{a,c\}$ and $\bar{f}'(P_2^{l+1}) = b$ by SM. However, if $\bar{f}(P_2^{l+1}) = c$, then $\bar{f}'(P_2^{l+1}) = c$ since $A(P_1, P_1') = \{a,b\}$, which is a contradiction. Therefore, $\bar{f}(P_2^{l+1}) = a$ and $\bar{f}'(P_2^{l+1}) = b$.

Since $\bar{f}(P_2) = a$ and $\bar{f}'(P_2) = b$, $\bar{f}(P_2') = a$ and $\bar{f}'(P_2') = b$. Finally, we can apply the same process to agent 3, 4, ..., and n to complete the proof. $\square$

Lemma 4. Let $f : \mathcal{D}^n \to A$ be a unanimous scf where $\mathcal{D}$ is a weakly connected domain without restoration. If f is G-LOBIC, then for every agent $i \in N$, his type P_i and $\boldsymbol{P}_{-i} \in \mathcal{D}^{n-1}$, $f(P_i', \boldsymbol{P}_{-i}) = f(P_i, \boldsymbol{P}_{-i})$ for any adjacent type P_i' of P_i such that $P_i'(1) = P_i(1)$.

Proof. Suppose on the contrary that for some agent (say, agent 1), $f(P_1', \boldsymbol{P}_{-1}) \neq f(P_1, \boldsymbol{P}_{-1})$ for some adjacent preferences P_1, P_1' with $a^* \equiv P_1(1) = P_1'(1)$ and some $\boldsymbol{P}_{-1} \in \mathcal{D}^{n-1}$. Let $A(P_1, P_1') = \{a,b\}$ for some $a, b \in A$ and $a \neq b$. Then by SM, either $[f(P_1, \boldsymbol{P}_{-1}) = a$ and $f(P_1', \boldsymbol{P}_{-1}) = b]$ or $[f(P_1, \boldsymbol{P}_{-1}) = b$ and $f(P_1', \boldsymbol{P}_{-1}) = a]$ holds. Without loss of generality, assume the former case and let $a \equiv P_1(k)$ and $b \equiv P_1(k+1)$. Then, we can modify the preference orderings of other agents contingent on their relative rankings between a, b and a^*. For simplicity, for any $l \in \{0,...,h+1\}$, let $\bar{f}(P_2^l) \equiv f(P_1, P_2^l, \boldsymbol{P}_{-\{1,2\}})$ and $\bar{f}'(P_2^l) \equiv f(P_1', P_2^l, \boldsymbol{P}_{-\{1,2\}})$.

Case 1: If $P_2(1) = a^*$, then take $P_2' = P_2$. Then $\bar{f}(P_2') = a$ and $\bar{f}'(P_2') = b$.

Case 2: If $P_2(1) \neq a^*$ and $a \, P_2 \, b$, take $P_2' = P_1$ so that ranking between a and b for P_2' matches that of P_2. Then by Lemma 3, $\bar{f}(P_2') = a$ and $\bar{f}'(P_2') = b$.

Case 3: Analogously, if $P_2(1) \neq a^*$ and $b \, P_2 \, a$, take $P_2' = P_1'$. Then $\bar{f}(P_2') = a$ and $\bar{f}'(P_2') = b$.

We can apply the same procedure to the preference orderings of agent 3, 4, ..., and n to get $f(P_1, P_2', P_3', ..., P_{n-1}', P_n') = a \neq a^*$ and $f(P_1', P_2', P_3', ..., P_{n-1}', P_n') = b \neq a^*$ where $P_j(1) = P_j'(1) = a^*$ for all $j \in N$, which contradicts unanimity of f. $\square$

Now, we use the lemmas above to show that $f(\bar{P}_1, \boldsymbol{P}_{-1}) = f(P_1, \boldsymbol{P}_{-1})$, which completes the proof of Theorem 1. It suffices to show that the equality holds if $f(P_1, \boldsymbol{P}_{-1}) \neq a^*$ since by this fact, $f(P_1, \boldsymbol{P}_1) = a^* \neq f(\bar{P}_1, \boldsymbol{P}_{-1})$ is not possible.

Suppose $f(P_1, \boldsymbol{P}_{-1}) \neq a^*$. Since $\mathcal{D}$ is weakly connected without restoration, there exists a path connecting P_1 and $\bar{P}_1$ $(P_1 = P_1^0, P_1^1, ..., P_1^h, P_1^{h+1} = \bar{P}_1)$ that is without $\{a^*, f(P_1, \boldsymbol{P}_{-1})\}$-restoration. If $P_1^l(1) = a^*$ for every $1 \leq l \leq h$, i.e., alternative a^* is never swapped in the sequence, then we can apply Lemma 4 to every swap of the sequence so that $f(\bar{P}_i, \boldsymbol{P}_{-1}) = f(P_1, \boldsymbol{P}_{-1})$. On the other hand, it is possible that there exists $1 \leq l' \leq h$ such that $P_1^{l'}(1) \neq a^*$. Since the path is without $\{a^*, f(P_1, \boldsymbol{P}_{-1})\}$-restoration, we have $a^* \, P_1^l f(P_1, \boldsymbol{P}_{-1})$ for every $1 \leq l \leq h$. However, Lemma 2 and Lemma 4 imply that for any $0 \leq l \leq h$, $f(P_1^l, \boldsymbol{P}_{-1}) \neq f(P_1^{l+1}, \boldsymbol{P}_{-1})$ is only possible when $A(P_1^l, P_1^{l+1}) = \{P_1^l(1), P_1^{l+1}(1)\}$ and $f(P_1^l, \boldsymbol{P}_{-1}) \in A(P_1^l, P_1^{l+1})$. Therefore, $f(P_1^{l+1}, \boldsymbol{P}_{-1}) = f(P_1^l, \boldsymbol{P}_{-1})$ for every $1 \leq l \leq h$ and thus $f(\bar{P}_1, \boldsymbol{P}_{-1}) = f(P_1, \boldsymbol{P}_{-1})$.

Finally, suppose $f(P_1, \boldsymbol{P}_{-1}) = a^*$ and assume that $f(\bar{P}_1, \boldsymbol{P}_{-1}) \neq a^*$ for the sake of contradiction. Then however, we can apply the argument above to $\bar{P}_1$, which results in $f(P_1, \boldsymbol{P}_{-1}) = f(\bar{P}_1, \boldsymbol{P}_{-1}) \neq a^*$.

$\square$

Proof of Proposition 1.

We show that for any profile of generic priors $\{\mu_i\}_{i \in N}$, f is LOBIC with respect to $\{\mu_i\}_{i \in N}$ if and only if it is OBIC with respect to $\{\mu_i\}_{i \in N}$.

It suffices to show that LOBIC implies OBIC. Suppose that f is LOBIC with respect to $\{\mu_i\}_{i \in N}$ and fix $i \in N$, $P_i, P_i' \in \mathcal{D}$, $a \in A$. Since $\mathcal{D}$ is connected without restoration, there exists a path $(P_i = P_i^0, P_i^1, ..., P_i^h, P_i^{h+1} = P_i')$ connecting P_i and P_i' without restoration. We show that for any $l \in \{0, ..., h\}$, if $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i^l)$, then $\pi_i^f(B(a, P_i), P_i) \geq \pi_i^f(B(a, P_i), P_i^{l+1})$. There are two possible cases depending on the intersection of $A(P_i^l, P_i^{l+1})$ and $B(a, P_i)$.

Case I: Suppose $A(P_i^l, P_i^{l+1}) \cap B(a, P_i) = \emptyset$ or $A(P_i^l, P_i^{l+1})$. Then, $\pi_i^f(B(a, P_i), P_i^l) = \pi_i^f(B(a, P_i), P_i^{l+1})$ by SM.

Case II: Suppose $A(P_i^l, P_i^{l+1}) = \{x, y\} \subset A$ and $\{x, y\} \cap B(a, P_i) = \{x\}$. Since the path is without $\{x, y\}$-restoration, $x \, P_i^l \, y$. Furthermore, since f is LOBIC with respect to $\{\mu_i\}_{i \in N}$, $\pi_i^f(B(x, P_i^l), P_i^l) \geq \pi_i^f(B(x, P_i^l), P_i^{l+1})$ holds, which implies that $\pi_i^f(x, P_i^l) \geq \pi_i^f(x, P_i^{l+1})$. Moreover, $\pi_i^f(b, P_i^l) = \pi_i^f(b, P_i^{l+1})$ holds for all $b \in B(a, P_i) \setminus \{x\}$ by SM. Therefore, $\pi_i^f(B(a, P_i), P_i^l) \geq \pi_i^f(B(a, P_i), P_i^{l+1})$.

Since $\pi_i^f(B(a,P_i),P_i) \geq \pi_i^f(B(a,P_i),P_i^1)$ holds by LOBIC, we have $\pi_i^f(B(a,P_i),P_i) \geq \pi_i^f(B(a,P_i),P_i')$ by induction.

$\square$

Proof of Theorem 2. For the equivalence between G-LOBIC and G-OBIC, it suffices to show that G-LOBIC implies G-OBIC. Suppose f is G-LOBIC with respect to $\{\mu_i\}_{i\in N}$. For a pair of preferences $P, P' \in \mathcal{D}$, consider a path $(P = P^0, P^1, ..., P^h, P^{h+1} = P')$ connecting P and P'. For an agent $i \in N$ and any alternative $a \in A$, $\pi_i^f(B(a,P),P) \geq \pi_i^f(B(a,P),P^1)$ holds by LOBIC of f with respect to $\{\mu_i\}_{i\in N}$. Proving that the following holds with any integer $k \in \{1,...,h\}$ completes the proof:

$$\pi_i^f(B(a,P),P) \geq \pi_i^f(B(a,P),P^k) \Rightarrow \pi_i^f(B(a,P),P) \geq \pi_i^f(B(a,P),P^{k+1}). \qquad (2)$$

Suppose (2) holds for some l $(1 \leq l \leq h)$. If $P^{l+1}(1) = P^l(1)$, then $\pi_i^f(B(a,P),P^{l+1}) = \pi_i^f(B(a,P),P^l)$ since f is tops-only by Theorem 1. If $P^{l+1}(1) \neq P^l(1)$, then we need only consider the case where $P^l(1) \notin B(a,P)$ and $P^{l+1}(1) \in B(a,P)$ since if otherwise, $\pi_i^f(B(a,P),P^{l+1}) \leq \pi_i^f(B(a,P),P^l)$ by SM of f (as in Proposition 1).

Let $x \equiv P^{l+1}(1)$. Since $\mathcal{D}$ is weakly connected without restoration, for any $y \notin B(a,P)$, there exists a path $(P = \bar{P}^0, \bar{P}^1, ..., \bar{P}^{\bar{h}}, \bar{P}^{\bar{h}+1} = P^{l+1})$ from P to P^{l+1} without $\{x,y\}$-restoration. Furthermore, since $x\,P\,y$ and $x\,P^{l+1}\,y$, there does not exist m $(1 \leq m \leq \bar{h})$ such that $P^m(1) = y$. Then, tops-onlyness and SM of f imply that $\pi_i^f(y,P) = \pi_i^f(y,\bar{P}^1) = ... = \pi_i^f(y,\bar{P}^{\bar{h}}) = \pi_i^f(y,P^{l+1})$. Therefore, $\pi_i^f(A \setminus B(a,P),P^{l+1}) = \pi_i^f(A \setminus B(a,P),P)$ and thus $\pi_i^f(B(a,P),P^{l+1}) = \pi_i^f(B(a,P),P)$.

For the equivalence between LDSIC and DSIC, it suffices to show that LDSIC implies DSIC. Suppose on the contrary that there exist an agent (say, agent 1), $P_1, P_1' \in \mathcal{D}$, and $\boldsymbol{P}_{-1} \in \mathcal{D}^{n-1}$ such that $f(P_1', \boldsymbol{P}_{-1})\,P_1\,f(P_1, \boldsymbol{P}_{-1})$. For simplicity, let $x \equiv f(P_1', \boldsymbol{P}_{-1})$ and $y \equiv f(P_1, \boldsymbol{P}_{-1})$. Then if $x\,P_1'\,y$, every path connecting P_1 and P_1' is with $\{x,y\}$-restoration since tops-onlyness and LDSIC of f imply that for some preference $\hat{P}_1$ in the path, $\hat{P}_1(1) = y$. This contradicts the assumption that $\mathcal{D}$ is weakly connected without restoration. Therefore, we can assume that $y\,P_1'\,x$. Consider a path $(P_1 = P_1^0, P_1^1, ..., P_1^h, P_1^{h+1} = P_1')$ from P_1 to P_1' without $\{x,y\}$-restoration which exists since $\mathcal{D}$ is weakly connected without restoration. Then there exists l $(0 \leq l \leq h)$ such that $A(P_1^l, P_1^{l+1}) = \{x,y\}$, $P_1^l(1) = x$, $f(P_1^l, \boldsymbol{P}_{-1}) = y$, and $f(P_1^{l+1}, \boldsymbol{P}_{-1}) = x$. Since $x\,P_1^l\,y$, this contradicts LDSIC of f.

$\square$

Proof of Lemma 1.

We first argue that f satisfies elementary monotonicity. Suppose not, i.e., that for an agent (say, agent 1), some $\boldsymbol{P}_{-1} \in \mathcal{D}^{n-1}$ and $P_1, P_1' \in \mathcal{D}$, P_1' is a (a,b)-swap of P_1 for some $a, b \in A$, $f(P_1, \boldsymbol{P}_{-1}) \neq a$ and $f(P_1', \boldsymbol{P}_{-1}) = a$.

Since f is tops-only by Theorem 1, $P_1(1) = a$ and $P_1(2) = b$. Also, $f(P_1, \boldsymbol{P}_{-1}) \in \{a, b\}$ by SM so $f(P_1, \boldsymbol{P}_{-1}) = b$. Now we modify the preference orderings for all $j \in N$ such that $P_j \neq P_1$ and $P_j \neq P_1'$. Denote the set of such agents by $N' \subset N$.

If $2 \notin N'$, then take $P_2' = P_2$. Otherwise, take $P_2' = P_1$ if $a\, P_2\, b$ and $P_2' = P_1'$ if $b\, P_2\, a$. In both cases, we have $f(P_1, P_2', \boldsymbol{P}_{-\{1,2\}}) = b$ and $f(P_1', P_2', \boldsymbol{P}_{-\{1,2\}}) = a$ by Lemma 4. If we repeat the same process for agent 3, 4, ... , and n, then $(P_1, \boldsymbol{P'}_{-1})$ and $(P_1', \boldsymbol{P'}_{-1})$ are top-2 profiles but do not satisfy elementary monotonicity. This contradicts weak elementary monotonicity of f. Next, Lemma 5 completes the proof:

Lemma 5 (Mishra 2016)**.** For any domain $\mathcal{D} \subseteq \mathcal{P}$, an scf $f : \mathcal{D} \to A$ is LDSIC if and only if it is G-LOBIC and satisfies elementary monotonicity.

$\square$

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