A strategic implementation of the Talmud rule based on the concede-and-divide algorithm

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Bankruptcy problem 1/4

- When a firm goes to bankrupt, how to divide its liquidation value among its creditors?
- This class of resource allocation problems (the so-called bankruptcy problems) was motivated by the two puzzles in the ancient Jewish document (the Talmud).
- This literature was initiated by O'Neill (1982).
- For a survey, see Thomson (2015).

Bankruptcy problem: motivation 2/4

Contested Garment Problem

Worth of the garment	Claimant 1	Claimant 2
	100	200
200	50	150

Bankruptcy problem: motivation 3/4

Estate Division Problem

Estate of the man	Wife 1	Wife 2	Wife 3
	100	200	300
100	$\frac{100}{3}$	$\frac{100}{3}$	$\frac{100}{3}$
200	50	75	75
300	50	100	150

- People are puzzled about this, and try to rationalize the numbers in Puzzle I (the contested garment problem) and Puzzle II (the estate division problem).
- However, almost all of them fail, including O'Neill (1982).
  - Instead of looking at the specific numerical examples, O'Neill (1982) give a general description for the bankruptcy problems.

The purpose 1/1

- We apply the Nash program to justify the Talmud rule by introducing a non-cooperative game in which bilateral negotiations are resolved by bilateral bargaining procedures.
  - Our design for each bilateral bargaining procedure is based on the concede-and-divide algorithm.

Nash program vs. implementation theory 1/1

- The **Nash program** is a research agenda whose goal is to provide a non-cooperative support for solutions of cooperative games.
    - It provides a justification for a certain payoff vector by an equilibrium of some non-cooperative game.
    - The only actors playing a role are players.
- The general theory of **implementation** concerns the identification of conditions on social choice rules and domains that allow implementability.
    - The designer and her own information play important roles.

The model: formal definition 1/1

$$\varphi\left(N \equiv \{1, \cdots, n\}, c \equiv (c_1, \cdots, c_n) \in \mathbb{R}_+^N, E \in \mathbb{R}_+ \text{ with } \sum_{i \in N} c_i \geq E\right)$$

$$= (x_1, \cdots, x_n) \in \mathbb{R}_+^N \text{ s.t. for each } i \in N,\ 0 \leq x_i \leq c_i, \text{ and } \sum_{i \in N} x_i = E.$$

- The first condition, $0 \leq x_i \leq c_i$, says that creditor i should not receive more than his claim (claims boundedness) and a negative award (non-negativity). The condition is called reasonableness.
- The second condition, $\sum_{i \in N} x_i = E$, says that a rule should allocate the entire resource. This condition is called efficiency or feasibility.

Geometric representation of the model 1/1

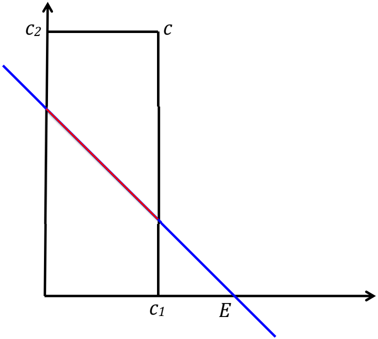


Figure: Bankruptcy problem

Applications of bankruptcy problems 1/1

- Contest garment problem (Puzzle I)
- Estate division problem (Puzzle II)
- Government (institution) budget reallocation problem
- Rationing (group buying) problem
- International (or emergency) aid problem
- Time allocation (supervisor-student) problem

General description of bankruptcy problems

The model can be applied to any situation in which price mechanism is not applicable and a limited resource (the total supply) is insufficient to fulfill the commensurable claims, needs, or demands, of some agents (or an individual) (the total demand).

The central rules 1/2

A number of bankruptcy rules have been proposed. Among them, the following rules are central in our analysis. They are:

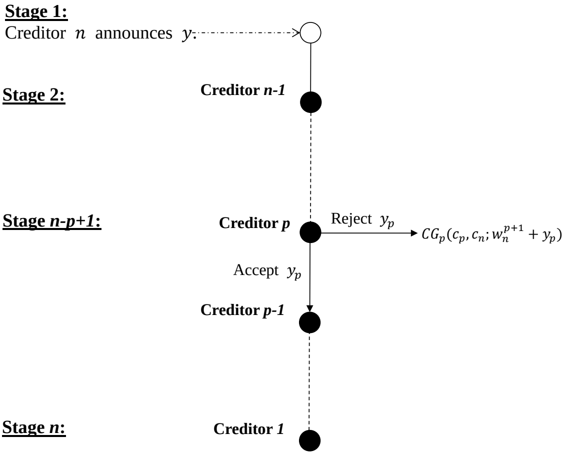
- The Constrained Equal Awards (CEA) rule (egalitarianism from the perspective of gains) assigns equal awards to all creditors subject to no one receiving more than his claim. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.
- The Constrained Equal Losses (CEL) rule (egalitarianism from the perspective of losses) assigns awards such that the loss (the difference between a creditor's claim and award) experienced by each creditor is equal subject to no one receiving a negative award. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

The central rules 2/2

- The Talmud (T) rule (Aumann and Maschler, 1985) rationalizes several numerical examples made in the Talmud, and is a "hybrid" of the CEA and CEL rules. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i(c,E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } \sum_{i \in N} \frac{c_i}{2} \geq E; \\ \frac{c_i}{2} + \max\left\{\frac{c_i}{2} - \lambda, 0\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} T_i\,(c, E) = E$.



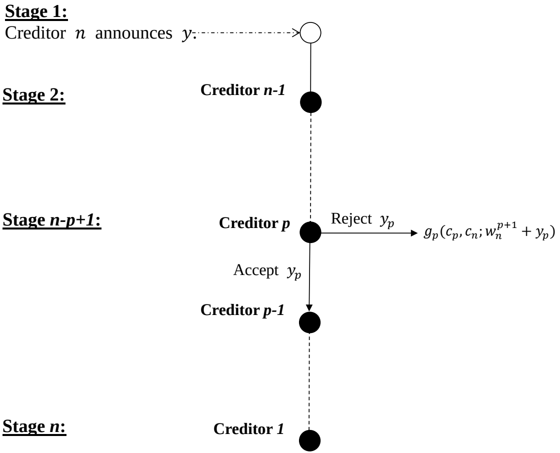
where $w_n^1 = y_n^1$ and,

for each $p = n-1, \ldots, 1$, $w_n^p = \begin{cases} w_n^{p+1} & \textit{if creditor } p \textit{ accepts } y_p; \\ CG_n(c_p, c_n; w_n^{p+1} + y_p) & \textit{otherwise.} \end{cases}$

Serrano (1995) shows that

Theorem: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, the unique Subgame Perfect Equilibrium (SPE) outcome of the game $\Gamma^{CG}(c, E)$ is $T\,(c, E)$.

The game in Dagan, Serrano and Volij (1997) 1/1



where $w_n = y_n$ and,

for each $p = n-1, \dots, 1$, $w_n^p = \begin{cases} w_n^{p+1} & \text{if creditor } p \text{ accepts } y_p; \\ g_n(c_p, c_n; w_n^{p+1} + y_p) & \text{otherwise.} \end{cases}$

The result in Dagan, Serrano and Volij (1997) 1/1

They show that

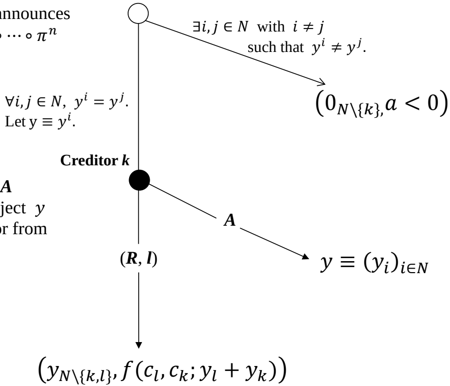
Theorem: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, the unique NE outcome of the game $\Gamma^g(c, E)$ is $g(c, E)$. Moreover, it can be supported by a pure strategy SPE. (g is bilaterally consistent, super-modular, and resource monotone.)

Stage 1:

Each creditor $i \in N$ announces (y^i, π^i). Let $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and $\pi(1) = k$.

Stage 2:

Creditor k either takes ***A*** (accept y) or ***(R, l)*** (reject y and choose one creditor from $N\backslash\{k\}$, say creditor l).



Chang and Hu (2008)'s result 1/1

They show that

Theorem: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, the unique NE outcome of the game $\Gamma^f(c, E)$ is $f(c, E)$. Moreover, it can be supported by a pure strategy SPE. (f is bilaterally consistent and conversely consistent.)

Remarks on existing results 1/2

- Asymmetric treatment and outcome uniqueness: In the games in Serrano (1995) and Dagan et al. (1997), creditor n (the creditor with the largest claim) is the only creditor to propose an awards vector. This is an asymmetric treatment among the creditors (not in line with a level playing field). Moreover, if the proposer is not creditor n, the outcome uniqueness of his result does not hold.
- Undesirability of resolving bilateral negotiations: Invoking division rules to resolve bilateral negotiations is traditional in the literature but not preferable such as Serrano (1995), Dagan et al. (1997), and Chang and Hu (2008). Since the purpose of the Nash program is to justify cooperative solutions through non-cooperative procedures; ideally no cooperative solution should get involved in the details of non-cooperative procedures. Thus, it would be better if bilateral negotiations were resolved by non-cooperative procedures.

Remarks on existing results 2/2

- Harsh outside punishment: In Chang and Hu (2008), if a creditor proposes an awards vector different from others, the proposer receives a negative award and all other creditors zero.
- Unanimous proposals: In Chang and Hu (2008), by introducing a harsh punishment, creditors are forced to announce the same equilibrium strategies and thus their game only focuses on unanimous announcement on awards vectors.
- Privilege of regret: In Chang and Hu (2008), in Stage 2, the coordinator is given privilege of regretting her proposed awards vector in the previous stage due to the harsh punishment. It seems not natural.

Our first goal 1/1

- Our first goal is to strategically justify (or implement) the Talmud rule by introducing a "natural and reasonable" game in which creditors are treated symmetrically and bilateral negotiations are resolved by non-cooperative procedures.

Our second goal 1/1

- It can be seen that the half-claim vector $\frac{c}{2}$ plays an important role in the Talmud rule.
- Aumann and Maschler (1985) justify the vector by invoking legal conventions in the Talmud and psychological presumption (namely, more than half is like the whole and less than half is like nothing).
- However, there is no strategic interpretation of the vector. Our second goal is to fill this gap.

Concede-and-divide algorithm 1/4

- The idea of underlying our game is inspired by the concede-and-divide algorithm (Aumann and Maschler, 1985).
- They mention that for two-creditor problems, the awards vector prescribed in the Talmud for some numerical examples can be obtained by the following concede-and-divide algorithm.
  - To introduce the algorithm, we need to define the minimal award of a creditor.
  - The minimal award of a creditor is the remaining endowment after the other has been fully reimbursed if this remaining endowment is positive; it is zero, otherwise.

Concede-and-divide algorithm 2/4

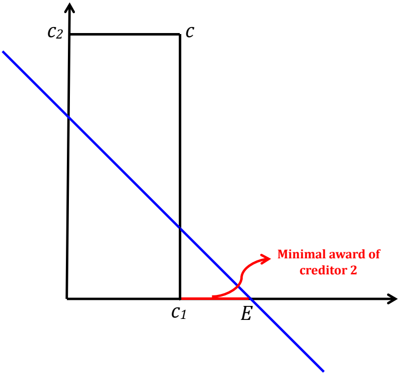


Figure: Geometric illustration of minimal awards

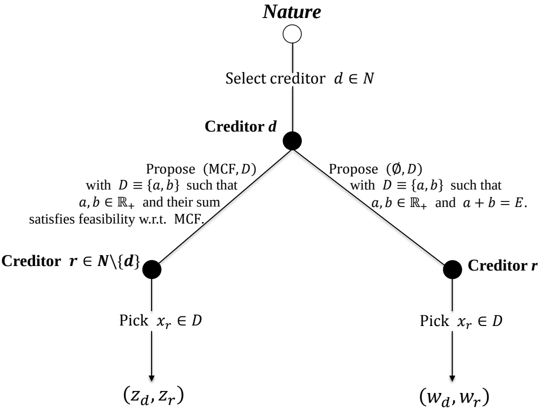
Concede-and-divide algorithm 3/4

- The algorithm assigns first each creditor her minimal award and then divide the remaining endowment equally.

Concede-and-divide algorithm 4/4

- This algorithm actually suggests a two-creditor non-cooperative procedure (or a two-creditor game) that involves a "divide-and-choose mechanism" and the following Minimal Concession First (MCF) process with respect to the minimal awards.
  - To introduce the process, let the perspective of gains be given.
  - The MCF process suggests first assigning each creditor her minimal award and next dividing the residual endowment.

Two-creditor minimal-awards concession game 1/1



where $z_r = max\{E - c_d, 0\} + x_r$ and $z_d = E - z_r$;

$w_r = x_r$ and $w_d = E - w_r$.

Inapplicability of the above concession game 1/2

Example

Let $N \equiv \{1,2\}$ and let $\left(\tilde{c} \equiv (\tilde{c}_1, \tilde{c}_2), \tilde{E}\right) = ((3,5),4)$. We exhibit an NE outcome of $\bar{\Omega}'_T(\tilde{c}, \tilde{E})$ that is not the Talmud outcome $\left(\frac{3}{2}, \frac{5}{2}\right)$. Consider the strategy profile $\bar{\sigma}^{T'} \equiv \left(\bar{\sigma}_1^{T'}, \bar{\sigma}_2^{T'}\right)$: each creditor $i \in N$ takes one of the following strategies. Let $j \in N \setminus \{i\}$.

- ***i*** **is the divider:** She proposes

$$(p^{T',i}, D^{T',i}) = \begin{cases} (\emptyset, \{2,2\}) & \text{if } i = 1 \text{ (the creditor with the smallest claim);} \\ \left(\text{MCF}, \left\{\frac{3}{2}, \frac{3}{2}\right\}\right) & \text{if } i = 2 \text{ (the creditor with the largest claim).} \end{cases}$$

- ***i*** **is the responder:** Given creditor j's proposal (p, D), creditor i picks max D.

Inapplicability of the above concession game 2/2

Example (continued)

Clearly, $\bar{\sigma}^{T'}$ is an SPE of $\bar{\Omega}'_T(\tilde{c}, \tilde{E})$, and is an NE. If *Nature* chooses creditor 1 as the divider, then by following $\bar{\sigma}^{T'}$, the game ends up with outcome $(2, 2) \neq \left(\frac{3}{2}, \frac{5}{2}\right)$. However, if *Nature* chooses creditor 2 as the divider, then by following $\bar{\sigma}^{T'}$, the game ends up with the Talmud outcome $\left(\frac{3}{2}, \frac{5}{2}\right)$. $\square$

Intuition behind the example 1/1

- After the decision on whether or not the MCF is conducted, the divide-and-choose mechanism is adopted to perform a division of the corresponding endowment.
  - Note that the perspective of gains is exogenously given in the above game.
- Given these facts, if creditor 1 (the small creditor) is the divider, she will not conduct the MCF process.
- The game ends up with an outcome that is not the Talmud outcome.
- However, if creditor 2 (the big creditor) is the divider, she will conduct the MCF process. The game ends up with the Talmud outcome.

Overcome the inapplicability and treat creditors symmetrically 1/1

- One way to recover the unique Talmud outcome of the procedure $\bar{\Omega}'_T(c, E)$ is to drop *Nature* and designate creditor 2 as the divider.
- However, by doing so, the strategy spaces of the creditors are different. Thus, the creditors are not treated symmetrically (not in line with a level playing field).

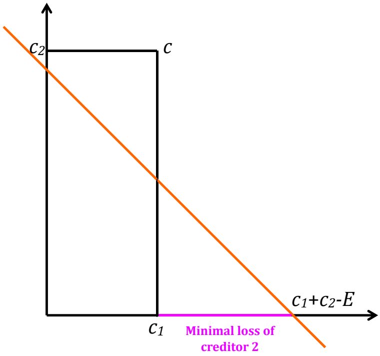
Dual of the concede-and-divide algorithm 1/5

- Alternatively, we could think of the dual of the concede-and-divide algorithm suggested by Aumann and Maschler (1985) to come up with the dual of the procedure $\bar{\Omega}'_T(c, E)$.
- They point out that a bankruptcy problem can be seen either from the perspective of gains observed by creditors, or from the perspective of losses they incur.
  - Namely, (c, E) is the problem seen from the perspective of gains.
  - $((c, \sum c_k - E))$ is the problem seen from the perspective of losses.

Dual of the concede-and-divide algorithm 2/5

- The two problem are dual to each other.
- Thus, we can define the dual of the minimal award of a creditor, called the minimal loss of a creditor.
- The minimal loss of a creditor is the remaining deficit after the other has been fully experienced her maximal loss (namely, her claim) if this remaining deficit is positive; it is zero, otherwise.

Dual of the concede-and-divide algorithm 3/5



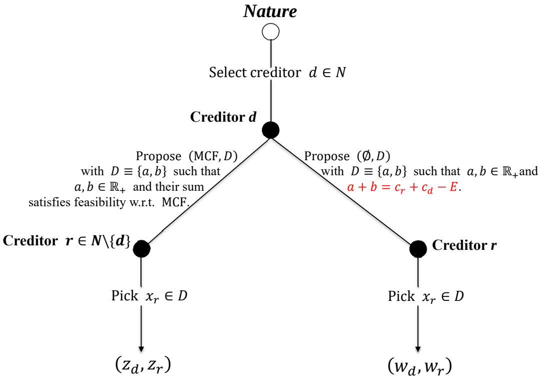
- The dual of the concede-and-divide algorithm is defined as follows:
  - each creditor first experiences her minimal loss;
  - next incurs a further loss obtained by dividing the residual deficit equally (if any);
  - her award is obtained by subtracting her total loss (namely, the sum of her minimal loss and an equal share of the residual deficit) from her claim.

  The dual of the concede-and-divide algorithm suggests a procedure that involves the Minimal Concession First (MCF) process with respect to the minimal losses.

The dual of the concede-and-divide algorithm 5/5

- To introduce the process, let the perspective of losses be given.
- The MCF process suggests first letting each creditor experience her minimal loss and next dividing the residual deficit.
- Thus, the dual of the procedure $\bar{\Omega}'_T(c, E)$ can be defined accordingly.

Two-creditor minimal-awards concession game 1/1



where

$$z_r = c_r - max\{(c_r + c_d - E) - c_d, 0\} - x_r$$

and

$$z_d = E - z_r;$$

$$w_r = c_r - x_r$$

and

$$w_d = E - w_r.$$

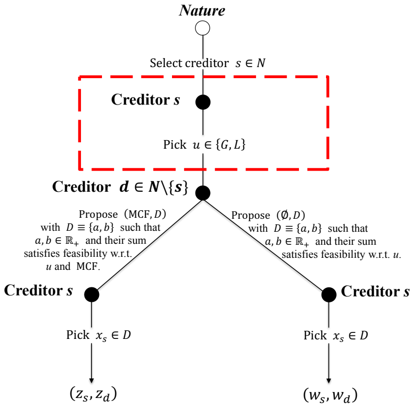
Two-creditor procedure for the Talmud rule 1/3

- One may wonder whether the dual of the procedure $\bar{\Omega}'_T(c, E)$ could help overcome the inapplicability of the procedure $\bar{\Omega}'_T(c, E)$.
- However, it can be shown that a similar argument for the inapplicability of the procedure $\bar{\Omega}'_T(c, E)$ can also be applied to the dual of the $\bar{\Omega}'_T(c, E)$.

Two-creditor procedure for the Talmud rule 2/3

- To treat creditors symmetrically and recover the unique Talmud outcome, we introduce the following procedure $\bar{\Omega}_T$ in which the choice between the perspective of gains and the perspective of losses is determined endogenously rather than exogenously.

Two-creditor procedure for the Talmud rule 3/3



where

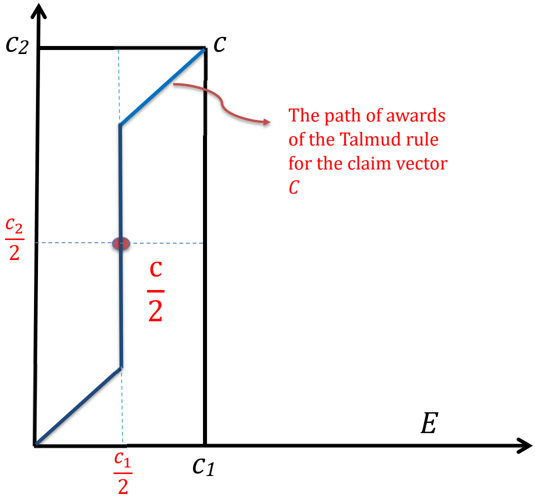
$$z_s = \begin{cases} max\{E - c_d, 0\} + x_s & if\ u = G; \\ c_s - max\{(c_s + c_d - E) - c_d, 0\} - x_s & if\ u = L, \end{cases} \text{ and } z_d = E - z_s;$$

$$w_s = \begin{cases} x_s & if\ u = G; \\ c_s - x_s & if\ u = L, \end{cases} \text{ and } w_d = E - w_s.$$

We show that

Proposition: Let $N \in \mathcal{N}$ with $|N| = 2$ and $(c, E) \in \mathcal{B}^N$. The unique NE outcome of $\bar{\Omega}_T(c, E)$ is $T(c, E)$. Moreover, it can be supported by a pure strategy SPE.

A strategic interpretation of the half-claim vector



A strategic interpretation of the half-claim vector

- Given that the divide-and-choose mechanism is adopted to perform a division of the corresponding endowment or deficit, creditor 1 (the small creditor) prefers the perspective of gains to the perspective of losses; creditor 2 (the big creditor) has the reverse preference.
- Thus, the perspective setter would pick a perspective that is beneficial to her.

A strategic interpretation of the half-claim vector 3/17

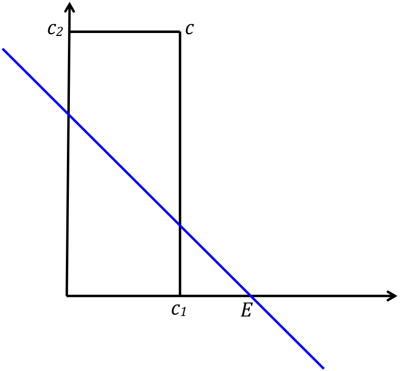


Figure: Creditor 1 is the perspective setter

A strategic interpretation of the half-claim vector

- To balance the advantage given to the setter, the other, called the divider, is allowed to choose either “to conduct”, or “not to conduct” the MCF process.

A strategic interpretation of the half-claim vector

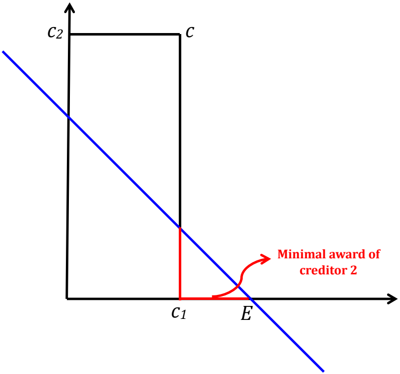
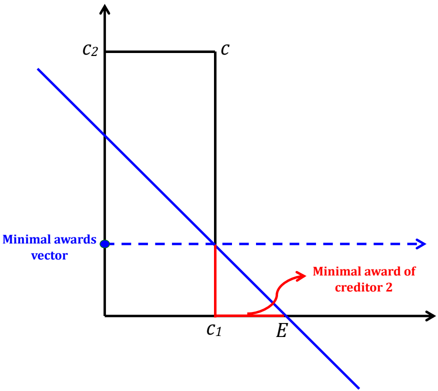


Figure: Creditor 1 is the perspective setter and creditor 2 is the divider

A strategic interpretation of the half-claim vector



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A strategic interpretation of the half-claim vector

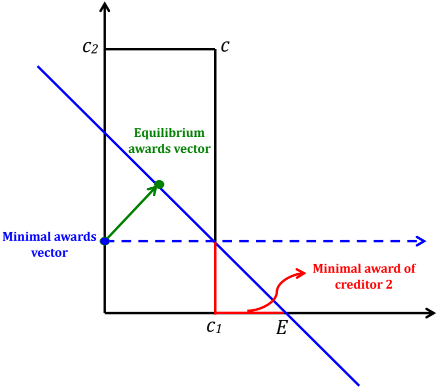


Figure: Creditor 1 is the perspective setter and creditor 2 is the divider

A strategic interpretation of the half-claim vector

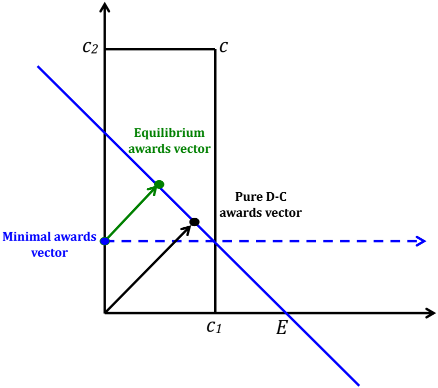


Figure: Creditor 1 is the perspective setter and creditor 2 is the divider

A strategic interpretation of the half-claim vector

- Thus, if the perspective of gains is chosen, the big creditor prefers "conducting" to "not conducting" the MCF process.

A strategic interpretation of the half-claim vector

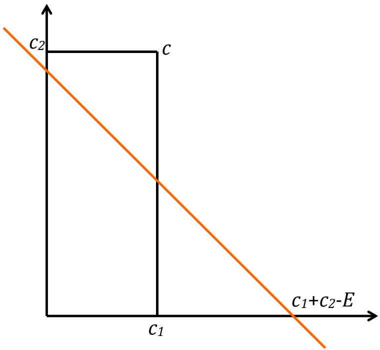


Figure: Creditor 2 is the perspective setter

A strategic interpretation of the half-claim vector

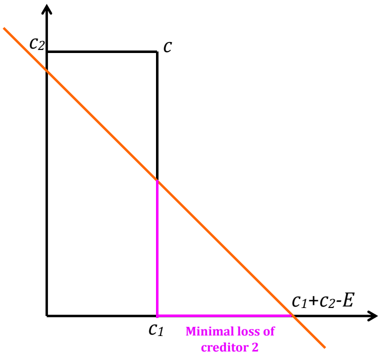


Figure: Creditor 2 is the perspective setter and creditor 1 is the divider

A strategic interpretation of the half-claim vector

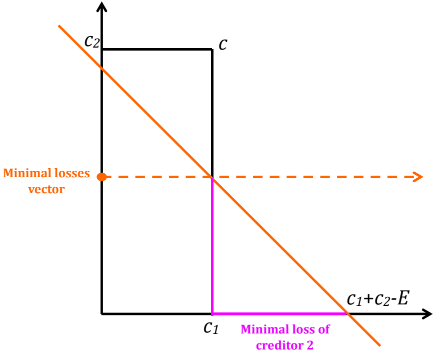


Figure: Creditor 2 is the perspective setter and creditor 1 is the divider

A strategic interpretation of the half-claim vector

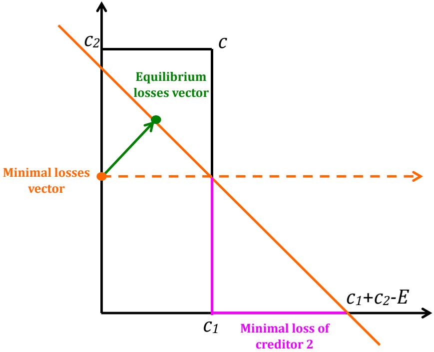


Figure: Creditor 2 is the perspective setter and creditor 1 is the divider

A strategic interpretation of the half-claim vector

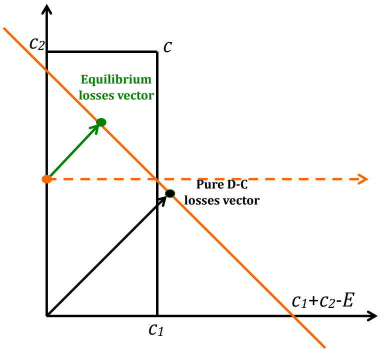


Figure: Creditor 2 is the perspective setter and creditor 1 is the divider

A strategic interpretation of the half-claim vector

- Thus, if the perspective of losses is chosen, the small creditor prefers "conducting" to "not conducting" the MCF process.

A strategic interpretation of the half-claim vector

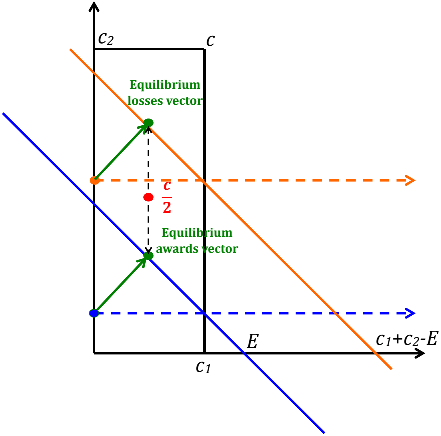


Figure: Relation between equilibrium awards and equilibrium losses

A strategic interpretation of the half-claim vector

- Thus, the half-claim vector is a consequence of balancing advantages between the creditors and exploiting the divide-and-choose mechanism.

A strategic implementation of the Talmud rule 1/3

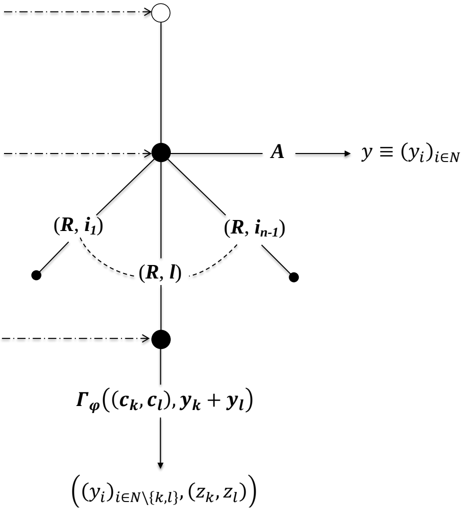
- We now extend our base result to more than two creditors by introducing the following tree-stage extensive form game.

A strategic implementation of the Talmud rule 2/3

Stage 1:
Each creditor $i \in N$ announces (y^i, π^i). Let $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and $\pi(1) = k$. Let y be the proposal. If for each $i, h \in N \backslash \{k\}$, $y^i = y^h$, then $y = y^i$; otherwise, $y = y^k$.

Stage 2:
Creditor k either takes $\boldsymbol{A}$ (accepts y) or $(\boldsymbol{R}, \boldsymbol{l})$ (rejects y and chooses one creditor from $N \backslash \{k\}$, say creditor l).

Stage 3:
Each creditor $i \in N \backslash \{k, l\}$ receives y_i, and creditors k and l play the two-creditor game $\Gamma_\varphi((c_k, c_l), y_k + y_l)$. Let (z_k, z_l) be an outcome of $\Gamma_\varphi((c_k, c_l), y_k + y_l)$.



A strategic implementation of the Talmud rule 3/3

Replace φ with T in the above game Ω_φ and call the resulting game Ω_T. We show that

Theorem: Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. The unique NE outcome of $\Omega_T(c, E)$ is $T(c, E)$. Moreover, it can be supported by a pure strategy SPE.

Concluding remarks 1/1

Our paper contributes to the literature as follows:

- Opening up black boxes of bilateral negotiations
- Obtaining an exact strategic implementation of the Talmud rule rather than in expected term
- Offering a strategic interpretation of the half-claim vector in the Talmud rule

Thank you!!