On the Maximal Domain Theorem*

Yi-You Yang†

April 28, 2016

Abstract

The maximal domain theorem by Gul and Stacchetti (J. Econ. Theory **87** (1999), 95-124) shows that for markets with indivisible objects and sufficiently many agents, the set of gross substitutable preferences is a largest set for which the existence of a competitive equilibrium is guaranteed. This implies that no relaxation of the gross substitutability can guarantee an equilibrium to exist. However, we note that there is a flaw in their proof, and give an example to show that a claim used in the proof may fail to be true. We correct the proof and improve the result by showing that even there are only two agents in the market, if the preferences of one agent are not gross substitutable, then gross substitutable preferences can be found for another agent such that no competitive equilibrium exists. Moreover, we introduce a weaker condition, the notion of implicit gross substitutability, which can guarantee the existence of a competitive equilibrium when the preferences of some agent are monotone.

Keywords: Maximal domain; competitive equilibrium; gross substitutability.

*Support by National Science Council of Republic of China under grant NSC 102-2410-H-156-001 is gratefully acknowledged.

†Department of Finance and Actuarial Science, Aletheia University, New Taipei City 25103, Taiwan. E-mail address: yyyang@mail.au.edu.tw

1 Introduction

An essential issue for markets with heterogeneous indivisible objects is under which conditions an efficient allocation of objects can be supported by a system of competitive prices as an equilibrium outcome.[1] A sufficient condition for the existence of a competitive equilibrium is the gross substitutability (GS) condition, which requires that objects are substitutes in the sense that the demand of each agent for an object does not decrease when prices of some other objects increase. Kelso and Crawford [7] introduce a price adjustment procedure and prove that under gross substitutable preferences, such procedure will give rise to a competitive equilibrium.

Gul and Stacchetti [3] study markets with monotone preferences by adopting a less restrictive condition, the weak gross substitutability (WGS) condition, which requires that agents view objects as substitutes for each other when prices are non-negative. Based on the Kelso-Crawford price adjustment procedure, they note that under the monotonicity assumption, WGS preferences are sufficient for the existence of a competitive equilibrium. Moreover, they prove that the WGS condition is also necessary in the maximal domain sense: for a market with sufficiently many agents, if the preferences of some agent violate the WGS condition, then WGS preferences can be found for other agents such that no competitive equilibrium exists.

Nevertheless, we note that there is a flaw in the proof of Gul-Stacchetti maximal domain theorem, and present an example to show that a claim used in the proof may fail to be true. To correct the proof, we give an equivalent characterization of the GS condition[2] and an alternative maximal domain result which shows that if the preferences

[1]A sampling of relevant works includes Kelso and Crawford [7], Bikhchandani and Mamer [2], Ma [8], Beviá et al. [1], Gul and Stacchetti [3, 4], Sun and Yang [10], and Teytelboym [11].

[2]See Theorem 3 for details.

of some agent fail the GS condition, we can construct GS preferences for another agent such that no competitive equilibrium exists in the two-agent market. This implies that even for markets with few agents, no relaxation of the GS condition (or of the WGS condition together with the monotonicity assumption[3]) can guarantee the existence of a competitive equilibrium.

On the one hand, our result improves upon the Gul-Stacchetti maximal domain theorem, but on the other hand, it makes it seem more difficult to give new existence results with conditions weaker then the GS condition. One way to circumvent the difficulty is to consider the markets in which not all agents have monotone preferences. It should be noted that, while monotonicity of preferences is a commonly used assumption in the literature, there are numerous economic situations in which monotonicity is not always satisfied.[4] For instance, an extra bed might be a burden for an agent with a small house. We introduce the notion of implicit gross substitutability (IGS), which requires that allowing agents to dispose of undesirable objects for free will make objects become substitutes, and thus exhibits substitutability in an implicit way. We prove that the IGS condition is weaker than the WGS condition, and yet is sufficient for the existence of a competitive equilibrium when there exists an agent with monotone preferences.

The rest of the paper is organized as follows. In Section 2, we recall the Gul-Stacchetti maximal domain theorem and give an example to show that there is a flaw in the proof. In Section 3, we give an alternative proof and prove new maximal domain results with a characterization of the gross substitutability. Finally, we provide a new existence theorem based on the IGS condition in Section 4, and present a proof in the Appendix.

[3]We prove that under monotonicity, GS and WGS are equivalent. See Lemma 2 (b) for details.

[4]See Manelli [9] and Hara [5, 6] for discussions on markets without the monotonicity assumption.

2 Gross substitutability as a maximal domain

Consider an economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of heterogeneous indivisible objects. Let $p = (p_a) \in \mathbb{R}^{|\Omega|}$ be a price vector, where p_a denotes the price of object $a \in \Omega$, which is allowed to be negative. For any bundle of objects $A \subseteq \Omega$, let $\chi_A \in \mathbb{R}^{|\Omega|}$ denote the characteristic price vector that has price 1 for objects $a \in A$ and price 0 for objects $a \notin A$. We assume that agents' net utility functions are quasilinear in prices in the sense that each agent i's utility of consuming bundle $A \subseteq \Omega$ at price level p is

$$u_i(A, p) \equiv v_i(A) - p(A),$$

where $v_i : 2^\Omega \to \mathbb{R}$ is a valuation function satisfying $v_i(\emptyset) = 0$ and $p(A)$ is a shorthand for $\sum_{a \in A} p_a$. We also assume that agents are not subject to any budget constraints, and hence we can represent such a trading economy by $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$.

A valuation function v_i is called *monotone* if $v_i(B) \leq v_i(A)$ for $B \subseteq A \subseteq \Omega$. The *monotone cover* of v_i is the valuation function $\hat{v}_i$ defined by

$$\hat{v}_i(A) = \max\{v_i(C) : C \subseteq A\} \text{ for } A \subseteq \Omega.$$

Clearly, a valuation function v_i is monotone if and only if $v_i = \hat{v}_i$.

A *competitive equilibrium* for economy $\mathcal{E}$ is a pair $\langle p; \mathbf{X} \rangle$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is a partition of objects among all agents and p is a price vector such that for all $i \in N$,

$$X_i \in D_{v_i}(p) \equiv \arg\max_{A \subseteq \Omega} u_i(A, p).$$

In that case, $\mathbf{X}$ is called an equilibrium allocation and p is called an equilibrium price vector. The possibility that $X_i = \emptyset$ for some agent i is allowed.

A crucial condition for the guaranteed existence of a competitive equilibrium is the *gross substitutability.* Formally, a valuation function $v_i : 2^\Omega \to \mathbb{R}$ is called *gross substitutable* (GS) if for any vector $p \in \mathbb{R}^{|\Omega|}$, the following condition holds:

$$A \in D_{v_i}(p), p' \geq p \Rightarrow \exists B \in D_{v_i}(p') \text{ such that } \{a \in A : p_a = p'_a\} \subseteq B. \tag{1}$$

Moreover, we say that v_i is *weakly gross substitutable* (WGS) if condition (1) holds for all non-negative vectors $p \in \mathbb{R}_+^{|\Omega|}$. Note that WGS is strictly weaker than GS. Consider the function $v_i : 2^\Omega \to \mathbb{R}$ given by $\Omega = \{a, b, c\}$ and

$$v_i(A) = \begin{cases} 2, & \text{if A=\{a\},} \\ 1, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that v_i is weakly gross substitutable, but violates the GS condition.

Kelso and Crawford [7] introduce a price adjustment procedure and show that under gross substitutable preferences, such procedure will give rise to a competitive equilibrium. More precisely, a direct application of Theorem 2 of Kelso and Crawford [7] leads to the following result.

Theorem 1 (Kelso-Crawford) *For the economy $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$, there exists a competitive equilibrium if one of the following conditions holds:*

(a) *each agent i's valuation function v_i satisfies the GS condition.*

(b) *each agent i's valuation function v_i is monotone and satisfies the WGS condition.*

On the other hand, Theorem 2 of Gul and Stacchetti [3] shows that when there are sufficiently many agents and each agent's preferences are assumed to be monotone, the set of WGS preferences is a maximal domain for which the existence of a competitive equilibrium is guaranteed.

Theorem 2 (Gul-Stacchetti) *Let* $v_1 : 2^{\Omega} \to \mathbb{R}$ *be a monotone valuation function that violates the WGS condition. Then there exists an n-agent economy* $\mathcal{E} = \langle \Omega; (v_1, \ldots, v_n) \rangle$ *such that* v_i *is monotone and satisfies the WGS condition for* $i = 2, \ldots, n$*, but no competitive equilibrium exists in* $\mathcal{E}$*.*[5]

To prove the above maximal domain theorem, Gul and Stacchetti [3, pp. 122-123] claim that if there exist a pair of bundles $A, B \subseteq \Omega$ and a vector $p \in \mathbb{R}^{|\Omega|}$ such that the following conditions hold:

(i) $|A \backslash B| > 1$ and $B \backslash A = \{b\}$ for some $b \in \Omega$;

(ii) $u_1(B, p) > u_1(A, p)$;

(iii) $C \subseteq \Omega$ and $u_1(C, p) > u_1(A, p) \Rightarrow |A \backslash C| + |C \backslash A| \geq |A \backslash B| + 1$;

then no competitive equilibrium exists in the economy $\langle \Omega; (v_1, v_2, v_3, v_{a_1}, \ldots, v_{a_r}) \rangle$ given by $\{a_1, \ldots, a_r\} = \Omega \backslash [A \cup B]$,

$$v_2(C) = \begin{cases} 0, & \text{if } C \cap (A \backslash B) = \emptyset, \\ \max\{p_a + v_1(\Omega) + 1 : a \in C \cap (A \backslash B)\}, & \text{otherwise,} \end{cases}$$

$$v_3(C) = \begin{cases} 0, & \text{if } C \cap [(A \backslash B) \cup \{b\}] = \emptyset, \\ \max\{p_a + v_1(\Omega) + 1 : a \in C \cap [(A \backslash B) \cup \{b\}]\}, & \text{otherwise,} \end{cases}$$

[5] According to Gul and Stacchetti's proof, the number of agents required to construct such an economy $\mathcal{E}$ is endogenous, and might be very large.

and,

$$v_{a_j}(C) = \begin{cases} v_1(\Omega) + 1, & \text{if } a_j \in C, \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, r$.

However, the following example shows that the claim is not correct.

Example 1 *Let $\Omega = \{a, b, c\}, A = \{a, c\}, B = \{b\}$, and let $p \in \mathbb{R}^{|\Omega|}$ be the vector such that $p_a = p_c = 2$ and $p_b = 1$. Consider the economy $\mathcal{E} = \langle \Omega; (v_1, v_2, v_3) \rangle$ given by*

$$v_1(C) = \begin{cases} 7, & \textit{if } C = \{a, c\} \textit{ or } \{a, b, c\}, \\ 5, & \textit{if } C = \{b\}, \textit{ or } \{a, b\} \textit{ or } \{b, c\}, \\ 3, & \textit{if } C = \{a\} \textit{ or } \{c\}, \\ 0, & \textit{if } C = \emptyset, \end{cases}$$

and

$$v_2(C) = \begin{cases} 10, & \textit{if } C \cap A \neq \emptyset, \\ 0, & \textit{otherwise,} \end{cases} \qquad v_3(C) = \begin{cases} 10, & \textit{if } C \cap A \neq \emptyset, \\ 9, & \textit{if } C = \{b\}, \\ 0, & \textit{if } C = \emptyset. \end{cases}$$

Note that conditions (i)-(iii) are satisfied, but the allocation $X_1 = \{b\}, X_2 = \{a\}, X_3 = \{c\}$ can be supported by prices $p_a = p_c = 2$ and $p_b = 1$ as an equilibrium allocation.

3 A correct proof of Theorem 2

Our approach relies on the notion of improvability, which requires that any suboptimal bundle $A \subseteq \Omega$ at price level $p \in \mathbb{R}^{|\Omega|}$ can be strictly improved by either removing an object from it, or adding some objects to it, or doing both. It should be noted that our improvability condition is similar to but apparently weaker than the single improvement (SI) condition by Gul and Stacchetti [3]. Formally, a valuation function $v_i : 2^\Omega \to \mathbb{R}$ is said to be *improvable* if for any price vector $p \in \mathbb{R}^{|\Omega|}$ and for any bundle $A \in 2^\Omega \backslash D_{v_i}(p)$, there exists a bundle $B \subseteq \Omega$ such that $|A \backslash B| \leq 1$ and $u_i(B,p) > u_i(A,p)$. We begin our analysis with a useful lemma.

Lemma 1 *Let $v_i : 2^\Omega \to \mathbb{R}$ be a valuation function. If v_i is not improvable, then there exist a price vector $p \in \mathbb{R}^{|\Omega|}$ and a pair of bundles $A, B \in D_{v_i}(p)$ such that $|A \backslash B| \geq 2$ and $A \subseteq C \subseteq A \cup B$ for all $C \in D_{v_i}(p) \backslash \{B\}$.*

Proof. Assume that v_i is not improvable. By definition, there exist a price vector $q \in \mathbb{R}^{|\Omega|}$ and a bundle $A \notin D_{v_i}(q)$ such that $|A \backslash C| \geq 2$ for all $C \in \Gamma_i(A,q) \equiv \{C \subseteq \Omega : u_i(C,q) > u_i(A,q)\}$. Let $B \in \Gamma_i(A,q)$ be a bundle satisfying

(i) $|A \backslash B| \leq |A \backslash C|$ for all $C \in \Gamma_i(A,q)$, and

(ii) $|C \backslash A| \geq |B \backslash A|$ for all $C \in \Gamma_i(A,q)$ with $A \cap C = A \cap B$.

Consider the vector

$$p = q + \varepsilon \chi_{\Omega \backslash (A \cup B)} - \varepsilon \chi_{A \cap B} - \frac{u_1(B,q) - u_1(A,q)}{|A \backslash B|} \chi_{A \backslash B}.$$

Then there exists a large number $\varepsilon > 0$ such that

$$A \in D_{v_i}(p) = \{B\} \cup \{C \subseteq \Omega : u_i(C,q) = u_i(A,q), A \subseteq C \subseteq A \cup B\}.$$

This completes the proof. ■

The following result shows that the notion of improvability is equivalent to the GS condition.

Theorem 3 *A valuation function* $v_i : 2^\Omega \to \mathbb{R}$ *satisfies the GS condition if and only if it is improvable.*

Proof. See Appendix A. ■

We then prove that even for markets with only two agents, the existence of a competitive equilibrium cannot be guaranteed by any relaxation of the GS condition.

Theorem 4 *Let* $v_1 : 2^\Omega \to \mathbb{R}$ *be a valuation function that violates the GS condition. Then there exists a GS valuation function* v_2 *such that no competitive equilibrium exists in the two-agent economy* $\mathcal{E} = \langle \Omega; (v_1, v_2) \rangle$.

Proof. Since v_1 violates the GS condition, the combination of Theorem 3 and Lemma 1 implies that v_1 is not improvable, and hence there exist a price vector $p \in \mathbb{R}^{|\Omega|}$ and a pair of bundles $A, B \in D_{v_i}(p)$ such that $|A \backslash B| \geq 2$ and $A \subseteq C \subseteq A \cup B$ for all $C \in D_{v_i}(p) \backslash \{B\}$.

Let $M = 1 + \sum_{C \in 2^\Omega} |v_1(C)| + \sum_{a \in \Omega} |p_a|$ and let $\bar{p} \in \mathbb{R}^{|\Omega|}$ be the vector given by

$$\bar{p}_a = \begin{cases} p_a, & \text{if } a \in B, \\ p_a - \delta, & \text{if } a \in A \backslash B, \\ M, & \text{otherwise,} \end{cases}$$

where

$$\delta = \frac{1}{|A \backslash B|} \min\{u_1(A, p) - u_1(C, p) : C \notin D_{v_1}(p)\} > 0.$$

Let v_2 be the valuation function given by

$$v_2(C) = \bar{p}(C) + \begin{cases} M, & \text{if } C \cap (A \backslash B) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, v_2 is gross substitutable since it is the sum of an additive function and a unit-demand function.[6]

We are going to prove that no competitive equilibrium exists in the economy $\mathcal{E} = \langle \Omega; (v_1, v_2) \rangle$. Suppose, to the contrary, that there exists an equilibrium $\langle p^*; (X_1, X_2) \rangle$ for $\mathcal{E}$. Since the equilibrium allocation (X_1, X_2) must be efficient, we have $X_1 \subseteq A \cup B, X_2 \cap (A \backslash B) \neq \emptyset$. This implies $X_1 \notin D_{v_1}(p)$ and hence $u_1(X_1, p) + \delta |A \backslash B| \leq u_1(B, p)$. Moreover, let $p' \in \mathbb{R}^{|\Omega|}$ be the vector given by

$$p'_a = \begin{cases} p_a, & \text{if } a \in B, \\ p^*_a, & \text{otherwise,} \end{cases}$$

then $\langle p'; (X_1, X_2) \rangle$ is also a competitive equilibrium for $\mathcal{E}$. We consider two cases.

Case I. $X_1 \backslash B \neq \emptyset$. Note that for each $a \in X_1 \backslash B \subseteq A \backslash B$, we have $p'_a \geq v_2(X_2 \cup \{a\}) - v_2(X_2) = \bar{p}_a = p_a - \delta$. This implies

$$\begin{aligned} u_1(X_1, p') &\leq u_1(X_1, p) + \delta |X_1 \backslash B| < u_1(X_1, p) + \delta |A \backslash B| \\ &\leq u_1(B, p) = u_1(B, p'), \end{aligned}$$

[6] Recall that a valuation function $v_i : 2^\Omega \to \mathbb{R}$ is *additive* if there exists a vector $p \in \mathbb{R}^{|\Omega|}$ such that $v_i(C) = p(C)$ for all $C \subseteq \Omega$, and that a monotone function v_i is *unit-demand* if $v_i(C) = max_{a \in C} v_i(\{a\})$ for all $C \subseteq \Omega$. One can easily check that the sum of an additive function and a unit-demand function is gross substitutable.

contracting the fact $X_1 \in D_{v_1}(p')$.

Case II. $X_1 \subseteq B$. Then we have $A \backslash B \subseteq X_2$. Since $|A \backslash B| \geq 2$, it follows that for each $a \in A \backslash B$, $p'_a \leq v_2(X_2) - v_2(X_2 \backslash \{a\}) = \bar{p}_a = p_a - \delta$. This implies

$$u_1(A, p') \geq u_1(A, p) + \delta |A \backslash B| > u_1(X_1, p) = u_1(X_1, p').$$

This is also impossible. ■

The following result implies the Gul-Satcchetti maximal domain theorem (Theorem 2) and shows that even for markets with few agents, no relaxation of the WGS condition, together with the monotonicity, can ensure the existence of a competitive equilibrium.

Theorem 5 *Let $v_1 : 2^{\Omega} \to \mathbb{R}$ be a monotone valuation function that violates the WGS condition. Then for any positive integer $n \geq 2$, there exists a sequence of monotone and WGS valuation functions, $\{v_2, \ldots, v_n\}$, such that no competitive equilibrium exists in the economy $\mathcal{E} = \langle \Omega; (v_1, v_2, \ldots, v_n) \rangle$.*[7]

Proof. Since v_1 violates the WGS condition, it violates the GS condition as well. By Theorem 4, there exists a GS valuation function w_2 such that no competitive equilibrium exists in the economy $\langle \Omega; (v_1, w_2) \rangle$.

It is not difficult to see that the monotone cover $\hat{w}_2$ of w_2 satisfies the GS condition. Let w_3 be the valuation function given by $w_3(A) = 0$ for $A \subseteq \Omega$, and let w_4 be an arbitrary GS valuation function. By Theorem 1, we know that there exists a competitive equilibrium $\langle p; (X_2, X_3, X_4) \rangle$ for the economy $\langle \Omega; (w_2, w_3, w_4) \rangle$. For each bundle $A \subseteq \Omega$, let A' denote a subset of A such that $\hat{w}_2(A) = w_2(A') = \hat{w}_2(A')$. Then for any $A \subseteq \Omega$, we

[7]In contrast to the Gul-Stacchetti maximal domain theorem, the number of agents in this theorem is exogenously given.

have

$$
\begin{aligned}
\hat{w}_2(X_2 \cup X_3) - p(X_2 \cup X_3) &\geq [w_2(X_2) - p(X_2)] + [w_3(X_3) - p(X_3)] \\
&\geq [w_2(A') - p(A')] + [w_3(A \backslash A') - p(A \backslash A')] \\
&= \hat{w}_2(A) - p(A).
\end{aligned}
$$

This implies that $\langle p; (X_2 \cup X_3, X_4) \rangle$ is a competitive equilibrium for economy $\langle \Omega; (\hat{w}_2, w_4) \rangle$. Since w_4 is an arbitrary GS valuation function, the result of Theorem 4 implies that $\hat{w}_2$ satisfies the GS condition.

Consider the economy $\mathcal{E} = \langle \Omega; (v_1, \ldots, v_n) \rangle$, where $v_2 = \hat{w}_2$ and $v_i = w_3$ for $i \geq 3$. We are going to prove that no competitive equilibrium exists in $\mathcal{E}$. Suppose, to the contrary, that there is a competitive equilibrium $\langle q; (Y_1, \ldots, Y_n) \rangle$ for $\mathcal{E}$. Since each agent's valuation function is monotone, we have $q_a \geq 0$ for all $a \in \Omega$, and without loss of generality, we may assume that $Y_i = \emptyset$ for $i \geq 3$.

Let Y_2' be a subset of Y_2 such that $\hat{w}_2(Y_2) = w_2(Y_2') = \hat{w}_2(Y_2')$. Then for any $A \subseteq \Omega$, we have

$$
\begin{aligned}
w_2(A) - q(A) \leq v_2(A) - q(A) \leq v_2(Y_2) - q(Y_2) &= v_2(Y_2') - q(Y_2') - q(Y_2 \backslash Y_2') \\
\leq v_2(Y_2') - q(Y_2') = w_2(Y_2') - q(Y_2'),&
\end{aligned}
$$

which implies $Y_2' \in D_{w_2}(q)$ and $q_a = 0$ for all $a \in Y_2 \backslash Y_2'$. Since v_1 is monotone, it follows that $Y_1 \cup (Y_2 \backslash Y_2') \in D_{v_1}(q)$, contradicting to the fact that no competitive equilibrium exists in $\langle \Omega; (v_1, w_2) \rangle$. ■

Finally, we close this section with an analogue of Theorem 5, which shows that for markets without the monotonicity assumption, the set of GS preferences is a maximal domain for which the existence of a competitive equilibrium is guaranteed.

Theorem 6 *Let $v_1 : 2^\Omega \to \mathbb{R}$ be a valuation function that violates the GS condition. Then for any positive integer $n \geq 2$, there exists a sequence of GS valuation functions, $\{v_2, \ldots, v_n\}$, such that no competitive equilibrium exists in the economy $\langle \Omega; (v_1, v_2, \ldots, v_n) \rangle$.*

The proof of Theorem 6 requires the following lemma, which implies that under the monotonicity assumption, WGS and GS are equivalent.

Lemma 2 *Let $v_1 : 2^\Omega \to \mathbb{R}$ be a WGS valuation function.*

(a) *The monotone cover $\hat{v}_1$ of v_1 satisfies the GS condition.*

(b) *v_1 satisfies the GS condition if it is monotone.*

Proof. (a) Let v_1 be a WGS valuation function. Consider the price adjustment procedure of Kelso and Crawford [7] for the economy $\mathcal{E} = \langle \Omega; (v_1, v_2, v_3) \rangle$, where v_2 is the valuation function given by $v_2(A) = 0$ for all $A \subseteq \Omega$ and v_3 is an arbitrary GS valuation function. Since v_2 is monotone and each valuation function satisfies WGS, it follows that each object will receive at least one offer at the initial zero price vector $\mathbf{0} \in \mathbb{R}^{|\boldsymbol{\Omega}|}$ and the procedure will terminate at a competitive equilibrium $\langle p; (X_1, X_2, X_3) \rangle$ such that $p \in \mathbb{R}_+^{|\Omega|}$ and $p_a = 0$ for $a \in X_2$. For any bundle $A \subseteq \Omega$, let A' be a subset of A such that $\hat{v}_1(A) = v_1(A')$. Note that $v_2(X_2) - p(X_2) \geq 0$. This implies that for any $A \subseteq \Omega$, $\hat{v}_1(X_1 \cup X_2) - p(X_1 \cup X_2) \geq [v_1(X_1) - p(X_1)] + [v_2(X_2) - p(X_2)] \geq v_1(A') - p(A') = \hat{v}_1(A) - p(A) + p(A \backslash A') \geq \hat{v}_1(A) - p(A)$, and hence $\langle p; (X_1 \cup X_2, X_3) \rangle$ is a competitive equilibrium for the economy $\langle \Omega; (\hat{v}_1, v_2) \rangle$. Together with Theorem 4, it follows that $\hat{v}_1$ satisfies GS.

(b) Note that if v_1 is monotone, then $v_1 = \hat{v}_1$. By the result of (a), we immediately obtain the desired result. ■

We are now ready to prove Theorem 6.

Proof of Theorem 6. Let $q \in \mathbb{R}^{|\Omega|}$ be the vector given by $q_a = \min_{a \in A \subseteq \Omega}[v_1(A) - v_1(A \backslash \{a\})]$ for $a \in \Omega$. Then the valuation function w_1 defined by

$$w_1(A) = v_1(A) - q(A) \text{ for } A \subseteq \Omega$$

is monotone, but violates the GS condition.

The combination of Lemma 2 (b) and Theorem 5 implies that w_1 violates the WGS condition, and hence there exists a sequence of monotone and GS valuation functions $\{w_2, \ldots, w_n\}$ such that no competitive equilibrium exists in $\langle \Omega; (w_1, \ldots, w_n) \rangle$. For $i = 2, \ldots, n$, let v_i be the valuation function defined by $v_i(A) = w_i(A) + q(A)$ for $A \subseteq \Omega$. Clearly, v_i satisfies the GS condition for $i = 2, \ldots, n$, but no competitive equilibrium exists in $\langle \Omega : (v_1, \ldots, v_n) \rangle$. ■

4 Beyond the gross substitutability

The foregoing maximal domain theorems make it seem difficult to establish existence results with relaxations of the GS condition. In this section, we try to circumvent the difficulty with the notion of implicit gross substitutability (IGS), which is inspired by the idea of "free disposal" condition. We are going to see that the IGS condition is weaker than the WGS condition, and yet is sufficient to guarantee a competitive equilibrium to exist when the preferences of some agent are known to be monotone.

Monotonicity of preferences is a commonly used assumption in the economic literature. This assumption can be justified by offering free disposal of unwanted objects. In that case, possessing more objects does not make any agent worse off, and each agent i's original valuation function v_i would thereby be replaced by its monotone cover $\hat{v}_i$.

A valuation function $v_i : 2^\Omega \to \mathbb{R}$ is called *implicitly gross substitutable* (IGS) if its monotone cover $\hat{v}_i$ is gross substitutable. Roughly speaking, the IGS condition requires that allowing agents to dispose of undesirable objects for free will make objects become substitutes for each other, and thus exhibits substitutability in an implicit way. The result of Lemma 2 (a) shows that IGS is weaker than WGS.

We now give our new existence result, Theorem 8, in which we try to extend the Kelso-Crawford equilibrium existence theorem (Theorem 1) with the IGS condition. Our idea is based on a more general observation which shows that when there exists an agent with monotone preferences, the existence of a competitive equilibrium is irrelevant to whether agents are allowed to dispose of undesirable objects for free.

Theorem 7 *Let $\mathcal{E} = \langle \Omega; (v_i, i \in N)\rangle$ be an economy and denote $\hat{\mathcal{E}} \equiv \langle \Omega; (\hat{v}_i, i \in N)\rangle$. If v_1 is monotone, then the following results hold:*

(a) *Each equilibrium allocation for $\mathcal{E}$ is also an equilibrium allocation for $\hat{\mathcal{E}}$.*

(b) *Each equilibrium price vector for $\hat{\mathcal{E}}$ is also an equilibrium price vector for $\mathcal{E}$.*

(c) *$\mathcal{E}$ has a competitive equilibrium if and only if $\hat{\mathcal{E}}$ has a competitive equilibrium.*

Proof. (a) Assume that $\langle p; \mathbf{X}\rangle$ is a competitive equilibrium for $\mathcal{E}$. We are going to prove that $\mathbf{X}$ is an equilibrium allocation for $\hat{\mathcal{E}}$. Let $p' \in \mathbb{R}_+^{|\Omega|}$ be the price vector given by

$$p'_a = \begin{cases} p_a, & \text{if } p_a \geq 0, \\ 0, & \text{if } p_a < 0. \end{cases}$$

We first prove that $\langle p'; \mathbf{X}\rangle$ is a competitive equilibrium for $\mathcal{E}$. Let $\bar{A} = \{a \in \Omega : p_a < 0\}$. In case there exists $a \in \bar{A}\backslash X_1$, since v_1 is monotone, we have

$$v_1(X_1 \cup \{a\}) - p(X_1 \cup \{a\}) \geq v_1(X_1) - p(X_1) - p_a > v_1(X_1) - p(X_1),$$

violating the fact $X_1 \in D_{v_1}(p)$. This implies $\bar{A} \subseteq X_1$, and hence we have $X_i \in D_{v_i}(p')$ for $i \neq 1$ and for each bundle $A \subseteq \Omega$,

$$\begin{aligned} v_1(X_1) - p'(X_1) &= [v_1(X_1) - p(X_1)] + p(\bar{A}) \geq [v_1(A \cup \bar{A}) - p(A \cup \bar{A})] + p(\bar{A}) \\ &= v_1(A \cup \bar{A}) - p'(A \cup \bar{A}) \geq v_1(A) - p'(A). \end{aligned}$$

We next prove that $\hat{v}_i(X_i) = v_i(X_i)$ for all $i \in N$. In case there exists an agent $i \neq 1$ such that $\hat{v}_i(X_i) > v_i(X_i)$, there exists a proper subset B of X_i such that $\hat{v}_i(X_i) = v_i(B) = \hat{v}_i(B)$. Together with the fact $p_a \geq 0$ for all $a \in X_i$, we have $v_i(B) - p(B) > v_i(X_i) - p(B) \geq v_i(X_i) - p(X_i)$. Since $X_i \in D_{v_i}(p)$, this is impossible.

We are now ready to prove that $\langle p'; \mathbf{X}\rangle$ is also a competitive equilibrium for $\hat{\mathcal{E}}$. In case there exists an agent $j \neq 1$ such that $\hat{v}_j(X_j) - p'(X_j) < \hat{v}_j(C) - p'(C)$ for some bundle $C \subseteq \Omega$. Since $X_j \in D_{v_j}(p')$ and $\hat{v}_j(X_j) = v_j(X_j)$, we have

$$v_j(C) - p'(C) \leq v_j(X_j) - p'(X_j) = \hat{v}_j(X_j) - p'(X_j) < \hat{v}_j(C) - p'(C).$$

This implies $v_j(C) < \hat{v}_j(C)$ and $\hat{v}_j(C) = v_j(C')$ for some proper subset C' of C. Hence, we have

$$v_j(C') - p'(C') \geq \hat{v}_j(C) - p'(C) > v_j(X_j) - p'(X_j),$$

contradicting to the fact $X_j \in D_{v_j}(p')$.

(b) Assume that $\langle p; \mathbf{X} \rangle$ is a competitive equilibrium for $\hat{\mathcal{E}}$. Note that since all agents in $\hat{\mathcal{E}}$ have monotone preference, the equilibrium price vector p must be non-negative. We are going to construct a competitive equilibrium $\langle p; \mathbf{Y} \rangle$ for $\mathcal{E}$ such that $Y_i \subseteq X_i$ for $i \neq 1$, and $Y_1 = [\cup_{i \neq 1}(X_i \backslash Y_i)] \cup X_1$.

For each $i = 2, \ldots, n$, we choose $Y_i \subseteq X_i$ such that $\hat{v}_i(X_i) = v_i(Y_i) = \hat{v}_i(Y_i)$. Since $X_i \in D_{\hat{v}_i}(p)$, we have

$$\hat{v}_i(X_i) - p(X_i) \geq \hat{v}_i(Y_i) - p(Y_i) = \hat{v}_i(X_i) - p(Y_i) \geq \hat{v}_i(X_i) - p(X_i).$$

This implies $p_a = 0$ for all $a \in X_i \backslash Y_i$, and for any bundle $A \subseteq \Omega$,

$$v_i(Y_i) - p(Y_i) = \hat{v}_i(X_i) - p(X_i) \geq \hat{v}_i(A) - p(A) \geq v_i(A) - p(A).$$

Let $Y_1 = [\cup_{i \neq 1}(X_i \backslash Y_i)] \cup X_1$. Since v_1 is monotone and $p_a = 0$ for all $a \in \cup_{i \neq 1}(X_i \backslash Y_i)$, it follows that for any bundle $A \subseteq \Omega$, we have

$$\begin{aligned} v_1(Y_1) - p(Y_1) &\geq v_1(X_1) - p(X_1) = \hat{v}_1(X_1) - p(X_1) \\ &\geq \hat{v}_1(A) - p(A) = v_1(A) - p(A). \end{aligned}$$

This completes the proof of (b). Finally, the combination of (a) and (b) yields the result of (c). ■

Theorem 8 *For the economy $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$, there exists a competitive equilibrium if v_1 is monotone and each agent i's valuation function v_i satisfies the IGS condition.*

Proof. Assume that v_1 is monotone and v_i satisfies IGS for $i = 1, \ldots, n$. This implies that $\hat{v}_i$ satisfies GS for $i = 1, \ldots, n$, and hence there exists a competitive equilibrium for the economy $\langle \Omega; (\hat{v}_1, \ldots, \hat{v}_n) \rangle$ by Theorem 1. Combining with Theorem 7, we obtain the desired result. ■

Appendix. Proof of Theorem 3

The proof of Theorem 3 requires the following lemma.

Lemma 3 *Suppose that the valuation function* $v_i : 2^\Omega \to \mathbb{R}$ *is improvable. Then for price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *with* $p' \geq p$ *and for* $A \in D_{v_i}(p) \backslash D_{v_i}(p')$*, there exists* $A^* \in \arg\min_{C \in D_{v_i}(p)} [p'(C) - p(C)]$ *such that* $\{a \in A : p'_a = p_a\} \subseteq A^*$.

Proof. Let $C^* \in \arg\min_{C \in D_{v_i}(p)} [p'(C) - p(C)]$ and let $X = \{a \in A \backslash C^* : p'_a > p_a\}$. In case $X = \emptyset$, we have $A \in \arg\min_{C \in D_{v_i}(p)} [p'(C) - p(C)]$ and the proof is done. Assume that $X = \{a_1, \ldots, a_r\} \neq \emptyset$. Since v_i is improvable and

$$A, C^* \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)}) = \{C \in D_{v_i}(p) : C \subseteq A \cup C^*\},$$

There exists a small positive number ε such that

1. $D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)} + \varepsilon \chi_{\{a_1\}}) = \{C \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)}) : a_1 \notin C\}$, and
2. $A \backslash \{a_1\} \subseteq A_1$ for some $A_1 \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)} + \varepsilon \chi_{\{a_1\}})$.

Inductively, we can construct a sequence of distinct bundles, $A_1, \ldots, A_r \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)})$, such that $A \backslash \{a_1, \ldots, a_j\} \subseteq A_j \subseteq A \cup C^*$ for $j = 1, \ldots, r$. Let $A^* = A_r$. Since $A \backslash X \subseteq A^* \subset A \cup C^*$ and $A^* \in D_{v_i}(p)$, it follows that $\{a \in A : p'_a = p_a\} \subseteq A^*$ and $\{a \in A^* \backslash C^* : p'_a > p_a\} = \emptyset$, and hence $A^* \in \arg\min_{C \in D_{v_i}(p)} [p'(C) - p(C)]$. ■

We are now ready to prove Theorem 3.

($\Rightarrow$) Let v_i be a GS valuation function. Suppose, to the contrary, that v_i is not improvable.

By Lemma 1, there exist a price vector $p \in \mathbb{R}^{|\Omega|}$ and a pair of bundles $A, B \in D_{v_i}(p)$ such that $|A \backslash B| \geq 2$ and $A \subseteq C \subseteq A \cup B$ for all $C \in D_{v_i}(p) \backslash \{B\}$. This implies $D_{v_i}(p + \chi_{\{a\}}) = \{B\}$. Hence, when the prices are increased from p to $p + \chi_{\{a\}}$, no bundles in $D_{v_i}(p + \chi_{\{a\}})$ contain object b, violating the GS condition.

($\Leftarrow$) Let $p^0, p' \in \mathbb{R}^{|\Omega|}$ be two distinct vectors such that $p' \geq p^0$ and let $A \in D_{v_i}(p^0) \backslash D_{v_i}(p')$. By Lemma 3, there exists $A_1 \in \arg\min_{C \in D_{v_i}(p^0)}[p'(C) - p^0(C)]$ such that $\{a \in A : p'_a = p^0_a\} \subseteq A_1$. In case $A_1 \in D_{v_i}(p')$, the proof is done. Otherwise, there exists a price vector $p^1 = t_1 p' + (1 - t_1)\, p^0$ for some positive number $t_1 < 1$ such that $A_1 \in D_{v_i}(p^1)$ but $A_1 \notin D_{v_i}(tp' + (1 - t)p^0)$ for $t > t_1$.

Since $A_1 \in D_{v_i}(p^1) \backslash D_{v_i}(p')$, there exists $A_2 \in \arg\min_{C \in D_{v_i}(p^1)}[p'(C) - p^1(C)]$ such that

$$\{a \in A : p'_a = p^0_a\} \subseteq \{a \in A_1 : p'_a = p^1_a\} \subseteq A_2$$

by Lemma 3 again. In case $A_2 \in D_{v_i}(p')$, the proof is done. Otherwise, there exists a price vector $p^2 = t_2 p' + (1 - t_2)p^1$ for some positive number $t_2 < 1$ such that $A_2 \in D_{v_i}(p^2)$ but $A_2 \notin D_{v_i}(tp' + (1 - t)p^1)$ for $t > t_2$. Note that $A_2 \neq A_1$ and $A_2 \notin D_{v_i}(p')$.

Since the set of objects is finite, we can inductively construct a sequence of distinct bundles $A_1, A_2, \ldots, A_r$ and a sequence of distinct price vectors $p^0 \leq \cdots \leq p^{r-1} \leq p^r = p'$ such that for $j = 1, \ldots, r$, we have $A_j \in \arg\min_{C \in D_{v_i}(p^{j-1})}[p'(C) - p^{j-1}(C)]$ and

$$\{a \in A : p'_a = p^0_a\} \subseteq A_j \in D_{v_i}(p^j).$$

This completes the proof.

References

[1] C. Beviá, M. Quinzii, and J. A. Silva, Buying several indivisible goods, Math. Soc. Sci. **37** (1999), 1-23.

[2] S. Bikhchandani and J. W. Mamer, Competitive equilibrium in an exchange economy with indivisibilities, J. Econ. Theory **74** (1997), 385-413.

[3] F. Gul and E. Stacchetti, Walrasian equilibrium with gross substitutes, J. Econ. Theory **87** (1999), 95-124.

[4] F. Gul and E. Stacchetti, The English auction with differentiated commodities, J. Econ. Theory **92** (2000), 66-95.

[5] C. Hara, Existence of equilibria in economies with bads, Econometrica **73** (2005), 647-658.

[6] C. Hara, Bargaining set and anonymous core without the monotonicity assumption, J. Math. Econ. **41** (2005), 545-556.

[7] A. S. Kelso and V. P. Crawford, Job matching, coalition formation, and gross substitutes, Econometrica **50** (1982), 1483-1504.

[8] J. Ma, Competitive equilibrium with indivisibilities, J. Econ. Theory **82** (1998), 458-468.

[9] A. M. Manelli, Monotonic preferences and core equivalence, Econometrica **59** (1991), 123-138.

[10] N. Sun and Z. Yang, Equilibria and indivisibilities: Gross substitutes and complements, Econometrica **74** (2006), 1385-1402.

[11] A. Teytelboym, Gross substitutes and complements: A simple generalization, Econ. Lett. **123** (2014), 135-138.