The Phillips curve in a matching model

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Abstract

Following ideas in Hume, monetary shocks are embedded in the Lagos-Wright (2005) model in a new way: there are only nominal shocks that are accomplished by way of individual transfers and there is sufficient noise in individual transfers so that realizations of those transfers do not permit the agents to deduce much about the aggregate realization. The last condition is achieved by assuming that the distribution of aggregate shocks is almost degenerate. For such rare shocks, aggregate output is increasing in the aggregate shock—our definition of the Phillips curve.

Key words: Phillips curve, matching model, imperfect information

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1   Introduction

In his Nobel lecture [9] entitled "Monetary Neutrality," Lucas begins by describing Hume's [5] views about the effects of changes in the money supply. Lucas emphasizes that Hume's views were dependent on how changes in the quantity of money come about. In order to get neutrality, Hume set out very special conceptual experiments which, when correct, amount to changes in monetary units. For some other kinds of changes, Hume says that there is a short-run Phillips curve:

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Accordingly we find that, in every kingdom into which money begins to flow in greater abundance than formerly, everything takes a new face: labour and industry gain life, the merchant bcomes more enterprising...

To account, then, for this phenomenon, we must consider, that though the high price of commodities be a necessary consequence of the encrease of gold and silver, yet it follows not immediately upon that encrease, but some time is required before the money circulates through the whole state, and makes its effect be felt on all ranks of people. At first, no alteration is perceived; by degrees the price rises, first of one commodity, then of another, till the whole at last reaches a just proportion with the new quantity of specie in the kingdom. In my opinion, it is only in this interval or intermediate situation, between the acquisition and rise of prices, that the encreasing quantity of gold and silver is favorable to industry. When any quantity of money is imported into a nation, it is not at first dispersed into many hands but is confined to the coffers of a few persons, who immediately seek to employ it to advantage. [Hume [5], page 37.]

Hume asserts that there is a positive association between increases in the stock of money and the level of real economic activity, which is our definition of the Phillips curve. He also offers what may at some time have been regarded as an explanation of it. A modern economist would not treat his discussion as an explanation, but might look to it for hints about modeling ingredients that when rigorously analyzed could conceivably constitute an explanation.

The passage from which the above excerpt comes contains at least two hints about modeling ingredients. First, changes in the quantity of money come about in a way that gives rise to changes in relative money holdings among people. In particular, the changes for individuals are not uniformly proportional to initial holdings as is required for neutrality. Second, trade seems to be occurring within small groups, rather than in a centralized market. That suggests the use of some sort of search/matching model. Given those ingredients, the passage hints at two conjectures that might be studied. One is that a change in relative money holdings has Phillips-curve type effects that dissipate over time through the effects of subsequent trades on those relative holdings. The other is that the change occurs in a way that is not seen by everyone when it occurs and that the Phillips-curve type effects dissipate when people learn about it. Although these are not mutually exclusive conjectures, we pursue only the second here.

In order to study it, we embed monetary shocks in Lagos-Wright [7] (LW) and assume that the aggregate monetary shocks are observed with a lag and are accomplished by

way of individual transfers in such a way that those transfers do not permit the agents to deduce much about the aggregate realization. That is, the individual transfers are very imperfect signals about the aggregate shock, where imperfection is modeled in the usual way: there is a fixed support for individual transfers and the aggregate shock determines the distribution over that fixed support. In order to make those signals almost uninformative, we assume for our main result that the distribution of the aggregate shock is close to a degenerate distribution. For such almost-degeneracy of aggregate shocks— that is, for rare shocks—we show that there is a Phillips curve.

We are not the first to use some version of LW to study the Phillips curve. Faig and Li [3] embed a version of the signal-extraction problem in Lucas [8] into that model. That signal-extraction problem involves a delicate confounding of real and nominal shocks and the sign of the slope of the Phillips curve depends on preferences. Our specification is closer to Hume and is conceptually simpler. People meet in pairs and do not see the transfers received or the trades in other meetings. Also, because people meet in pairs to trade, there is no price that aggregates the effect of the individual transfers and that allows people to deduce the aggregate shock. Notice, by the way, that pairwise meetings and the assumption that people do not see what is happening in other meetings is, arguably, the most attractive way of producing a necessary condition for essentiality of money—namely, some privacy of individual trading histories (see Wallace [13]).

After presenting that main result, we comment on robustness—in particular, on what might happen for other aggregate shock processes and what might happen if we abandon the special structure in LW in a way that leads to wealth effects.

2 The model

The background model is that in LW. Time is discrete and there are two stages at each date. In the first stage, the decentralized market (the DM), production and consumption occur in pairwise meetings that occur at random in the following way. Just prior to such meetings, each person looks forward to being a consumer (a buyer) who meets a random producer (seller) with probability σ, looks forward to being a producer who meets a random consumer with probability σ, and looks forward to a no-coincidence meeting with probability $1 - 2\sigma$, where $\sigma \leq 1/2$. In the DM, $u(y)$ is the utility of consuming and $c(y)$ is the disutility of producing, where u and c are twice differentiable, $u(0) = c(0) = 0$, u is strictly concave, c is convex, and $u'(0) - c'(0)$ converges to infinity. We also assume that $y^* = \arg\max_y [u(y) - c(y)]$ exists. In the second stage all agents can consume and produce and meet in the centralized market (the CM), where the utility of consuming

is z and where negative z is production. At each stage production is perishable and the discount factor between dates is $\beta \in (0, 1)$.

In our version, the gross growth rate of the money stock follows an *iid* process with finite support S, where $S = \{s_1, s_2, ..., s_N\}$ with $1 \leq s_n < s_{n+1}$. We let π denote the distribution over S. The changes in the stock of money are accomplished by random proportional transfers to individuals. We let $\tau \in T = \{\tau_1, \tau_2, ..., \tau_I\}$ be the set of possible gross proportional transfers to a person, where $1 \leq \tau_i < \tau_{i+1}$. We let $\mu_s(\tau)$ be the probability that an agent receives the transfer $\tau \in T$ conditional on the aggregate state s. (As this suggests, conditional on s, agents receive independently drawn transfers.) We assume that μ_s satisfies the following conditions. First, the individual transfers aggregate to s. That is,

$$\sum_{i=1}^{I} \mu_s(\tau_i)\tau_i = s \tag{1}$$

for each $s \in S$. Second, except for one of the neutrality results, we assume that μ_s has full support for all $s \in S$ and that it satisfies the following strict version of first-order stochastic dominance: for any $m > n$ and for all $i = 1, ..., I - 1$,

$$\sum_{k=1}^{i} \mu_{s_m}(\tau_k) < \sum_{k=1}^{i} \mu_{s_n}(\tau_k). \tag{2}$$

The weak inequality of version of (2) is first-order stochastic dominance.[1]

There are two versions of the model; one with an information lag and one without such a lag. The sequence of actions when there is an information lag is as follows. After people leave the CM, the growth rate of the stock of money, $s \in S$, is realized, but is not observed. Then, agents meet at random in pairs. Then each agent receives a proportional money transfer, a draw from $\mu_s(\tau)$. Within a meeting, both the pre-transfer and post-transfer money holdings are common knowledge. In a meeting where the transfers received are τ_i and τ_j, the common posterior about s is given by Baye's rule:

$$p_{ij}(s) = \frac{\pi(s)\mu_s(\tau_i)\mu_s(\tau_j)}{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)}. \tag{3}$$

If the meeting is between a buyer and seller, then the buyer makes a take-it-or-leave-it offer. After meetings, agents learn s and enter the next CM.[2] If an agent leaves the pairwise trade stage, the DM, with m amount of money, then the agent enters the next

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[1] Condition (1) and the full support assumption about μ_s imply that the range of S is a strict subset of the range of T.

[2] In a somewhat different context, Araujo and Shevchenko [1] study the complications that arise when the information lag is longer.

CM with m/s amount of money. (This is the standard way of normalizing inflation so that the per capita quantity of money and the price of money can be constant.) Then, in price-taking trade, the good trades for money at the next CM.

The sequence of actions when there is no information-lag is identical except that the growth rate of the stock of money, s, is observed when it is realized. In that case, $p_{ij}(s)$ in (3) is replaced by a distribution that is degenerate on the realized s.[3]

We model the transfers as proportional to money holdings in order to isolate the effect of heterogeneous transfers. As we show below, if $\mu_s(\tau)$ is degenerate—which by (1) implies that $\tau = s$ for some τ—then S and π do not matter. Otherwise, both matter, whether or not there is an information-lag.

3 Stationary equilibrium

We start with existence of stationary monetary equilibrium. The real objects in a stationary monetary equilibrium are output in buyer-seller meetings—denoted y_{ij}, where τ_i is the transfer received by the buyer and τ_j is that received by the seller—and the price of money in the CM—denoted v.

Proposition 1. In each version of the model, either with or without an information-lag, there exists a unique valued-money stationary equilibrium. In that equilibrium $y_{ij} < y^*$ for at least some (i, j). Moreover, y_{ij} and v are continuous in π.

The proof, which follows, is familiar from other expositions of LW. The strategy is guess-and-verify: we guess that the value function entering the CM is affine, and that the rate-of-return of money from one CM to next CM is equal to one. Given that guess, the equilibrium condition reduces to the solution to an optimal saving decision in the CM, a decision which is common to everyone.

Proof. Assume that there is an information-lag. (As we remark at the end, the no-information-lag version is a special case.) We refer to the current centralized market as

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[3]As may be evident, the information-lag version and the no information-lag version are special cases of a more general specification. The buyer's transfer and the seller's transfer play very different roles in the model. Both symmetrically affect the posterior over the aggregate realization in the information-lag version. In addition, the buyer's transfer affects spending directly in both versions. Therefore, the information-lag version is equivalent to a setting in which only the buyer realizes a transfer, but the pair in a meeting see the buyer's transfer and also see the buyer's transfer in one other randomly chosen meeting. But, then, what if the pair in a meeting see the buyer's transfer and that in l randomly chosen meetings? Obviously, as $l \to \infty$, there is convergence to the no-information-lag version.

the CM. We guess that the continuation value of entering the next CM with m' amount of money is $v'm' + A$, where v' is the price of money and A is a constant. We also guess that $v' = v$, the price of money in the current CM.

A person who enters the DM with m amount of money is a buyer with probability σ, a seller with probability σ, or is in a no-coincidence meeting with probability $1 - 2\sigma$. Given the assumed continuation value in the next CM, we start by considering the buyer's problem, the only significant one in the DM,

Consider a buyer who enters the DM with m amount of money and is in meeting (i, j), one in which the buyer receives transfer τ_i and the seller receives transfer τ_j. The problem of the buyer is to choose output, y_{ij}, and the amount of money to offer, d_{ij}, to maximize $u(y_{ij}) + v'(m\tau_i - d_{ij})E_{ij}(1/s)$ subject to $m\tau_i - d_{ij} \geq 0$ and

$$c(y_{ij}) \leq v'd_{ij}E_{ij}(1/s). \tag{4}$$

Here, $E_{ij}(1/s) = \sum_{n=1}^{N} p_{ij}(s_n)/s_n$, where $p_{ij}(s_n)$ is given by (3) and where $E_{ij}(1/s) = E_{ji}(1/s)$. Because the value function in the next CM is affine, the seller's money holding appears in (4) only by way of $E_{ij}(1/s)$. At a solution to this problem, (4) holds at equality. (If not, then increase y_{ij}.) Substituting (4) at equality into the objective, the objective becomes $u(y_{ij}) - c(y_{ij})$. Therefore, the solution has two branches depending on whether the constraint $m\tau_i - d_{ij} \geq 0$ is binding. The nonbinding case has $y_{ij} = y^*$ and d_{ij} given by (4) at equality. The binding case has $d_{ij} = m\tau_i$ and y_{ij} given by (4) at equality. That is,

$$y_{ij} = \begin{cases} y^* \text{ if } c(y^*) \leq v'm\tau_i E_{ij}(1/s) \\ \\ c^{-1}(v'm\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > v'm\tau_i E_{ij}(1/s) \end{cases} \tag{5}$$

and

$$d_{ij} = \begin{cases} c(y^*)/v'E_{ij}(1/s) \text{ if } c(y^*) \leq v'm\tau_i E_{ij}(1/s) \\ \\ m\tau_i \text{ if } c(y^*) > v'm\tau_i E_{ij}(1/s) \end{cases}. \tag{6}$$

Therefore, the buyer's payoff is $u(y_{ij}) + (m\tau_i - d_{ij})v'E_{ij}(1/s) + A$.

Now suppose the person with m is not a buyer but realizes transfer τ_i when the trading partner realizes transfer τ_j. This person's payoff is the same whether the person is a seller or is in a no-coincidence meeting because sellers receive no gains from trade—(4) at equality with the roles of i and j reversed. Thus, the person's payoff as a seller or in a no-coincidence meeting is $v'm\tau_i E_{ij}(1/s) + A$.

Therefore, ignoring constant terms, both money brought into the CM and the constant A, the CM problem is

$$\max_{m \geq 0} -zm + \beta \sum_{i,j} \gamma_{ij} \left\{ \sigma \left[u(y_{ij}) - v' d_{ij} E_{ij}(1/s) \right] + v' m \tau_i E_{ij}(1/s) \right\}, \tag{7}$$

where $\gamma_{ij} = \sum_n \pi(s_n) \mu_{s_n}(\tau_i) \mu_{s_n}(\tau_j)$, which by the law of iterated expectations, is the unconditional probability that the person receives transfer τ_i and is in a meeting with a person who receives τ_j. Now, because $\sum_{i,j} \gamma_{ij} \tau_i E_{ij}(1/s) = 1$ (see Lemma 1 in the appendix), this becomes

$$\max_{m \geq 0} -vm + \beta v' m + \beta \sigma \sum_{i,j} \gamma_{ij} \left[u(y_{ij}) - v' d_{ij} E_{ij}(1/s) \right]. \tag{8}$$

Letting $x = mv$, real saving, and imposing $v = v'$, an equivalent choice problem is

$$\max_{x \geq 0} G(x) \equiv -(\beta^{-1} - 1)x + \sigma \sum_{i,j} \gamma_{ij} \{ u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x) E_{ij}(1/s) \}, \tag{9}$$

where, by (5) and (6),

$$\tilde{y}_{ij}(x) = \begin{cases} y^* \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ c^{-1}(x\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases} \tag{10}$$

and

$$v\tilde{d}_{ij}(x) = \begin{cases} c(y^*)/E_{ij}(1/s) \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ x\tau_i \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases}. \tag{11}$$

The rest of the proof appears in the appendix. There, it is shown that G is differentiable with derivative denoted $G'(x)$. It is then shown that $G'(0) > 0$ and that $G'(x) < 0$ for $x \geq \bar{x}$, where

$$\bar{x} = \frac{\min_{i,j} [\tau_i E_{ij}(1/s)]}{c(y^*)}. \tag{12}$$

It follows that the choice of x can be limited to the compact domain $[0, \bar{x}]$, that G has a maximum, and that any maximum occurs in the interval $(0, \bar{x})$. Then it is shown that $G'(x)$ is strictly decreasing on $(0, \bar{x})$, which implies that G is strictly concave on $(0, \bar{x})$. That implies that the maximum, denoted $\hat{x}$, is unique. And, because $\bar{x}$ is continuous in π, the Theorem of the Maximum implies that $\hat{x}$ is continuous in π. Because y_{ij} is continuous in x, it also is continuous in π. If the per capita quantity of money is unity, another normalization, we get $v = \hat{x}$ by equating real saving to the real value of money. To see that $y_{ij} < y^*$ for some (i, j), consider two cases. If $\min_{i,j} [\tau_i E_{ij}(1/s)] = \tau_k E_{kl}(1/s)$ for all

(k, l), then $\hat{x} < \bar{x}$ implies $y_{kl} < y^*$ for all (k, l). Otherwise, $y_{kl} < y^*$ for all (k, l) such that $\min_{i,j}[\tau_i E_{ij}(1/s)] < \tau_k E_{kl}(1/s)$. The proof for the no-information-lag version is identical except that $E_{ij}(1/s)$ is replaced by $1/s$.∎

Two neutrality results follow from the above exposition.

Corollary 1. If economies 1 and 2 are identical except that $S^2 = \alpha S^1$ and $T^2 = \alpha T^1$ for some $\alpha > 1$, then both economies have the same real equilibria.

Proof. It is immediate that $E_{ij}^2(1/s) = (1/\alpha)E_{ij}^1(1/s)$. From that it follows that α does not appear in (9)-(12).∎

It follows that we can without loss of generality impose $\tau_1 = 1$, as we do in some examples below.

Corollary 2. If there is no heterogeneity in realized individual transfers—meaning that for each s, $\mu_s(\tau) = 1$ for some τ—then the equilibrium is the same as that of an economy with a degenerate π (no aggregate uncertainty).

Proof. Let s be the realized aggregate shock. By the hypothesis, there exists $\tau(s)$ such that $\mu_s[\tau(s)] = 1$. It follows that $E_{ij}(1/s) = 1/s$ and from (1) that $\tau(s) = s$. Therefore, using (10), and (11), $u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)$ does not depend on s and (9) reduces to the special case of no randomness.∎

4   The Phillips Curve under near degeneracy

Now we turn to our main result, the existence of a Phillips curve. Proposition 1 establishes the existence of a unique stationary monetary equilibrium. For any $\pi \in \Delta(S)$, let $y_{ij}^1(\pi)$ be the corresponding equilibrium DM output for meeting type (τ_i, τ_j) with an information lag, and let $y_{nij}^0(\pi)$ be that without an information lag for meeting type (s_n, τ_i, τ_j). Then, the respective aggregate DM outputs are

$$Y^1(s_n, \pi) = \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)y_{ij}^1(\pi) \tag{13}$$

and

$$Y^0(s_n, \pi) = \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)y_{nij}^0(\pi) = \sigma \sum_i \mu_{s_n}(\tau_i)y_{ni}^0(\pi), \tag{14}$$

where the second equality in (14) holds because output does not depend on the seller's transfer when there is no information lag.

We say that a Phillips curve exists in version i if $Y^i(s_n, \pi)$ is strictly increasing in s_n. The entire discussion is about output in the DM, because output in the CM does not

depend on the realized aggregate shock. We start with a discussion of what we can say about aggregate DM output when there is no information-lag.

The following corollary (of Proposition 1) gives a sufficient condition for aggregate output to be constant when there is no information-lag.

Corollary 3. If (i) $c(y) = y$ and (ii) β is small enough so that $y_{ni}^0 < y^*$ for all n and i, then $Y^0(s_n, \pi)$ does not depend on s_n.

Proof. Let x_π be the equilibrium real balance in the CM when there is no information-lag. Under the assumptions,

$$Y^0(s_n, \pi) = \sigma \sum_i \mu_{s_n}(\tau_i) y_{ni}^0(\pi) = \sigma \sum_i \mu_{s_n}(\tau_i) \tau_i x_\pi / s_n = \sigma x_\pi \left[\sum_i \mu_{s_n}(\tau_i) \tau_i \right] / s_n = \sigma x_\pi,$$
(15)

where the second inequality follows from the second line of (10) and $E_{ij}(1/s) = 1/s$, and where the last equality follows from (1).■

Now we present an example that shows that curvature in $c(y)$ is enough to make $Y^0(s_n, \pi)$ nonmonotone in s_n. It implies that nothing general can be said about the Phillips curve in the absence of an information-lag.[4]

Example 1. Consider the model without an information-lag. Suppose that $c(y) = y^2$, $T = \{1, \tau\}$, and that β is small enough so that $y_{ni}(\pi) < y^*$ for all n, i. Then, $Y^0(s_n, \pi)$ is nonmonotone in s_n. (The proof is in the appendix.)

Now, we turn to what happens with an information lag. According to (13), aggregate output is a weighted average of meeting specific outputs. The meeting specific outputs, represented by the matrix $[y_{ij}^1]_{i,j}$, do not depend on s. And according to (2), the higher is s, the more weight is placed on those components of the matrix with higher transfers. Intuitively, a Phillips curve would be obtained if meetings with higher transfers are associated with higher outputs. But that is delicate. Although high transfers to buyers tend to increase spending in the meetings, high transfers (no matter whether to buyers or to sellers) also suggest that the aggregate shock is high and, therefore, tend to offset the higher spending effect. Our main Phillips-curve result limits the informational role of transfers by describing what happens in a neighborhood of a degenerate π. (In the next section on robustness, we present some examples that go beyond near-degeneracy.)

Proposition 2. Let $\tilde{\pi} \in \Delta(S)$ and $\tilde{n}$ be such that $\tilde{\pi}(s_{\tilde{n}}) = 1$. There is a neighborhood

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[4]As is well-known, for some purposes, $c(y) = y$ can be regarded as a normalization, which treats output in a DM meeting as the disutility of production. However, we want to take seriously that output in the definition of the Phillips curve is production and not the disutility of production.

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of $\tilde{\pi}$ such that for all π in that neighborhood: (i) $Y^1(s, \pi)$ is strictly increasing in s; and (ii) $Y^1(s, \pi) - Y^0(s, \pi) > 0$ for all $s > s_{\tilde{n}}$ and $Y^1(s, \pi) - Y^0(s, \pi) < 0$ for all $s < s_{\tilde{n}}$.

Proof. (i) Because Y^1 is continuous in π, it suffices to show that $Y^1(s, \tilde{\pi})$ is strictly increasing in s. For $\pi = \tilde{\pi}$, $p_{ij}(s_{\tilde{n}}) = 1$. This implies that $E_{ij}(1/s) = 1/s_{\tilde{n}}$ for all i, j. Therefore, $y_{ij}^1(\tilde{\pi})$ does not depend on j and, as implied by Proposition 1, $y_{1j}^1(\tilde{\pi}) < y^*$. Also, by (10), $y_{ij}^1(\tilde{\pi})$ is weakly increasing in i and is not constant. Hence, by our strict stochastic dominance assumption (see (2), $Y^1(s, \tilde{\pi})$ is strictly increasing in s.

(ii) Because $Y^1(s, \pi)$ and $Y^0(s, \pi)$ are continuous in π, it is enough to establish the inequalities for $\pi = \tilde{\pi}$. For $\pi = \tilde{\pi}$, the equilibrium real balance, $\hat{x}$, does not depend on whether or not there is an information-lag. Therefore, $y_{ij}^1(\tilde{\pi}) = y_{\tilde{n}ij}^0(\tilde{\pi})$. By (10) with $E_{ij}(1/s) = 1/s$, $y_{nij}^0(\tilde{\pi}) < y_{n'ij}^0(\tilde{\pi})$ for all $n > n'$ and all i, j such that $y_{nij}^0(\tilde{\pi}) < y^*$ (which necessarily holds for $i = 1$). Therefore, for $n > \tilde{n}$,

$$
\begin{aligned}
&[Y^1(s_n, \tilde{\pi}) - Y^0(s_n, \tilde{\pi})] \\
=\ & \sigma \sum_{i,j} \mu_{s_n}(\tau_i) \mu_{s_n}(\tau_j) [y_{ij}^1(\tilde{\pi}) - y_{nij}^0(\tilde{\pi})] \\
=\ & \sigma \sum_{i,j} \mu_{s_n}(\tau_i) \mu_{s_n}(\tau_j) [y_{\tilde{n}ij}^0(\tilde{\pi}) - y_{nij}^0(\tilde{\pi})] > 0.
\end{aligned}
\tag{16}
$$

Similarly, for $n < \tilde{n}$,

$$
\begin{aligned}
&[Y^1(s_n, \tilde{\pi}) - Y^0(s_n, \tilde{\pi})] \\
=\ & \sigma \sum_{i,j} \mu_{s_n}(\tau_i) \mu_{s_n}(\tau_j) [y_{ij}^1(\tilde{\pi}) - y_{nij}^0(\tilde{\pi})] \\
=\ & \sigma \sum_{i,j} \mu_{s_n}(\tau_i) \mu_{s_n}(\tau_j) [y_{\tilde{n}ij}^0(\tilde{\pi}) - y_{nij}^0(\tilde{\pi})] < 0,
\end{aligned}
\tag{17}
$$

which completes the proof.∎

Notice that part (ii) of Proposition 2 says that the output effects of (rare) shocks are larger when there is an information-lag. There is no claim about whether there is a Phillips curve when there is no information-lag. Also, it is obvious that Proposition 2 holds when the pair in a meeting see any finite number of individual transfers from other meetings.

5   Robustness

There are special aspects of the model and special aspects of the process for aggregate shocks in Proposition 2. Here we discuss whether the Phillips curve result will survive if we depart from some of them.

5.1 More general *iid* shocks

Here we provide some examples that consider more general *iid* processes in the model with an information lag. In all these examples, we assume a two-point support for individual transfers—namely, $T = \{1, \tau\}$—that $c(y) = y$, and that β is sufficiently small so that the buyer constraint is always binding in equilibrium.[5] A two-point support for T is not crucial for these examples but is convenient because condition (1) then determines $\mu_s(\tau)$ and condition (2) is implied.

We start with an example for which there is a Phillips curve when the pair in a meeting get a single signal; that is, when a common transfer is received by both the buyer and the seller.

Example 2. Assume that the buyer and seller in each meeting receive the same transfer and that π is such that $E(s) = (1 + \tau)/2$. Then $Y^1(s, \pi)$ is affine and strictly increasing in s. (The proof is in the appendix.)

Notice that the only distributional assumption in Example 2 is that $E(s)$ is at the midpoint of the support of T. However, as the following counter-example to monotonicity shows, the assumption of a single signal is important.

Example 3. For each $N \in \mathbb{N}$, let $S_N = \{1 + \varepsilon, 1 + 2\varepsilon, ..., 1 + N\varepsilon\}$ with $\varepsilon = (\tau - 1)/(N + 1)$ and let

$$\pi(1 + \varepsilon) = \pi(\tau - \varepsilon) = \frac{1}{2}(1 - \delta),$$

and

$$\pi(1 + n\varepsilon) = \frac{\delta}{N - 2} \text{ for all } 2 \leq n \leq N - 1$$

for some small $\delta > 0$. Then, there exist (large) $\bar{N}$ and (small) $\bar{\delta} > 0$ such that if $N > \bar{N}$ and if $\delta < \bar{\delta}$, then $Y^1(s, \pi)$ is not monotone in s. (The proof is in the appendix.)

It is straightforward to verify that $E(s) = (1 + \tau)/2$ in Example 3. Hence, aside from having two signals, Example 3 is a special case of Example 2. In example 2, there are only two types of meetings, one with a common transfer of τ and the other with a common transfer 1. In Example 3, in contrast, there are four kinds of meetings, which implies that aggregate output is a quadratic function of s. As N gets large and δ gets small, π gets concentrated on two points that are near 1 and τ. Therefore, if both the buyer and the seller receive τ, then they are almost certain that the state is near τ, while if both receive 1, then they are almost certain that the state is near 1. If they were certain, then output would be the same in those two meetings. If they are almost certain, then output when

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[5] These examples satisfy the assumptions of Corollary 3. Therefore, total output is independent of s when there is no information lag.

both receive τ is slightly larger than when both receive 1. When one receives τ and the other receives 1, their interim belief is almost identical to the prior belief, and average output over those two meetings is higher than that for the other two kinds of meetings. Hence, aggregate output is increasing in the measure of meetings with mixed transfers, a measure which is increasing in s for small s and decreasing in s for large s.

Although Example 3 tells us that we will not get a Phillips curve without strengthening the general assumptions we make, it is very special in two senses. First, if we amend it by assuming that the range of S is sufficiently far from the end points in T, and if the right end of T is not too large, then monotonicity is restored.

Example 4. Suppose that $T = \{1, \tau\}$ with $\tau < 6$, and that π is symmetric with $E(s) = (1 + \tau)/2$. If $S \subset [0.3\tau + 0.7, 0.7\tau + 0.3]$, then $Y^1(s, \pi)$ is strictly increasing. (The proof is in the appendix.)

Second, Example 3 is a counter-example only because our definition of the Phillips curves calls for $Y^1(s, \pi)$ to be monotone in s—even over parts of S that occur with very low probability. A different approach would define the Phillips curve probabilistically—for example, as a positive correlation between total output and s. Defined as a correlation, Example 3 fails as a counter-example because the endpoints of S occur with arbitrarily high probability, and output at the high endpoint is larger than at the low endpoint for N large. For example, if $u(y) = \sqrt{y}$, $\beta = 1/1.05$, $\sigma = 0.3$, $\tau = 1.2$, and π is as specified in Example 3 with $N = 200$ and $\delta = 0.05$, then $Y^1(s, \pi)$ is non-monotone in s but the correlation between $Y^1(s, \pi)$ and s is 0.99.

5.2 Markov aggregate shocks

With Markov shocks, the state entering a CM is the realized s from the previous period. Then the guess is that there is a stationary equilibrium in which the price of money in the CM depends on the realized aggregate shock from the previous period, denoted $v(s)$. If so, then people in the CM in state s face a return distribution between that CM and the next CM, where the realized return is $v(s')/v(s)$ and s' is the realized aggregate shock in the current period. The stationary equilibrium conditions implied by optimal saving choices give rise to N simultaneous equations in $v(s)$ for $s \in S$. In that case, existence requires a fixed point argument and the well-known challenge is to choose a domain for $v(s)$ for $s \in S$ that excludes $v(s) = 0$ for all $s \in S$.

However, if we assume that the Markov process is nearly degenerate—has a transition matrix for s which has a column all of whose elements are near unity—then our result in

Proposition 2 applies by way of the implicit function theorem. In addition, our technique for getting a Phillips curve should also apply for a Markov process that is nearly degenerate in a different sense—namely, has a transition matrix that is close to the identity matrix.

5.3   Wealth effects

Barro [2] (pages 2 and 3) expresses the following concern about the robustness of the Phillips curve in Lucas [8]. The transfer in Lucas goes entirely to potential consumers—as it does in Faig and Li [3]. If both consumers and producers receive transfers, then richer producers may want to produce less and that could offset the greater spending by consumers. We have transfers going to both consumers and producers, but the structure of LW precludes wealth effects on producers. In particular, the transfer that the seller receives matters for the trade in the DM only because it is a signal about the aggregate shock.[6]

There are generalizations of LW that have wealth effects. The LW structure imposes a periodic quasi-terminal condition: when the CM stage occurs, the economy restarts from that stage with a degenerate distribution of money holdings and, hence, with no state variable. If there are many DM stages before the CM stage in each period, then there are wealth effects before the last DM stage because the continuation value of money will be strictly concave at least over some of its domain. Indeed, with a large number of DM stages, the model at intermediate DM stages resembles the divisible-money versions of the Shi [11] and Trejos and Wright [12] matching models studied by Zhu [14] and Molico [10], versions without aggregate shocks. With aggregate shock realizations at each DM stage and even with the realized aggregate shock at a stage announced just prior to the next stage, such a model combines both the role of an information lag and the role of shocks on the evolution of the distribution of money holdings.

As might be expected, not much is known about such a version of the model. Some very preliminary investigation of a version with two DM stages suggests that the implied seller wealth effects do tend to weaken the kind of Phillips curve effects found for the one DM-stage model. However, the implications seem to depend on the details of the model.

---

[6]There seems to be no work that explores an incomplete-information theory of the Phillips curve and allows for producer wealth effects (c.f. Hellwig [4]). Katzman *et. al.* [6] claim that their specification avoids Barro's concern because they have transfers going to both consumers and producers. However, their results are for $\{0, 1\}$ money holdings and depend on a condition on the distribution of money holdings—namely, that fewer than half the population has money. That condition is troublesome because it seems to have no analogue when money is divisible.

6 Concluding remark

Although we established a Phillips curve only in the LW model and for rare aggregate shocks, the general message is that it is not hard to get a Phillips curve if one follows some of the ideas in Hume; namely, heterogeneous transfers of money, small group trade, and incomplete information about aggregate transfers. Although Hume wrote in the 1740's, economists were not able to work with those ideas until at least the 1980's. One can only wonder how different the history of macroeconomics would have been if those ideas had been analyzed much earlier.

7 Appendix

Here we give the missing proofs. We begin with a lemma that is used in the proof of Proposition 1.

Lemma 1. $\sum_{i,j} \gamma_{ij} \tau_i E_{ij}(1/s) = 1$.

Proof. We have

$$
\sum_{i,j} \gamma_{ij} \tau_i E_{ij}(1/s)
$$

$$
= \sum_{i,j} \gamma_{ij} \tau_i \left[\sum_{s'} p_{ij}(s') \frac{1}{s'} \right]
$$

$$
= \sum_{s,i,j} \gamma_{ij} \tau_i \left[\pi(s) \mu_s(\tau_i) \mu_s(\tau_j) / \gamma_{ij} \right] \frac{1}{s}
$$

$$
= \sum_{s,i,j} \gamma_{ij} \pi(s) \mu_s(\tau_i) \mu_s(\tau_j) \frac{1}{s}
$$

$$
= \sum_{s,j} \pi(s) \mu_s(\tau_j) \frac{1}{s} \left[\sum_i \mu_s(\tau_i) \tau_i \right]
$$

$$
= \sum_{s,j} \pi(s) \mu_s(\tau_j) = 1. \blacksquare
$$

Completion of Proof of Proposition 1.

In order to complete the proof, we need to establish the two claims about G set out below. We start by repeating the definition of G.

$$\max_{x \geq 0} G(x) \equiv -(\beta^{-1} - 1)x + \sigma \sum_{i,j} \gamma_{ij}\{u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\}, \qquad (18)$$

where, by (5) and (6),

$$\tilde{y}_{ij}(x) = \begin{cases} y^* \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ c^{-1}(x\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases} \qquad (19)$$

and

$$v\tilde{d}_{ij}(x) = \begin{cases} c(y^*)/E_{ij}(1/s) \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ x\tau_i \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases}. \qquad (20)$$

Claim 1. G is differentiable with derivative denoted G'. Moreover, $G'(0) > 0$ and $G'(x) < 0$ for all $x \geq \bar{x}$, where

$$\bar{x} = \frac{\min_{i,j}[\tau_i E_{ij}(1/s)]}{c(y^*)}. \qquad (21)$$

Proof. For existence of G', it suffices to show that $u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)$ is differentiable for each (i,j). There are two relevant cases. If $c(y^*) > x\tau_i E_{ij}(1/s)$, then $\tilde{y}_{ij}(x) < y^*$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\right] = \left[\frac{u'(\tilde{y}_{ij}(x))}{c'(\tilde{y}_{ij}(x))} - 1\right]\tau_i E_{ij}(1/s). \qquad (22)$$

If $c(y^*) > x\tau_i E_{ij}(1/s)$, then, $\tilde{y}_{ij}(x) = y^*$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\right] = 0. \qquad (23)$$

Because both derivatives coincide at $c(y^*) = x\tau_i E_{ij}(1/s)$, G is differentiable. Also, by (22), $G'(0) > 0$. And by (23), $G'(x) < 0$ for all $x \geq \bar{x}$. $\square$

Claim 2. $G'(x)$ is strictly decreasing on $(0, \bar{x})$.

Proof. For any $x \in (0, \bar{x})$, $G'(x)$ is a sum of terms, some of which are given by (22) and others of which are constant. Obviously, those given by (22) are strictly decreasing because $\tilde{y}_{ij}(x) < y^*$. Hence, the result follows. $\square$

Example 1.

Consider the model without an information-lag. Suppose that $c(y) = y^2$, $T = \{1, \tau\}$, and that β is small enough so that $y_{ni}(\pi) < y^*$ for all n, i. Then, $Y^0(s_n, \pi)$ is nonmonotone in s_n.

Proof. Let $x(\pi)$ be the equilibrium price for money in the CM. In this case, we have

$$y_{ni}(\pi) = \sqrt{\tau_i x(\pi)/s_n}.$$

Thus,

$$
\begin{aligned}
Y^0(s_n, \pi) &= \mu_{s_n}(1)y_{n1}(\pi) + \mu_{s_n}(\tau)y_{n2}(\pi) \\
&= \frac{\tau - s}{\tau - 1}\sqrt{x(\pi)/s_n} + \frac{s - 1}{\tau - 1}\sqrt{\tau x(\pi)/s_n} \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\tau - 1)}\left[(s_n - 1)\sqrt{\tau} + \tau - s_n\right] \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\tau - 1)}\left[(s_n + \sqrt{\tau})(\sqrt{\tau} - 1)\right] \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\sqrt{\tau} + 1)}(s_n + \sqrt{\tau}) \\
&= \frac{\sqrt{x(\pi)}}{\tau - 1}\left(\sqrt{s_n} + \sqrt{\frac{\tau}{s_n}}\right).
\end{aligned}
$$

Now, let $f_n = \sqrt{s_n} + \sqrt{\frac{\tau}{s_n}}$, and hence

$$f_n^2 = s_n + \frac{\tau}{s_n} + 2,$$

which is increasing in s_n if and only if $s_n \geq \sqrt{\tau}$. $\square$

For Examples 2, 3, and 4, recall that we assume an information lag, $c(y) = y$, $T = \{1, \tau\}$ and β small enough so that $y_{ij} < y^*$ in all meetings.

Example 2.

Assume that the buyer and seller in each meeting receive the same transfer. Suppose that π has $E(s) = (1 + \tau)/2$. Then, $Y^1(s, \pi)$ is affine and strictly increasing in s.

Proof. To keep track of the transfers, we use τ_h to denote τ and τ_ℓ to denote 1. Since the pair receive the same transfer, we use E_{τ_h} and E_{τ_ℓ} to denote the interim expectation with transfer τ_h or τ_ℓ. Thus,

$$E_{\tau_h}(1/s) = \frac{1 - \tau_\ell E(1/s)}{E(s) - \tau_\ell}, \quad E_{\tau_\ell}(1/s) = \frac{\tau_h E(1/s) - 1}{\tau_h - E(s)}.$$

Since the buyer constraint is always binding and $c(y) = y$, it follows that

$$Y^1(s, \pi) = \sigma\frac{x_\pi}{\tau_h - \tau_\ell}\left\{(\tau_h - s)\frac{\tau_h E(1/s) - 1}{\tau_h - E(s)}\tau_\ell + (s - \tau_h)\frac{1 - \tau_\ell E(1/s)}{E(s) - \tau_\ell}\tau_h\right\}, \quad (24)$$

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where x_π is the equilibrium real balance. This shows that the aggregate output is linear in s. By (24), $Y(s)$ is strictly increasing in s iff

$$[\tau_h - E(s)][E(s) - \tau_\ell] - \tau_h \tau_\ell [E(s)E(1/s) - 1] > 0.$$

Since $E(s) = \frac{1}{2}(\tau_h + \tau_\ell)$, this holds iff

$$\frac{1}{2}\left(\frac{1}{\tau_\ell} + \frac{1}{\tau_h}\right) > E(1/s).$$

Now,

$$
\begin{aligned}
E(1/s) &= \sum_{s \geq E(s)} \pi(s)\frac{1}{s} + \sum_{s < E(s)} \pi(s)\frac{1}{s} \\
&= \sum_{s \geq E(s)} \pi(s)\frac{2E(s) - s}{[s - E(s) + E(s)][E(s) - (s - E(s))]} \\
&\quad + \sum_{s < E(s)} \pi(s)\frac{2E(s) - s}{[E(s) - (E(s) - s)][E(s) + (E(s) - s)]} \\
&< \sum_{s \geq E(s)} \pi(s)\frac{2E(s) - s}{\tau_h \tau_\ell} + \sum_{s < E(s)} \pi(s)\frac{2E(s) - s}{\tau_h \tau_\ell} \\
&= \sum_{s \in S} \frac{\pi(s)[2E(s) - s]}{\tau_h \tau_\ell} = \frac{E(s)}{\tau_h \tau_\ell} = \frac{1}{2}\left[\frac{1}{\tau_h} + \frac{1}{\tau_\ell}\right].
\end{aligned}
$$

To see the strict inequality in the above equations, consider $s \geq E(s)$ and let $\varepsilon = s - E(s)$. Then, $s = E(s) + \varepsilon$ and $\varepsilon < \tau_h - E(s) = E(s) - \tau_\ell$. Thus,

$$
\begin{aligned}
&\frac{1}{[s - E(s) + E(s)][E(s) - (s - E(s))]} \\
&= \frac{1}{[E(s) + \varepsilon][E(s) - \varepsilon]} = \frac{1}{E(s)^2 - \varepsilon^2} < \frac{1}{E(s)^2 - [E(s) - \tau_\ell]^2} = \frac{1}{\tau_h \tau_\ell}.
\end{aligned}
$$

$\square$

Example 3.

For each $N \in \mathbb{N}$, let $S_N = \{1 + \varepsilon, 1 + 2\varepsilon, ..., 1 + N\varepsilon\}$ with $\varepsilon = (\tau - 1)/(N + 1)$ and let

$$\pi(1 + \varepsilon) = \pi(\tau - \varepsilon) = \frac{1}{2}(1 - \delta),$$

and

$$\pi(1 + n\varepsilon) = \frac{\delta}{N - 2} \text{ for all } 2 \leq n \leq N - 1$$

for some small $\delta > 0$. Then, there exist (large) $\bar{N}$ and (small) $\bar{\delta} > 0$ such that if $N > \bar{N}$ and if $\delta < \bar{\delta}$, then $Y^1(s, \pi)$ is not monotone in s.

Proof. Let $\delta = 0$. It is easy to verify the following.

$$
E_{\tau\tau}(1/s) = \frac{E_\pi(s) + E_\pi(1/s) - 2}{E_\pi(s^2) - 2E_\pi(s) + 1},
$$

$$
E_{\tau 1}(1/s) = E_{1\tau}(1/s) = \frac{(1+\tau) - E_\pi(s) - \tau E_\pi(1/s)}{(1+\tau)E_\pi(s) - E_\pi(s^2) - \tau},
$$

$$
E_{11}(1/s) = \frac{\tau^2 E_\pi(1/s) - 2\tau + E_\pi(s)}{\tau^2 - 2\tau E_\pi(s) + E_\pi(s^2)}.
$$

Now, assuming that the buyer constraint is always binding, we have

$$
Y(s_n, \pi) \propto \frac{(s_n - 1)^2}{(\tau - 1)^2} E_{\tau\tau}(1/s)\tau + \frac{(s_n - 1)(\tau - s_n)}{(\tau - 1)^2} E_{\tau 1}(1/s)(1 + \tau) + E_{11}(1/s)\frac{(\tau - s_n)^2}{(\tau - 1)^2}.
$$

Under $\pi(1 + \varepsilon) = \pi(\tau - \varepsilon) = 1/2$, we have

$$
E_\pi(s) = \frac{1}{2}(1 + \tau),
$$

$$
E_\pi(s^2) = \frac{1}{2}(\tau^2 - 2(\tau - 1)\varepsilon + 2\varepsilon^2 + 1),
$$

$$
E_\pi(1/s) = \frac{1}{2}\frac{\tau + 1}{(\tau - \varepsilon)(1 + \varepsilon)}.
$$

Taking $N \to \infty$, or, equivalently, $\varepsilon \to 0$, we have

$$
\lim_{\varepsilon \to 0} E_{\tau\tau}(1/s) = \lim_{\varepsilon \to 0} \frac{\frac{1}{2}(1 + \tau) + \frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)} - 2}{\frac{1}{2}(\tau^2 - 2(\tau - 1)\varepsilon + 2\varepsilon^2 + 1) - 2\frac{1}{2}(1 + \tau) + 1} = \frac{1}{\tau},
$$

$$
\lim_{\varepsilon \to 0} E_{\tau 1}(1/s) = \lim_{\varepsilon \to 0} \frac{(1 + \tau) - \frac{1}{2}(1 + \tau) - \tau\frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)}}{(1 + \tau)\frac{1}{2}(1 + \tau) - \frac{1}{2}(\tau^2 - 2(\tau - 1)\varepsilon + 2\varepsilon^2 + 1) - \tau} = \frac{1}{2}\frac{\tau + 1}{\tau},
$$

$$
\lim_{\varepsilon \to 0} E_{11}(1/s) = \lim_{\varepsilon \to 0} \frac{\tau^2\frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)} - 2\tau + \frac{1}{2}(1 + \tau)}{\tau^2 - 2\tau\frac{1}{2}(1 + \tau) + \frac{1}{2}(\tau^2 - 2(\tau - 1)\varepsilon + 2\varepsilon^2 + 1)} = 1.
$$

Thus, we have

$$
\lim_{\varepsilon \to 0} Y(s, \pi) \propto \frac{(s - 1)^2}{(\tau - 1)^2} + \frac{(s - 1)(\tau - s)}{(\tau - 1)^2}\frac{(1 + \tau)^2}{2\tau} + \frac{(\tau - s)^2}{(\tau - 1)^2}
$$

$$
= \frac{s + \tau + s\tau - s^2}{2\tau}.
$$

Therefore, $Y(s, \pi)$ is increasing iff $s \leq \frac{1+\tau}{2}$. For ε sufficiently small, the Phillips curve increases first (between $s = 1 + \varepsilon$ and $s = \frac{1+\tau}{2}$), and then decreases at the tail (between $s = \frac{1+\tau}{2}$ and $s = \tau - \varepsilon$). Since the function $Y(s, \pi)$ is continuous in ε and in δ (note that π implicitly depends on δ), the non-monotonicity holds for small ε and small δ as well. $\square$

Example 4.

Suppose that $T = \{1, \tau\}$ with $\tau < 6$, and that π is symmetric with $E(s) = (1+\tau)/2$. If $S \subset [0.3\tau + 0.7, 0.7\tau + 0.3]$, then $Y^1(s, \pi)$ is strictly increasing.

Proof. To keep track of the transfers, we use τ_h to denote τ and τ_ℓ to denote 1. Since $T = \{\tau_\ell < \tau_h\}$, we have

$$E_{\tau_h, \tau_h}(1/s) = \frac{E_\pi(s) + \tau_\ell E_\pi(1/s) - 2\tau_\ell}{E_\pi(s^2) - 2\tau_\ell E_\pi(s) + \tau_\ell^2},$$

$$E_{\tau_h, \tau_\ell}(1/s) = E_{\tau_\ell, \tau_h}(1/s) = \frac{(\tau_h + \tau_\ell) - E_\pi(s) - \tau_h \tau_\ell E_\pi(1/s)}{(\tau_h + \tau_\ell)E_\pi(s) - E_\pi(s^2) - \tau_h \tau_\ell},$$

$$E_{\tau_\ell, \tau_\ell}(1/s) = \frac{\tau_h^2 E_\pi(1/s) - 2\tau_h + E_\pi(s)}{\tau_h^2 - 2\tau_h E_\pi(s) + E_\pi(s^2)}.$$

Since buyer constraint is always binding and $c(y) = y$, we have

$$Y^1(s_n, \pi) \propto \frac{(s_n - \tau_\ell)^2}{(\tau_h - \tau_\ell)^2} E_{\tau_h, \tau_h}(1/s)\tau_h + \frac{(s_n - \tau_\ell)(\tau_h - s_n)}{(\tau_h - \tau_\ell)^2} E_{\tau_h, \tau_\ell}(1/s)(\tau_\ell + \tau_h) + \frac{(\tau_h - s_n)^2}{(\tau_h - \tau_\ell)^2} E_{\tau_\ell, \tau_\ell}(1/s)\tau_\ell.$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} Y^1(s, \pi) \propto \frac{2}{(\tau_h - \tau_\ell)^2}\left[A + B\left(s - \frac{\tau_h + \tau_\ell}{2}\right)\right],$$

where

$$A = \frac{\tau_h - \tau_\ell}{2}\left\{\frac{E_\pi(s) + \tau_\ell E_\pi(1/s) - 2\tau_\ell}{E_\pi(s^2) - 2\tau_\ell E_\pi(s) + \tau_\ell^2}\tau_h - \frac{\tau_h^2 E_\pi(1/s) - 2\tau_h + E_\pi(s)}{\tau_h^2 - 2\tau_h E_\pi(s) + E_\pi(s^2)}\tau_\ell\right\},$$

$$\begin{aligned}
B &= \frac{E_\pi(s) + \tau_\ell E_\pi(1/s) - 2\tau_\ell}{E_\pi(s^2) - 2\tau_\ell E_\pi(s) + \tau_\ell^2}\tau_h + \frac{\tau_h^2 E_\pi(1/s) - 2\tau_h + E_\pi(s)}{\tau_h^2 - 2\tau_h E_\pi(s) + E_\pi(s^2)}\tau_\ell \\
&\quad - \frac{(\tau_h + \tau_\ell) - E_\pi(s) - \tau_h \tau_\ell E_\pi(1/s)}{(\tau_h + \tau_\ell)E_\pi(s) - E_\pi(s^2) - \tau_h \tau_\ell}(\tau_h + \tau_\ell).
\end{aligned}$$

Now,

$$A = (\tau_h - \tau_\ell)^2\frac{\tau_h + \tau_\ell - 2\tau_h \tau_\ell E_\pi(1/s)}{(\tau_h - \tau_\ell)^2 + 4Var_\pi(s)},$$

$$B = 16(\tau_h - \tau_\ell)^2\frac{\tau_h \tau_\ell\left[E_\pi(s)E_\pi(1/s) - 1\right] - Var_\pi(s)}{(\tau_h - \tau_\ell)^4 - 16Var(s)^2}.$$

Now, $S = \{s_1, ..., s_N\}$ and consider the case where N is even. Since π is symmetric, for some $\varepsilon_1 > ... > \varepsilon_{N/2} > 0$, $s_n = E(s) - \varepsilon_n$ and $s_{N-n} = E(s) + \varepsilon_n$ with $\pi(s_{N-n}) = \pi(s_n)$ for all $n = 1, ..., N/2$. Let $t = \frac{\tau_h + \tau_\ell}{2}$ and let $e = \frac{\tau_h - \tau_\ell}{2}$. Since $S \subset [0.3\tau_h + 0.7\tau_\ell, 0.7\tau_h + 0.3\tau_\ell]$, it follows that $\varepsilon_n < 0.7e$ for all $n = 1, ..., N/2$.

Now, $Var_\pi(s) = \sum_{n=1}^{N/2} 2\pi(s_n)\varepsilon_n^2$. We also use π to denote the distribution over ε_n's.

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Then,

$$
\begin{aligned}
A &= 4e^2 \left\{ \frac{2t - 2(t-e)(t+e)E_\pi\left[\frac{t}{t^2-\varepsilon^2}\right]}{4e^2 + 4E_\pi(\varepsilon^2)} \right\} \\
&> 4e^2 \left\{ \frac{2t - 2(t-e)(t+e)\frac{t}{t^2-(0.7e)^2}}{4e^2 + 4(0.7e)^2} \right\} > 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
B &= 16(4e^2) \left\{ \frac{(t-e)(t+e)\left[tE_\pi\left[\frac{t}{t^2-\varepsilon^2}\right] - 1\right] - E_\pi(\varepsilon^2)}{16e^4 - 16E_\pi(\varepsilon^2)^2} \right\} \\
&= -4e^2 \left\{ \frac{E_\pi\left(\frac{e^2-\varepsilon^2}{t^2-\varepsilon^2}\varepsilon^2\right)}{e^4 - E_\pi(\varepsilon^2)^2} \right\} > -4e^2 \left\{ \frac{\frac{e^2-(0.7e)^2}{t^2-(0.7e)^2}(0.7e)^2}{e^4 - (0.7e)^4} \right\},
\end{aligned}
$$

where the second inequality follows from the fact that $\frac{e^2-\varepsilon^2}{t^2-\varepsilon^2}\varepsilon^2$ is increasing in ε as long as $\varepsilon < 0.7e$:

$$
\frac{\mathrm{d}}{\mathrm{d}(\varepsilon^2)}\left(\frac{e^2-\varepsilon^2}{t^2-\varepsilon^2}\varepsilon^2\right) = \frac{(e^2-2\varepsilon^2)(t^2-\varepsilon^2) + (e^2\varepsilon^2 - \varepsilon^4)}{(t^2-\varepsilon^2)^2} > 0
$$

if and only if

$$
e^2t^2 - 2t^2\varepsilon^2 + \varepsilon^4 > 0,
$$

which holds if $e^2 > 2\varepsilon^2$.

Now, since $A > 0$ and $B < 0$, $\frac{\mathrm{d}}{\mathrm{d}s}Y^1(s) > 0$ for all $s \in S$ if

$$
A + B(0.7e) > 0,
$$

which holds if

$$
\left\{ \frac{2t - 2(t-e)(t+e)\frac{t}{t^2-(0.7e)^2}}{4e^2 + 4(0.7e)^2} \right\} - \left\{ \frac{\frac{e^2-(0.7e)^2}{t^2-(0.7e)^2}(0.7e)^2}{e^4 - (0.7e)^4} \right\}(0.7e) > 0,
$$

which holds if

$$
\frac{1}{2}e^2\frac{1/2}{(1-1/4)(t^2-0.5e^2)}(t - 0.7e - 0.5t) > 0,
$$

which is equivalent to $0.5t > 0.7e$, that is, $\tau_h < 6\tau_\ell$. $\square$

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