Quality and Competition between Public and Private Firms

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Abstract

We study a multi-stage, quality-price game between a public firm and a private firm. The market consists of a set of consumers who have different quality valuations. A public firm aims to maximize social surplus, whereas the private firm maximizes profit. In the first stage, both firms simultaneously choose qualities. In the second stage, both firms simultaneously choose prices. Consumers' quality valuations follow a general distribution. Firms' unit production cost is an increasing and convex function of quality. There may be multiple equilibria. In some, the public firm chooses a low quality, and the private firm chooses a high quality. In others, the opposite is true. We characterize subgame-perfect equilibria, and provide conditions on consumer valuation distribution for first-best equilibrium qualities. Various policy implications are drawn.

Keywords: price-quality competition, quality, public firm, private firm, mixed oligopoly

JEL Classifications: D4, L1, L2, L3

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1 Introduction

In many markets such as health care, education, transportation, and utility, public and private firms jointly serve consumers. Product and service qualities are major concerns in these markets. These concerns stem from a fundamental point made by Spence (1975). Because a good's quality benefits all buyers, the social benefit of quality is the sum of consumers' valuations. At a social optimum, the average consumer valuation of quality should be equal to the marginal cost of quality. Yet, a profit-maximizing firm is only concerned with the consumer who is indifferent between buying and not. A firm's choice of quality will be one that equates this marginal consumer's valuation to the marginal cost of quality. This gives the classic Spence (1975) result: even when products are priced at marginal costs, their qualities will be inefficient. In this paper, we show that a mixed oligopoly, in which public and private firms interact, may be a mechanism for remedying this inefficiency.

We use a standard model of vertical product differentiation. In the first stage, two firms simultaneously choose product qualities. In the second stage, firms simultaneously choose product prices. The two firms have access to the same technology. The only difference from the textbook setup is that one firm is a social-surplus maximizing public firm, whereas the other remains a profit-maximizing private firm. Surprisingly, this single difference has many implications.

First, the model exhibits multiple equilibria: in some equilibria, the public firm's product quality is higher than the private firm's, but in others, the opposite is true. More important, equilibrium qualities may be socially efficient. In fact, we present general conditions on consumers' quality-valuation distribution in which qualities in low-public-quality equilibria are efficient, as well as general conditions in which qualities in high-public-quality equilibria are efficient. When equilibrium qualities are inefficient, for both public and private firms, deviations from the first best go in tandem. That is, either qualities in public and private firms are both below the corresponding first-best levels, or they are both above. Equilibrium qualities form a rich set, and we have constructed examples with many configurations.

Our analysis proceeds in the standard way. Given a subgame defined by a pair of qualities, we find the equilibrium prices. Then we solve for equilibrium qualities, letting firms anticipate that their quality choices will result in the corresponding equilibrium prices in the next stage. In the pricing subgame, qualities are given. The public firm's objective is to maximize social surplus, so its price best response must achieve the efficient allocation of consumers across the two firms. This requires that consumers fully internalize the cost difference between high and low qualities. Given the private firm's price, the public firm sets its price so that the difference in prices is exactly the difference in quality costs. The private firm's best response is the typical inverse demand elasticity rule.

When firms choose qualities, they anticipate the equilibrium prices in the next stage. Given the private firm's quality, the public firm chooses its quality to maximize the surplus of consumers that it will serve. It anticipates the efficient assignment of consumers across firms in the next stage, so it chooses quality for the best welfare of its own customers. The private firm, however, will try to manipulate the equilibrium prices through its quality.

Because the equilibrium price difference will be the quality cost difference, when the private firm chooses a quality different from the public firm's, it implements a larger price difference. Without any price response from the public firm, the private firm would have chosen the quality that would be optimal for the marginal consumer, just as we have stated above (Spence (1975)). A larger quality difference, however, would be preferred because that would raise the price. Because of the price manipulation, the private firm's equilibrium quality is one that maximizes the utility of an inframarginal consumer, not the utility of the marginal consumer who would just be indifferent between buying from the public and private firms.

In the first best, the socially efficient qualities are determined by equating average consumer valuations and marginal cost of quality. The surprise is that in contrast to private duopoly, the private firm's equilibrium quality choice may coincide with the first-best quality. In other words, the inframarginal consumer whose utility is being maximized by the private firm happens to have the average valuation among the private firm's customers.

The (sufficient) conditions for first-best equilibria refer to properties of consumers' quality-valuation distribution. In the class of equilibria where the public firm produces at a low quality, equilibrium qualities are first best when the valuation distribution has a linear hazard rate.[1] The linear hazard rate condition is equivalent to the private firm's marginal revenue function linear in consumer valuation. In the class of equilibria where the public firm produces at a high quality, equilibrium qualities are first best when the valuation distribution has a linear reverse hazard rate. The linear reverse hazard rate condition is equivalent to the private firm's marginal revenue function being linear in consumer valuation.

Although the hazard and reverse hazard rates have figured prominently in the information economics literature, we are unaware of any work that imposes linearity on them. We derive all distributions that possess the linearity properties. We wish to note that the uniform distribution, which has been used often to describe consumer valuations, has linear hazard *and* reverse hazard rates (but it is not the only distribution with this property—see Remark 5 below). Also, because hazard and reverse hazard rates can behave quite differently, for some distributions equilibria for one class (say, low quality at public) can be the first best but not equilibria in the other (say high quality at public), and vice versa.

We draw various policy implications from our results. Our use of a social-welfare objective function for the public firm can be regarded as making a normative point. If the public firm aims to maximize only consumer surplus, then it will subscribe to marginal-cost pricing. Then the equilibrium price difference between public and private firms will never be the quality cost difference because the private firm never prices at marginal cost. The first best is never achieved (even when hazard or reverse hazard rates are linear). A social-welfare objective does mean that the public firm tolerates high prices. However, our policy recommendation is that undesirable effects from high prices should be remedied by a tax credit or subsidy to consumers regardless of from where they purchase. This ensures that consumers face a price difference equal to the quality difference, a necessary condition for the first best.

Our research contributes to the literature of mixed oligopolies. We use the classical model of quality-price competition in Gabszewicz and Thisse (1979, 1986) and Shaked and Sutton (1982, 1983). Whereas profit-maximizing firms use quality differentiation to relax price competition, a social-surplus maximizing

[1] See Lemmas 3 and 6 below. If F denotes the distribution, and f the density, then the hazard rate is $\frac{1-F}{f}$, and the reverse hazard rate is $\frac{F}{f}$. By a function being linear, we mean that it has a constant slope and an intercept. This is often called affine linear in mathematics, but we trust that our abbreviation will not cause any confusion.

public firm does not. This difference has led to the mixed oligopoly literature, which revolves around the theme that the presence of a public firm may improve welfare.

Grilo (1994) studies a mixed duopoly in the quality, vertical differentiation framework. In her model, consumers' valuations of qualities follow a uniform distribution. The unit cost of production may be convex or concave in quality. The paper derives first-best equilibria. In a Hotelling, horizontal differentiation model with quadratic transportation cost, Cremer et al. (1991) show that a public firm improves welfare when the total number of firms is either two, or more than six. Also using a Hotelling model, Matsumura and Matsushima (2004) show that mixed oligopoly gives some cost-reduction incentives. In a Cournot model, Cremer et al. (1989) consider replacing some private firms by public enterprises, and nationalizing some private firms. In these models, public firms may discipline private firms.

For profit-maximizing firms, Cremer and Thisse (1991) show that, under very mild conditions on transportation costs, horizontal differentiation models are actually a special case of vertical product differentiation (see also Champsaur and Rochet (1989)). The isomorphism can be transferred to mixed duopolies. The key in the Cremer-Thisse (1991) proof is that demands in horizontal models can be translated into equivalent demands in vertical models. Firms' objectives are unimportant. Hence, results in horizontal mixed oligopolies do relate to vertical mixed oligopolies. In most horizontal differentiation models, consumers are assumed to be uniformly distributed on the product space, and the transportation or mismatch costs are quadratic. These assumptions translate to a uniform distribution of consumer quality valuations and a quadratic quality cost function in vertical differentiation models.

The first-best results in Grilo (1994) are related to the efficient equilibria in the two-firm case in Cremer et al. (1991) because both papers use the uniform distribution for consumer valuations. (Grilo (1994), however, does not use the quadratic transportation cost assumption.) By contrast, we use a general distribution for consumer valuation and a general quality cost function. In this general environment, we fully characterize equilibria. Our results simultaneoulsly reveal the limitation of the uniform distribution and which properties of the uniform distribution (linear hazard and reverse hazard rates) have been the driver of earlier results. Furthemore, when consumer valuations follow a uniform distribution, the issue of multiple equilibria is moot for a duopoly. As we show below, multiple equilibria are important for general distributions. We derive conditions on general distributions for first-best equilibria. Moreover, our equilibrium qualities translate to equilibrium locations under general consumer distributions on the Hotelling line and transportation costs. Therefore, our conditions on consumer valuation distributions for first-best qualities will be corresponding conditions on consumer location distributions on the Hotelling line for horizontal mixed oligopolies.

For private firms, Anderson et al. (1997) provide the first characterization for a general location distribution with quadratic transportation costs. Our techniques are consistent with those in Anderson et al. (1997), but we use a general cost function. A recent paper by Benassi et al. (2006) uses a symmetric trapezoid valuation distribution and explores consumers' nonpurchase options. Yurko (2011) has worked with lognormal distributions. Our monotone hazard and reverse hazard rate assumptions are valid under the trapezoid distribution, but invalid under lognormal distributions. In any case, our general characterization on the private oligopoly complements these recent advances.

Qualities in mixed provisions are often discussed in the education and health sectors. However, perspectives such as political economy, taxation, and income redistribution are incorporated, so public firms typically are assumed to have objective functions different from social welfare. Brunello and Rocco (2008) combine consumers voting and quality choices by public and private schools, and let the public school be a Stackelberg leader. Epple and Romano (1998) consider vouchers and peer effects but have used a competitive model for interaction between public and private schools. Grassi and Ma (2011, 2012) present models of publicly rationed supply and private firm price responses under public commitment and noncommitment. Our results here indicate that commitment may not be necessary, and imperfectly competitive markets may sometimes be efficient.

Privatization has been a policy topic in mixed oligopolies. Ishibashi and Kaneko (2008) set up a mixed duopoly with price and quality competition. The model has both horizontal (Hotelling) and vertical differentiation. However, all consumers have the same valuation on quality, and are uniformly distributed on the horizontal product space (as in Ma and Burgess (1993)). They show that the government should manipulate the objective of the public firm so that it maximizes a weighted sum of profit and social welfare, a form of partial privatization. (Using a Cournot model, Matsumura (1998) earlier demonstrates that partial privatization is a valuable policy.) Our model is richer on the vertical dimension, but consists of no horizontal differentiation. Our policy implication has a privatization component to it, but a simple social welfare objective for the public firm is sufficient.

Section 2 presents the model. Section 3 studies equilibria in which the public firm's quality is lower than the private firm's, and Section 4 studies the opposite case. In each section, we first derive subgame-perfect equilibrium prices, and then equilibrium qualities. We present a characterization of equilibrium qualities, and conditions for equilibrium qualities to be first best. Section 5 considers policies, and various robustness issues. We consider alternative preferences for the public firm. We also let cost functions of the firms be different. Then we let consumers have outside options. Finally we consider multiple private firms. Section 6 presents a benchmark private duopoly model. The last section draws some concluding remarks. Proofs are collected in the Appendix. Details of numerical computation are in the Supplement.

2 The model

2.1 Consumers

There is a set of consumers with total mass normalized at 1. Each consumer would like to receive one unit of a good or service. In our context, it is helpful to think of such goods and services as education, transportation, and health care including child care, medical, and nursing home services. The public sector often participates actively in these markets. In fact, in the literature, many papers are written for these specific markets; see, for example, Epple and Romano (1998) and Brunello and Rocco (2008).

A good has a quality, denoted by q, which is assumed to be positive. Each consumer has a valuation of quality v. This valuation varies among consumers. We let v be a random variable defined on the positive support $[\underline{v}, \overline{v}]$ with distribution F and strictly positive density f. We also assume that f is continuously differentiable

We will use two properties of the distribution, namely $[1-F]/f \equiv h$, and $F/f \equiv k$. We assume that h is decreasing, and that k is increasing, so $h'(v) < 0$ and $k'(v) > 0$. The assumptions ensure that profit functions, to be defined below, are quasi-concave, and are implied by f being logconcave (Anderson et al. (1997)). These monotonicity assumptions are satisfied by many common distributions such as the uniform, the exponential, the beta, etc. (Bagnoli and Bergstrom (2004)). We will call h the hazard rate, and k the reverse hazard rate, although the terminology used by economists varies.[2]

Valuation variations among consumers have the usual interpretation of preference diversity due to wealth, taste, or cultural differences. We may call a consumer with valuation v a type-v consumer, or simply consumer v. If a type-v consumer purchases a good with quality q at price p, his utility is $vq - p$. The quasi-linear utility function is commonly adopted in the literature (see, for example, the standard texts Tirole (1988) and Anderson et al. (1992)).

We assume that each consumer will buy a unit of the good. This can be made explicit by postulating that each good offers a sufficiently high benefit which is independent of v, or that the minimum valuation $\underline{v}$ is sufficiently high. The full-market coverage assumption is commonly used in the extant literature of product differentiation (either horizontal or vertical), but Delbono et al. (1996) and Benassi et al. (2016) have explored the implications of consumer outside options, and we defer to Subsection 5.4 to discuss more.

2.2 Public and private firms

There are two firms, Firm 1 and Firm 2, and they have access to the same technology. Production requires a fixed cost. The implicit assumption is that the fixed cost is so high that entries by many firms cannot be sustained. We focus on the case of a mixed oligopoly so we do not consider the rather trivial case of two public firms. Often a mixed oligopoly is motivated by a more efficient private sector, so in Subsection 5.3 we let firms have different technologies, and will explain how our results remain robust.

The variable, unit production cost of the good at quality q is $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and strictly convex function. A higher quality requires a higher marginal cost, and this marginal cost also increases in quality. We also assume that c is twice differentiable, and that it satisfies the usual

[2] In statistics $f/(1-F)$ is called the hazard rate. Suppose that the random variable x has distribution F and density f. Then $f(v)/(1-F(v))$ is the conditional density of $x = v$ given that $x \geq v$. For example, if x denotes the time of failure, the hazard rate measures the density of failure occurring at v given that failure has not occurred before v. We are unable to find a common usage for f/F in statistics. However, f/F is the conditional density of $x = v$ given that $x \leq v$. That is, this is the density of failure occurring at $x = v$ given that failure has occurred by v.

Inada conditions: $\lim_{q \to 0^+} c(q) = \lim_{q \to 0^+} c'(q) = 0$, so in both the first best and in the equilibria of the extensive forms to be analyzed, both firms will be active.

Firm 1 is a public firm, and run by a utilitarian regulator. Firm 1's objective is to maximize social surplus; the discussion of a general objective function for the public firm is deferred until Subsection 5.2. Firm 2 is a profit-maximizing private firm. Each firm chooses its product quality and price. We let p_1 and q_1 denote Firm 1's price and quality; similarly, p_2 and q_2 denote Firm 2's price and quality. Given these prices and qualities, each consumer buys from the firm that offers the higher utility. A consumer chooses a firm with a probability equal to a half if he is indifferent between them.

Consider any (p_1, q_1) and (p_2, q_2), and define $\widehat{v}$ by $\widehat{v}q_1 - p_1 = \widehat{v}q_2 - p_2$. Consumer $\widehat{v}$ is supposed to be just indifferent between purchasing from Firm 1 and Firm 2. If $\widehat{v} \in [\underline{v}, \overline{v}]$, then the demands for the two firms are as follows:

$$\begin{array}{ccl} \text{Demand for Firm 1} & \text{Demand for Firm 2} & \\ F(\widehat{v}) & 1 - F(\widehat{v}) & \text{if} \quad q_1 < q_2 \\ 1 - F(\widehat{v}) & F(\widehat{v}) & \text{if} \quad q_1 > q_2 \\ 1/2 & 1/2 & \text{if} \quad q_1 = q_2 \end{array} \tag{1}$$

We sometimes call consumer $\widehat{v}$ the indifferent or marginal consumer. (Otherwise, if $\widehat{v} \notin [\underline{v}, \overline{v}]$, or fails to exist, one firm will be unable to sell to any consumer.)

If Firm 1's product quality is lower than Firm 2's, its demand is $F(\widehat{v})$ when its price is sufficiently lower than Firm 2's price. Conversely, if Firm 2's price is not too high, then its demand is $1 - F(\widehat{v})$. If the two firms' product qualities are identical, then they must charge the same price if both have positive demands. In this case, all consumers are indifferent between them, and each firm receives half of the market. The demand functions exhibit discontinuity when firms offer products with identical qualities: any small price difference will cause demand to shift completely to the firm that offers the lower price.

2.3 Allocation, social surplus, and first best

An allocation consists of a pair of product qualities, one at each firm, and an assignment of consumers across the firms. The social surplus from an allocation is

$$\int_{\underline{v}}^{v} [xq_\ell - c(q_\ell)]f(x)\mathrm{d}x + \int_{v}^{\overline{v}} [xq_h - c(q_h)]f(x)\mathrm{d}x. \tag{2}$$

Here, the qualities at the two firms are q_ℓ and q_h, $q_\ell < q_h$. Those consumers with valuations between $\underline{v}$ and v get the good with quality q_ℓ, whereas those with valuations between v and $\overline{v}$ get the good with quality q_h. The first best is (q_ℓ^*, q_h^*, v^*) that maximizes (2), and is characterized by the following:

$$\frac{\displaystyle\int_{\underline{v}}^{v^*} xf(x)\mathrm{d}x}{F(v^*)} = c'(q_\ell^*) \tag{3}$$

$$\frac{\displaystyle\int_{v^*}^{\overline{v}} xf(x)\mathrm{d}x}{1 - F(v^*)} = c'(q_h^*) \tag{4}$$

$$v^* q_\ell^* - c(q_\ell^*) = v^* q_h^* - c(q_h^*). \tag{5}$$

The characterization of the first best in (3), (4), and (5) is standard. Those consumers with lower valuations should consume the good at a low quality (q_ℓ^*), and those with higher valuations should consume at a high quality (q_h^*). Therefore, for the first best, divide consumers into two groups: those with $v \in [\underline{v}, v^*]$ and those with $v \in [v^*, \overline{v}]$. The (conditional) average valuation of consumers in $[\underline{v}, v^*]$ is in the left-hand side of (3), and, in the first best, this is equal to the marginal cost of the lower first-best quality, the right-hand side of (3). A similar interpretation applies to (4) for those consumers with higher valuations. Finally, the division of consumers into the two groups is achieved by identifying consumer v^* who enjoys the same surplus from both qualities, and this yields (5).

As Spence (1975) has shown, quality is like a public good, so the total social benefit is the aggregate consumer benefit, and in the first best, the average valuation should be equal to the marginal cost of quality. As a result the indifferent consumer v^* actually receives too little surplus from q_ℓ because $v^* > c'(q_\ell)$, but too much from q_h because $v^* < c'(q_h)$.

2.4 Extensive form

We study subgame-perfect equilibria of the following game.

Stage 0: Nature draws consumers' valuations v and these are known to consumers only.

Stage 1: Firm 1 chooses a quality q_1; simultaneously, Firm 2 chooses a quality q_2.

Stage 2: Qualities in Stage 1 are common knowledge. Firm 1 chooses a price p_1; simultaneously, Firm 2 chooses a price p_2.

Stage 3: Consumers observe price-quality offers from both firms, and pick a firm for purchase.

An outcome of this game consists of firms' prices and qualities, (p_1, q_1) and (p_2, q_2), and the allocations of consumers across the two firms. Subgames at Stage 2 are defined by the firms' quality pair (q_1, q_2). Subgame-perfect equilibrium prices in Stage 2 are those that are best responses in subgames defined by (q_1, q_2). Finally, equilibrium qualities in Stage 1 are those that are best responses given that prices are given by a subgame-perfect equilibrium in Stage 2.

There are multiple equilibria. In one class of equilibria, in Stage 1 the public firm chooses low quality, whereas the private firm chooses high quality, and in Stage 2, the public firm sets a low price, and the private firm chooses a high price. In the other class, the roles of the firms, in terms of their ranking of qualities and prices, are reversed. However, because the two firms have different objectives, equilibria in these two classes yield different allocations.

3 Equilibria with low quality at public firm

In this section, we study equilibria when the public Firm 1's quality q_1 is lower than the private Firm 2's quality q_2.

3.1 Subgame-perfect equilibrium prices

Consider subgames in Stage 2 defined by (q_1, q_2) with $q_1 < q_2$. According to (1), each firm will have a positive demand only if $p_1 < p_2$, and there is $\widetilde{v} \in [\underline{v}, \overline{v}]$ with

$$\widetilde{v}q_1 - p_1 = \widetilde{v}q_2 - p_2 \qquad \text{or} \qquad \widetilde{v}(p_1, p_2; q_1.q_2) = \frac{p_2 - p_1}{q_2 - q_1}, \tag{6}$$

where we have emphasized that $\widetilde{v}$, the consumer indifferent between buying from Firm 1 and Firm 2, depends on qualities and prices. Expression (6) characterizes firms' demand functions. Firms' payoffs are:

$$\int_{\underline{v}}^{\widetilde{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\widetilde{v}}^{\overline{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x \tag{7}$$

$$[1 - F(\widetilde{v})][p_2 - c(q_2)]. \tag{8}$$

The expression in (7) is social surplus when consumers with valuations in $[\underline{v}, \widetilde{v}]$ buy from Firm 1, whereas others buy from Firm 2. The prices that consumers pay to firms are transfers, so do not affect social surplus. The expression in (8) is Firm 2's profit.

Firm 1 chooses its price p_1 to maximize (7) given the demand (6) and price p_2. Firm 2 chooses price p_2 to maximize (8) given the demand (6) and price p_1. Equilibrium prices, $(\widehat{p}_1, \widehat{p}_2)$, are best responses against each other.

Lemma 1 *In subgames (q_1, q_2) with $q_1 < q_2$, and $\underline{v} < \dfrac{c(q_2) - c(q_1)}{q_2 - q_1} < \overline{v}$, equilibrium prices $(\widehat{p}_1, \widehat{p}_2)$ are:*

$$\widehat{p}_1 - c(q_1) = \widehat{p}_2 - c(q_2) = (q_2 - q_1)\frac{1 - F(\widehat{v})}{f(\widehat{v})} \equiv (q_2 - q_1)h(\widehat{v}), \tag{9}$$

$$\textit{where} \qquad \widehat{v} = \frac{c(q_2) - c(q_1)}{q_2 - q_1}. \tag{10}$$

Lemma 1 says that the equilibrium price difference across firms is the same as the cost difference: $\widehat{p}_2 - \widehat{p}_1 = c(q_2) - c(q_1)$. Second, it says that Firm 2 makes a profit, and its price-cost margin is proportional to the quality differential and the hazard rate h.

We explain the result as follows. Firm 1's payoff is social surplus, so it seeks the consumer assignment to the two firms, $\widetilde{v}$, to maximize social surplus (7). This is achieved by getting consumers to fully internalize the cost difference between the high and low qualities. Therefore, given $\widehat{p}_2$, Firm 1 sets $\widehat{p}_1$ so that the price differential $\widehat{p}_2 - \widehat{p}_1$ is equal to the cost differential $c(q_2) - c(q_1)$. In equilibrium, the indifferent consumer is given by $\widehat{v}q_1 - c(q_1) = \widehat{v}q_2 - c(q_2)$, which indicates an efficient allocation in the quality subgame (q_1, q_2).

Firm 2 seeks to maximize profit. Given Firm 1's price $\widehat{p}_1$, Firm 2's optimal price follows the marginal-revenue-marginal-cost calculus. For a unit increase in p_2, the marginal loss is $[p_2 - c(q_2)]f(\widetilde{v})/(q_2 - q_1)$, whereas the marginal gain is $[1 - F(\widetilde{v})]$. Therefore, profit maximization yields $\widehat{p}_2 - c(q_2) = (q_2 - q_1)\dfrac{1 - F(\widehat{v})}{f(\widehat{v})}$. (This is also the standard inverse elasticity rule for the determination of Firm 2's price-cost margin.[3]) Putting firms' best responses together, we have Lemma 1.

The key point in Lemma 1 is that equilibrium market shares and prices can be determined separately. Once qualities are given, Firm 1 will aim for the socially efficient allocation, and it adjusts its price, given Firm 2's price, to achieve that. Firm 2, on the other hand, aims to maximize profit so its best response depends on Firm 1's price as well as the elasticity of demand. Firm 1 does make a profit, and we will return to this issue in Subsection 5.2.

To complete the characterization of price equilibria, we consider subgames (q_1, q_2) with $q_1 < q_2$, and either $\dfrac{c(q_2) - c(q_1)}{q_2 - q_1} < \underline{v}$ or $\overline{v} < \dfrac{c(q_2) - c(q_1)}{q_2 - q_1}$. In the former case, Firm 1 would like to allocate all consumers to Firm 2, whereas in the other case, Firm 1 would like to allocate all consumers to itself. In both cases, there are multiple equilibrium prices. They take the form of high values of $\widehat{p}_1$ when all consumers go to Firm 2, but low values of $\widehat{p}_1$ in the other. In any case, equilibria in the game must have two active firms, so these subgames cannot arise.[4]

The equilibrium prices $(\widehat{p}_1, \widehat{p}_2)$ in (9) and (10) formally establish three functional relationships, those that relate any qualities to equilibrium prices and allocation of consumers across firms. We can write them as $\widehat{p}_1(q_1, q_2)$, $\widehat{p}_2(q_1, q_2)$, and $\widehat{v}(q_1, q_2) \equiv \widetilde{v}(\widehat{p}_1(q_1, q_2), \widehat{p}_2(q_1, q_2); q_1.q_2)$. Applying the Implicit Function Theorem, we derive how equilibrium prices and market share change with qualities. As it turns out, we will only need to use the information of how $\widehat{p}_1(q_1, q_2)$ and $\widehat{p}_2(q_1, q_2)$ change with q_2:

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[3] Firm 2's demand is $1 - F(\widetilde{v})$. Hence, elasticity is $\dfrac{\mathrm{d}(1 - F(\widetilde{v}))}{\mathrm{d}p_2}\dfrac{p_2}{1 - F(\widetilde{v})} = -\dfrac{q_2 - q_1}{h(\widetilde{v})}p_2$.

[4] The discussion of equilibria of firms having identical qualities is in Subsection 4.1.

Lemma 2 *From the definition of* $(\widehat{p}_1, \widehat{p}_2)$ *and* $\widehat{v}$ *in (9) and (10), we have* $\widehat{v}$ *increasing in* q_1 *and* q_2*, and*

$$\frac{\partial \widehat{p}_1(q_1,q_2)}{\partial q_2} = h(\widehat{v}) + h'(\widehat{v})\left[c'(q_2) - \widehat{v}\right] \tag{11}$$

$$\frac{\partial \widehat{p}_2(q_1,q_2)}{\partial q_2} = c'(q_2) + h(\widehat{v}) + h'(\widehat{v})\left[c'(q_2) - \widehat{v}\right]. \tag{12}$$

Lemma 2 describes how the equilibrium consumer changes with qualities, and the strategic effect of Firm 2's quality on Firm 1's price. Consider a subgame (q_1, q_2). Figure 1 shows the determination of $\widehat{v}$. We have drawn the utility function of the marginal consumer $\widehat{v}$, whose utilities are $\widehat{v}q_1 - p_1 = \widehat{v}q_2 - p_2$. Suppose that q_1 increases. From Figure 1, consumer $\widehat{v}$ strictly prefers to buy from Firm 1, as does consumer $\widehat{v} + \epsilon$ for a small and positive value of ϵ. Next, suppose that q_2 increases, consumer also $\widehat{v}$ strictly prefers to buy from Firm 1. The point is that quality q_1 is too low for consumer $\widehat{v}$ but quality q_2 is too high. An increase in q_1 makes Firm 1 more attractive to consumer $\widehat{v}$, but an increase in q_2 makes Firm 2 less attractive to him.

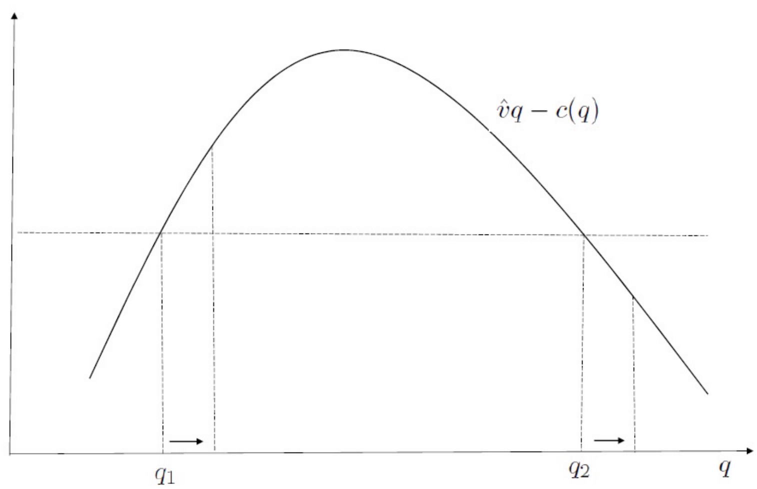

Figure 1: Qualities and the marginal consumer's utility

If Firm 2 increases its quality, it expects to lose market share. However, it does not mean that its profit must decrease. From (8), Firm 2's profit is increasing in Firm 1's price.[5] Hence if in fact Firm 1 raises its price against a higher q_2, Firm 2 may earn a higher profit. In any case, because h is decreasing, and

[5] The partial derivative of (8) with respect to p_1 is $\dfrac{f(\widehat{v})[p_2 - c(q_2)]}{q_2 - q_1} > 0$.

$c'(q_2) > \widehat{v}$, according to Lemma 2, an increase in q_2 may result in higher or lower equilibrium prices. The point is simply that Firm 2 can influence Firm 1's price response. Also, Firm 2's equilibrium price always increases at a higher rate than Firm 1's: $\partial\widehat{p}_2/\partial q_2 - \partial\widehat{p}_1/\partial q_2 = c'(q_2)$ (see (11) for $\partial\widehat{p}_1/\partial q_2$ and (12) for $\partial\widehat{p}_2/\partial q_2$).

3.2 Subgame-perfect equilibrium qualities

At qualities q_1 and q_2, the continuation equilibrium payoffs for Firms 1 and 2 are, respectively,

$$\int_{\underline{v}}^{\widehat{v}(q_1,q_2)} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\widehat{v}(q_1,q_2)}^{\overline{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x \tag{13}$$

$$[1 - F(\widehat{v}(q_1,q_2))][\widehat{p}_2(q_1,q_2) - c(q_2)], \tag{14}$$

where $\widehat{p}_2$ is Firm 2's equilibrium price and $\widehat{v}$ is the indifferent consumer from Lemma 1. Let $(\widehat{q}_1, \widehat{q}_2)$ be the equilibrium qualities. They are mutual best responses, given continuation equilibrium prices:

$$\widehat{q}_1 = \underset{q_1}{\operatorname{argmax}} \int_{\underline{v}}^{\widehat{v}(q_1,\widehat{q}_2)} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\widehat{v}(q_1,\widehat{q}_2)}^{\overline{v}} [x\widehat{q}_2 - c(\widehat{q}_2)]f(x)\mathrm{d}x \tag{15}$$

$$\widehat{q}_2 = \underset{q_2}{\operatorname{argmax}} [1 - F(\widehat{v}(\widehat{q}_1,q_2))][\widehat{p}_2(\widehat{q}_1,q_2) - c(q_2)]. \tag{16}$$

A change in quality q_1 has two effects on social surplus (13). First, it directly changes $vq_1 - c(q_1)$, the surplus of consumers who purchase the good at quality q_1. Second, it changes the equilibrium prices and the marginal consumer $\widehat{v}$ (hence market shares) in Stage 3. This second effect is second order because the equilibrium prices in Stage 3 maximize social surplus. Hence, the first-order derivative of (13) with respect to q_1 is $\int_{\underline{v}}^{\widehat{v}(q_1,q_2)}[x - c'(q_1)]f(x)\mathrm{d}x$.

Similarly, a change in quality q_2 has two effects on Firm 2's profit. First, it directly changes the marginal consumer's surplus $\widehat{v}q_2 - c(q_2)$. Second, it changes the equilibrium prices and the marginal consumer. We rewrite (16) as

$$[1 - F(\widehat{v}(q_1,q_2))]\,[\widehat{v}(q_1,q_2)q_2 - c(q_2) - \widehat{v}(q_1,q_2)q_1 + \widehat{p}_1(q_1,q_2)] \tag{17}$$

because

$$\widehat{v}(q_1,q_2) = \widetilde{v}(\widehat{p}_1(q_1,q_2), \widehat{p}_2(q_1,q_2); q_1.q_2) \equiv \frac{\widehat{p}_1(q_1,q_2) - \widehat{p}_2(q_1,q_2)}{q_1 - q_2} \tag{18}$$

which gives the channels for the influence of q_2 on prices. Firm 2's equilibrium price in Stage 3 maximizes profit, so the effect of q_2 on profit in (17) via $\widehat{v}(q_1, q_2)$ has a second-order effect. Therefore, the first-order derivative of (17) with respect to quality q_2 is $\widehat{v}(q_1, q_2) - c'(q_2) + \dfrac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2}$ (where we have omitted the factor $[1 - F(\widehat{v}(q_1, q_2))]$).

We set the first-order derivatives of social surplus with respect to q_1 and of profit with respect to q_2 to zero. Then we apply (11) in Lemma 2 to obtain the following.

Proposition 1 *Equilibrium qualities* ($\widehat{q}_1$, $\widehat{q}_2$), *and the marginal consumer* $\widehat{v}$ *solve the following three equations in* q_1, q_2, *and* v

$$
\begin{aligned}
\frac{\int_{\underline{v}}^{v} x f(x) dx}{F(v)} &= c'(q_1) \\
v + \frac{h(v)}{1 - h'(v)} &= c'(q_2) \\
v q_2 - c(q_2) &= v q_1 - c(q_1).
\end{aligned}
$$

As we have explained, given Firm 2's quality and the continuation equilibrium prices, Firm 1's return to quality q_1 consists of its own consumers. Hence $\widehat{q}_1$ equates the conditional average valuation of consumers in $[\underline{v}, \widehat{v}]$, $\dfrac{\int_{\underline{v}}^{\widehat{v}} x f(x) \mathrm{d}x}{F(\widehat{v})}$, and the marginal cost $c'(q_1)$. This is the first equation.

Firm 2's quality will affect Firm 1's price in Stage 2. If this were not the case (imagine that $\partial \widehat{p}_1 / \partial q_2$ were 0), the profit-maximizing quality would be the optimal level for the marginal consumer: $\widehat{v} = c'(q)$, reminiscent of the basic property of quality in Spence (1975). By raising quality from one satisfying $\widehat{v} = c'(q)$, Firm 2 may also raise Firm 1's price, hence its own profit. This is a first-order gain. The optimal tradeoff is now given by $\widehat{v} + \dfrac{\partial \widehat{p}_1(\widehat{q}_1, \widehat{q}_2)}{\partial q_2} = c'(\widehat{q}_2)$. We use (11) to simplify, and show that. Firm 2 sets its quality to be efficient for a consumer with valuation $\widehat{v} + \dfrac{h(\widehat{v})}{1 - h'(\widehat{v})}$. This is the second equation.

According to Proposition 1, the only difference between equilibrium qualities and those in the first best stems from how Firm 2 chooses its quality. Firm 2's consumers have average valuation $\dfrac{\int_{\widehat{v}}^{\overline{v}} x f(x) \mathrm{d}x}{1 - F(\widehat{v})}$, which should be set to the marginal cost of Firm 2's quality for social efficiency. Can this average valuation be equal to $\widehat{v} + \dfrac{h(\widehat{v})}{1 - h'(\widehat{v})}$? Our next result gives a class of valuation distributions for which the answer is affirmative. First, we present a mathematical lemma, which, through a simple application of integration by parts, allows us to write the conditional expectation of valuations in terms of hazard rate and the density.

Lemma 3 *For any distribution F (and its corresponding density f and hazard rate $h \equiv (1-F)/f$), we have*

$$\frac{\int_{v}^{\overline{v}} x f(x)dx}{1-F(v)} \equiv v + \frac{\int_{v}^{\overline{v}} f(x)h(x)dx}{f(v)h(v)}. \tag{19}$$

Proposition 2 *Suppose that the hazard rate h is linear; that is, $h(x) = \alpha - \beta x$, $x \in [\underline{v}, \overline{v}]$, for some α and $\beta \geq 0$. Then for any v*

$$v + \frac{h(v)}{1-h'(v)} = \frac{\int_{v}^{\overline{v}} x f(x)dx}{1-F(v)} \equiv v + \frac{\int_{v}^{\overline{v}} f(x)h(x)dx}{f(v)h(v)}. \tag{20}$$

Equilibrium qualities and market shares are first best.

Proposition 2 exhibits a set of consumer-valuation distributions for which the quality-price competition game yields first-best equilibrium qualities. We have managed to write the conditional average in terms of the hazard rate in Lemma 3, and this is $\widehat{v} + \dfrac{\int_{\widehat{v}}^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(\widehat{v})h(\widehat{v})}$. When the hazard rate is linear, $\dfrac{h(\widehat{v})}{1-h'(\widehat{v})} \equiv \dfrac{\int_{\widehat{v}}^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(\widehat{v})h(\widehat{v})}$, Firm 2's profit maximization incentive aligns with the social incentive. The following remark gives the economic interpretation for the linear hazard rate.

Remark 1 *When Firm 2 sells to high-valuation consumers, its marginal revenue is linear in consumer valuation if and only if $h(v)$ is linear.*

As far as we know, the linearity of the hazard rate has never been used in the theoretical literature such as auction design, regulation, and screening and pricing under asymmetric information. The Myerson virtual cost and Laffont-Tirole information rent adjustments almost always involve the hazard rate (see, for instance, Myerson (1997), Laffont and Tirole (1993)), but no linearity assumption has been used before. By contrast, in the empirical literature (such as labor economics), the linear hazard model has been very popular, although our assumption has no direct bearing on the estimation of, and inference from, such models.

Many distributions satisfy the linear hazard rate assumption. They include the popular uniform and exponential distributions. Essentially, the condition $[1-F(v)]/f(v) = \alpha - \beta v$ requires that all distributions conditional on the random variable exceeding v to be quite similar. For example, if the lower parts of the uniform or exponential distributions are removed, the remaining distributions still are uniform or exponential. In any case, $[1 - F(v)]/f(v) = \alpha - \beta v$ defines a differential equation, and we solve for all distributions that have linear hazard rates.

Remark 2 *Suppose that $h(x) = \alpha - \beta x$. Then if $\beta = 0$, f is the exponential distribution $f(x) = \dfrac{A}{\alpha}\exp(-\dfrac{x}{\alpha})$, with $\overline{v} = \infty$, and $A = \exp(\dfrac{\underline{v}}{\alpha})$, so when $\underline{v} = 0$, $f(x) = \dfrac{1}{\alpha}\exp(-\dfrac{x}{\alpha})$ for $x \in \mathbb{R}_+$. If $\beta > 0$, then $f(x) = \left[\dfrac{(\alpha - \beta x)^{(1-\beta)}}{(\alpha - \beta \underline{v})}\right]^{\frac{1}{\beta}}$, with $\alpha - \beta\overline{v} = 0$. For the uniform distribution, we have $h(x) = \overline{v} - x$ (so $\alpha = \overline{v}$, and $\beta = 1$).*

Next, we show that firms' equilibrium qualities must be either simultaneously excessive or deficient. It cannot be an equilibrium for one firm's quality higher than the first best while the other firm's quality lower than the first best.

Proposition 3 *Let an equilibrium be written as $(\widehat{q}_1, \widehat{q}_2, \widehat{v})$, corresponding to Firm 1's quality, Firm 2's quality, and the marginal consumer. If the equilibrium is not first best, either*

$$(\widehat{q}_1, \widehat{q}_2, \widehat{v}) < (q_\ell^*, q_h^*, v^*) \quad \textit{or} \quad (\widehat{q}_1, \widehat{q}_2, \widehat{v}) > (q_\ell^*, q_h^*, v^*).$$

That is, when equilibrium qualities are not first best, either both firms have equilibrium qualities lower than the corresponding first-best levels, or both have equilibrium qualities correspondingly higher.

The proposition can be explained as follows. Firm 1 aims to maximize social surplus. If Firm 2 chooses $q_2 = q_h^*$, Firm 1's best response is to pick $q_1 = q_\ell^*$. Next, Firm 1's best response is increasing in q_2. This stems from the properties of $\widehat{v}(q_1, q_2)$, the efficient allocation of consumers across the two firms. Quality q_1 is too low for consumer $\widehat{v}$, whereas quality q_2 is too high. If q_2 increases, consumer $\widehat{v}$ would become worse off buying from Firm 2, so actually $\widehat{v}$ increases. This also means that Firm 1 should raise its quality because it now serves consumers with higher valuations. In other words, if Firm 2 raises its quality, Firm 1's best response is to raise quality. Therefore, Firm 1's quality is higher than the first best q_ℓ^* if and only if Firm 2's quality is higher than the first best q_h^*.

We have constructed a number of examples to verify that equilibrium qualities can be either below or above the first best. However, it is more effective if we discuss these examples after we have presented the other class of equilibria in which the public firm chooses a higher quality than the private firm. The examples are presented in Subsection 4.3. Also, at this point we already can draw various policy implications from the results, but will defer those discussions until after we have presented the other class of equilibria. See Subsections 5.1 and 5.2.

4 Equilibria with high quality at public firm

In this class of equilibria Firm 1's quality is higher than Firm 2's, $q_1 > q_2$. Because the two firms have different objectives, equilibria in this class are not isomorphic to those in the previous section. However, many definitions and proof procedures that have been used previously can be applied analogously, so where appropriate we will omit proofs.

4.1 Subgame-perfect equilibrium prices

When $q_1 > q_2$, the firms have positive demand only if $p_1 > p_2$. The definition for demand in (1) continues to apply. We rewrite the definition of the indifferent consumer $\widetilde{v}$:

$$\widetilde{v}q_1 - p_1 = \widetilde{v}q_2 - p_2 \qquad \text{or} \qquad \widetilde{v}(p_1, p_2; q_1.q_2) = \frac{p_1 - p_2}{q_1 - q_2}. \tag{21}$$

The firms' payoffs are respectively:

$$\int_{\underline{v}}^{\widetilde{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x + \int_{\widetilde{v}}^{\overline{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x \tag{22}$$

$$F(\widetilde{v})[p_2 - c(q_2)]. \tag{23}$$

The expressions in (22) and (23) are social surplus and Firm 2's profit, and are similar to those in (7) and (8). Here, consumers with low valuations buy from the low-quality-low-price private firm, whereas consumers with high valuations buy from the high-quality-high-price public firm.

Firm 1 chooses price p_1 to maximize social surplus (22) given the demand function (21) and price p_2. Firm 2 chooses price p_2 to maximize profit (23) given the demand function (21) and price p_1. Equilibrium prices, $(\widehat{p}_1, \widehat{p}_2)$, are best responses against each other. The following lemma is the characterization of equilibrium prices and consumer allocation. Its proof is similar to that of Lemma 1, and omitted.

Lemma 4 *In subgames* (q_1, q_2) *with* $q_1 > q_2$*, and* $\underline{v} < \dfrac{c(q_1) - c(q_2)}{q_1 - q_2} < \overline{v}$*, equilibrium prices* $(\widehat{p}_1, \widehat{p}_2)$ *are:*

$$\widehat{p}_1 - c(q_1) = \widehat{p}_2 - c(q_2) = (q_1 - q_2)\frac{F(\widehat{v})}{f(\widehat{v})} \equiv (q_1 - q_2)k(\widehat{v}), \tag{24}$$

$$\text{where} \qquad \widehat{v} = \frac{c(q_1) - c(q_2)}{q_1 - q_2}. \tag{25}$$

Lemma 4 presents the equilibrium prices and consumer allocations in subgames with $q_1 > q_2$. Their properties parallel those in Lemma 1. Firm 1 implements the socially efficient consumer allocation by setting a price differential equal to the cost differential: $\widehat{p}_1 - \widehat{p}_2 = c(q_1) - c(q_2)$. Firm 2's profit maximization follows the usual marginal-revenue-marginal-cost tradeoff. We use the reverse hazard rate, $k = F/f$, to obtain (24). Finally, for subgames (q_1, q_2) with $q_1 > q_2$, and either $\dfrac{c(q_1) - c(q_2)}{q_1 - q_2} < \underline{v}$ or $\overline{v} < \dfrac{c(q_1) - c(q_2)}{q_1 - q_2}$ one firm will be inactive, so these subgames are irrelevant.

The equilibrium prices and allocation in (24) and (25) depend on the qualities, so we write them as $\widehat{p}_1(q_1, q_2)$, $\widehat{p}_2(q_1, q_2)$, and $\widehat{v}(q_1, q_2)$. We totally differentiate these three functions to obtain how prices and allocation change with Firm 2's quality. The following lemma presents these results. The proof follows the same steps as those in Lemma 2, and is omitted.

Lemma 5 *From the definition of* $(\widehat{p}_1, \widehat{p}_2)$ *in (24) and (25), we have* $\widehat{v}$ *increasing in* q_1 *and* q_2*,*

$$\frac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2} = -k(\widehat{v}) + k'(\widehat{v})\left[\widehat{v} - c'(q_2)\right] \tag{26}$$

$$\frac{\partial \widehat{p}_2(q_1, q_2)}{\partial q_2} = c'(q_2) - k(\widehat{v}) + k'(\widehat{v})\left[\widehat{v} - c'(q_2)\right]. \tag{27}$$

Lemma 5 shows how Firm 2's quality will alter equilibrium prices and allocation. Unlike subgames where Firm 2's quality is higher than Firm 1's, Firm 2's market share increases with both q_1 and q_2. However, the effect of a higher quality q_2 on prices may be ambiguous, but the effect of q_2 on $\widehat{p}_2$ is larger than that on $\widehat{p}_1$ by $c'(q_2)$.

Finally, we consider subgames where both firms have chosen the same qualities, $q_1 = q_2$. According to (1), the firms share the market equally if they charge the same price; otherwise, the firm that charges the lower price gets all consumers. However, Firm 1's objective is social surplus, which, for $q_1 = q_2$, is $\int_{\underline{v}}^{\overline{v}} [vq_1 - c(q_1)]\mathrm{d}v$, irrespective of prices. Any price can be a best response for Firm 1. Clearly, Firm 2 prefers its price (and Firm 1's price) to be as high as possible. Here, we select the equilibrium in which the price is at marginal cost $c(q_1)$. Our reason for this selection is to main continuity. In both (9) and (24), the price-cost margin tends to zero as q_1 and q_2 tend to each other.

4.2 Subgame-perfect equilibrium qualities

Given qualities q_1 and q_2, Firm 1 and Firm 2 have, respectively, the continuation equilibrium payoffs

$$\int_{\underline{v}}^{\widehat{v}(q_1,q_2)} [xq_2 - c(q_2)]f(x)\mathrm{d}x + \int_{\widehat{v}(q_1,q_2)}^{\overline{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x \tag{28}$$

$$F(\widehat{v}(q_1,q_2))[\widehat{p}_2(q_1,q_2) - c(q_2)], \tag{29}$$

where $\widehat{p}_2$ is Firm 2's equilibrium price and $\widehat{v}$ is the indifferent consumer (see Lemma 4). Let $(\widehat{q}_1,\widehat{q}_2)$ be the equilibrium qualities. They are mutual best responses, given continuation equilibrium prices:

$$\widehat{q}_1 = \underset{q_1}{\operatorname{argmax}} \int_{\underline{v}}^{\widehat{v}(q_1,\widehat{q}_2)} [x\widehat{q}_2 - c(\widehat{q}_2)]f(x)\mathrm{d}x + \int_{\widehat{v}(q_1,\widehat{q}_2)}^{\overline{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x \tag{30}$$

$$\widehat{q}_2 = \underset{q_2}{\operatorname{argmax}} F(\widehat{v}(\widehat{q}_1,q_2))[\widehat{p}_2(\widehat{q}_1,q_2) - c(q_2)]. \tag{31}$$

We apply the same method to characterize equilibrium qualities. Changing q_1 in Firm 1's payoff in (30) only affects the second integral there because the effect via the first integral is second order by the Envelope Theorem. To study the effect of changing q_2 on Firm 2's payoff, we use the definition of $\widehat{v}$ to rewrite profit in (31) as

$$F(\widehat{v}(\widehat{q}_1,q_2))[\widehat{v}(\widehat{q}_1,q_2)q_2 - c(q_2) - \widehat{v}(\widehat{q}_1,q_2)\widehat{q}_1 + \widehat{p}_1(\widehat{q}_1,q_2)].$$

Hence, changing q_2 has only two effects: the direct effect on the surplus of the marginal consumer $\widehat{v}q - c(q_2)$, and the effect on Firm 1's equilibrium price, because any effect on the marginal consumer is second order according to the Envelope Theorem. We obtain the first-order conditions

$$\begin{aligned} \int_{\widehat{v}(q_1,q_2)}^{\overline{v}} [x - c'(q_1)]f(x)\mathrm{d}x &= 0 \\ \widehat{v}(\widehat{q}_1,q_2) - c'(q_2) + \frac{\partial \widehat{p}_1(\widehat{q}_1,q_2)}{\partial q_2} &= 0. \end{aligned}$$

Then we apply Lemma 5, and use $-k(\widehat{v}) + k'(\widehat{v})\,[\widehat{v} - c'(q_2)]$ to substitute for $\partial\widehat{p}_1/\partial q_2$. To sum up, we present the characterization in the next Proposition (proof omitted).

Proposition 4 *Equilibrium qualities* $(\widehat{q}_1, \widehat{q}_2)$, *and the marginal consumer* $\widehat{v}$ *solve the following three equations in* q_1, q_2, *and* v

$$\begin{aligned} \frac{\int_v^{\overline{v}} x f(x)dx}{1-F(v)} &= c'(q_1) \\ v - \frac{k(v)}{1+k'(v)} &= c'(q_2) \\ vq_2 - c(q_2) &= vq_1 - c(q_1). \end{aligned}$$

The intuitions behind Proposition 4 are similar to those in Proposition 1 in the previous section. Firm 1 chooses q_1 to maximize the surplus of those consumers with valuations higher than $\widehat{v}$. The marginal consumer is $\widehat{v}$ but Firm 2 chooses the quality that is efficient for a lower type $\widehat{v} - \dfrac{k(\widehat{v})}{1+k'(\widehat{v})}$. Firm 2's lower quality serves to use product differentiation to create a bigger cost differential, and hence a bigger price differential between the two firms.

Again, the difference between the equilibrium qualities and the first best stems from Firm 2's quality choice. We can identify a class of distributions for which Firm 2's profit incentive aligns with the social incentive. First, we present a mathematical result that relates the reverse hazard rate and conditional expectations.

Lemma 6 *For any distribution* F *(and its corresponding density* f *and reverse hazard rate* $k \equiv F/f$*), we have*

$$\frac{\int_{\underline{v}}^{v} x f(x)dx}{F(v)} \equiv v - \frac{\int_v^{\overline{v}} f(x)k(x)dx}{f(v)k(v)}. \tag{32}$$

Proposition 5 *Suppose that the reverse hazard rate* k *is linear; that is,* $k(x) = \gamma + \delta x$, $x \in [\underline{v}, \overline{v}]$, *for some* γ *and* $\delta \geq 0$. *Then for any* v

$$v - \frac{k(v)}{1+k'(v)} = \frac{\int_{\underline{v}}^{v} x f(x)dx}{F(v)} \equiv v + \frac{\int_{\underline{v}}^{v} f(x)k(x)dx}{f(v)k(v)}. \tag{33}$$

Equilibrium qualities and market shares are first best.

We also present the following relationship between the linear reverse hazard rate and the private firm's marginal revenue.

Remark 3 *When Firm 2 sells to low-valuation consumers, its marginal revenue is linear in consumer valuation if and only if $k(v)$ is linear.*

The linear reverse hazard rate in Proposition 5 may look similar to the earlier condition for the first best in Proposition 2, but in fact, hazard rate and the reverse hazard rate can behave rather differently. For example, the exponential distribution has a constant hazard rate (see Remark 2), but the reverse hazard rate is nonlinear.[6] As another example, a "triangular" distribution has a linear reverse hazard rate, but its hazard rate is nonlinear (see Example 1 below). We present all distributions that have linear reverse rates in the following.

Remark 4 *Suppose that $k(x) = \gamma + \delta x$. Then $\delta > 0$, and $f(x) = \left[\dfrac{(\gamma+\delta x)^{1-\delta}}{(\gamma+\delta\overline{v})}\right]^{\frac{1}{\delta}}$ with $\gamma + \delta\underline{v} = 0$. For the uniform distribution, $\gamma = -\underline{v}$ and $\delta = 1$.*

For the uniform distribution, both hazard and reverse hazard rates are linear, but this is not the only one. The following characterizes those distributions whose hazard and reverse hazard rates are both linear.

Remark 5 *Finally, if $h(v) = \alpha - \beta v$ and $k(v) = \gamma + \delta v$ for the same distribution, we have $f(v) = [\beta(\overline{v} - v) + \delta(v - \underline{v}]^{-1}$ and $F(v) = \delta(v - \underline{v})\,[\beta(\overline{v} - v) + \delta(v - \underline{v}]^{-1}$.*

When the equilibrium is not first best, the distortion in equilibria with higher public qualities exhibits the same pattern as in equilibria with lower public qualities: Proposition 3 holds verbatim for the class of high-public-quality equilibria. (The proof parallels that for Proposition 3, and is omitted.[7]) Either both firms simultaneously produce qualities higher than first best, or both simultaneously produce qualities lower.

4.3 Examples and comparison between equilibrium and first-best qualities

We now present three sets of examples to illustrate the different types of equilibria. All examples use the same quadratic cost function $c = \frac{1}{2}q^2$, but different distributions. The *Mathematica* programs for the

[6] Suppose that x has the exponential density $\dfrac{1}{\alpha}\exp(-\dfrac{x}{\alpha})$ on $\mathbb{R}_+$, $\alpha > 0$, then $h(v) = \alpha$, and $k(v) = \alpha\left[\exp(\dfrac{v}{\alpha}) - 1\right]$.

[7] The proof of Proposition 3 can just be repeated here. The only difference is that the cross partial derivative of Firm 1's objective function now becomes $-[\widehat{v} - c'(q_1)]\partial\widehat{v}/\partial q_2$. This is positive because now in an equilibrium $\widehat{v} < c'(q_1)$ whereas $\partial\widehat{v}/\partial q_2$ remains positive.

computations are in the Supplement.

Example 1 illustrates Proposition 5 and considers distributions for which either the hazard rate or the reverse hazard rate is linear. Examples 2 and 3 consider distributions for which neither the hazard nor the reverse hazard rates are linear and therefore represent the equilibrium outcomes in which the qualities can be higher or lower than the first best.

Example 1 *Two triangular distributions:* $f(v) = 2v$ *and its reverse* $f(v) = 2(1-v)$*, for* $v \in [0,1]$*.*

In the first triangular distribution, we have :

$$f(v) = 2v \qquad\qquad F(v) = v^2$$

$$h(v) = \frac{1-v^2}{2v} \qquad\qquad k(v) = \frac{v}{2},$$

so the hazard rate is not linear, but the reverse hazard rate is. Proposition 5 says that when the public firm's quality is higher than the private firm's, equilibrium qualities are first best, but this may not be true when the public firm's quality is lower. The following presents the first best and the equilibria:

				Welfare
First best	$q_l^* = 0.4120$	$q_h^* = 0.8241$	$v^* = 0.6180$	0.2423
Low public quality	$\widehat{q}_1 = 0.3849$	$\widehat{q}_2 = 0.7698$	$\widehat{v} = 0.5774$	0.2416
High public quality	$\widehat{q}_1 = q_h^*$	$\widehat{q}_2 = q_l^*$	$\widehat{v} = v^*$	0.2423

When the public Firm 1 chooses a low quality, equilibrium qualities are all below the first best, and there is a small welfare loss.

In the second triangular distribution, we have

$$f(v) = 2 - 2v \qquad\qquad F(v) = 2v - v^2$$

$$h(v) = \frac{1-v}{2} \qquad\qquad k(v) = \frac{2v - v^2}{2(1-v)},$$

so the hazard rate is linear but the reverse hazard rate is not. Equilibrium qualities are first best when the public firm chooses a low quality (Proposition 2). The following presents the first best and equilibria:

				Welfare
First best	$q_l^* = 0.1760$	$q_h^* = 0.5880$	$v^* = 0.3820$	0.0756
Low public quality	$\widehat{q}_1 = q_l^*$	$\widehat{q}_2 = q_h^*$	$\widehat{v} = v^*$	0.0756
High public quality	$\widehat{q}_1 = 0.6151$	$\widehat{q}_2 = 0.2302$	$\widehat{v} = 0.4227$	0.0749

In this example, when the public Firm 1 chooses a high quality, equilibrium qualities are all above the first best.

Example 2 *Two exponential distributions:* $f(v) = \dfrac{[\exp(-v/\alpha)]/\alpha}{1-\exp(-\overline{v}/\alpha)}$ *and its reverse* $f(v) = \dfrac{[\exp(-(\overline{v}-v)/\alpha)]/\alpha}{1-\exp(-\overline{v}/\alpha)}$, *for* $\alpha > 0$, *and* $v \in [0, \overline{v}]$.

In the first exponential, we have

$$f(v) = \frac{[\exp(-v/\alpha)]/\alpha}{1-\exp(-\overline{v}/\alpha)} \qquad F(v) = \frac{1-\exp(-v/\alpha)}{1-\exp(-\overline{v}/\alpha)}$$

$$h(v) = \alpha\left[1-\exp(-(\overline{v}-v)/\alpha)\right] \qquad k(v) = \alpha\left[\exp(v/\alpha)-1\right],$$

so neither the hazard rate nor the reverse hazard rate are linear. We have computed the first best and equilibria for $\alpha = 20$ and $\overline{v} = 100$:

				Welfare
First best	$q_l^* = 11.1172$	$q_h^* = 46.9151$	$v^* = 29.0162$	299.857
Low public quality	$\widehat{q}_1 = 11.4546$	$\widehat{q}_2 = 49.0791$	$\widehat{v} = 30.2668$	299.617
High public quality	$\widehat{q}_1 = 61.4085$	$\widehat{q}_2 = 28.9712$	$\widehat{v} = 45.1898$	191.835

In both equilibria, firms' qualities are higher than the first best. Moreover, the equilibrium with the public firm producing a lower quality has a higher equilibrium welfare.

In the second exponential distribution, we have

$$f(v) = \frac{[\exp(-(\overline{v}-v)/\alpha)]/\alpha}{1-\exp(-\overline{v}/\alpha)} \qquad F(v) = \frac{\exp(-(\overline{v}-v)/\alpha)-\exp(-\overline{v}/\alpha)}{1-\exp(-\overline{v}/\alpha)}$$

$$h(v) = \alpha\left[\exp((\overline{v}-v)/\alpha)-1\right] \qquad k(v) = \alpha\left[1-\exp(v/\alpha)\right].$$

Again, neither the hazard rate nor the reverse hazard rate are linear. We use the same values of α and $\overline{v}$, and compute the first best and equilibria:

				Welfare
First best	$q_l^* = 53.0849$	$q_h^* = 88.8828$	$v^* = 70.9838$	3367.69
Low public quality	$\widehat{q}_1 = 38.5915$	$\widehat{q}_2 = 71.0288$	$\widehat{v} = 54.8102$	3259.67
High public quality	$\widehat{q}_1 = 88.5454$	$\widehat{q}_2 = 50.9209$	$\widehat{v} = 69.7332$	3367.45

In both equilibria, firms' qualities are lower than the first best. However, the equilibrium in which the public firm produces a higher quality yields a higher welfare.

Example 3 *A Beta distribution:* $f(v) = \dfrac{v^{(\alpha-1)}(1-v)^{(\beta-1)}}{\displaystyle\int_0^1 x^{(\alpha-1)}(1-x)^{(\beta-1)}\,dx}$*, for* $\alpha, \beta > 0$*, and* $v \in [0,1]$*.*

The Beta distribution with parameters α and β (as in the above expression) constitutes a big class. For some values of α and β, its hazard or reverse hazard rates are linear (for example a Beta distribution with $\alpha = \beta = 1$ is the uniform distribution). We have computed the equilibria two different sets of parameters: $\alpha = 5$, $\beta = 2$, and $\alpha = 2$, $\beta = 5$. The densities are illustrated in the following diagram.

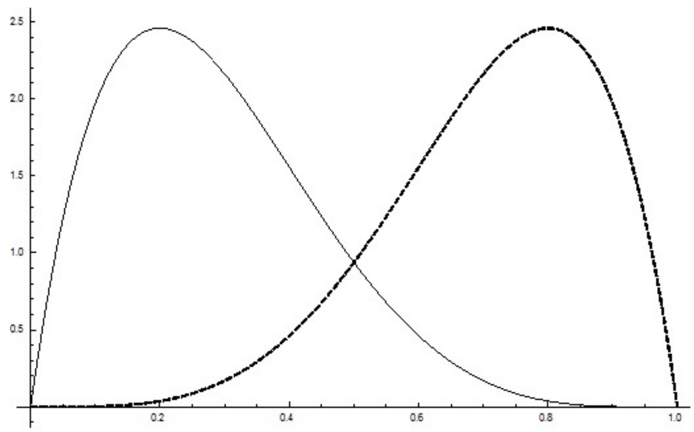

Figure 2: Beta densities when parameters are $\alpha = 5$, $\beta = 2$ (solid) and $\alpha = 2$, $\beta = 5$ (dashed).

For $\alpha = 5$ and $\beta = 2$, we have

$$f(v) = 30v^4(1-v) \qquad\qquad F(v) = 6v^5 - 5v^6$$

$$h(v) = \frac{1-5v^5}{30v^4} \qquad\qquad k(v) = \frac{6v-5v^2}{30(1-v)}$$

and the hazard and reverse hazard rates are not linear. The first best and equilibria are as follow:

				Welfare
First best	$q_l^* = 0.5476$	$q_h^* = 0.8182$	$v^* = 0.6829$	0.2638
Low public quality	$\widehat{q}_1 = 0.5006$	$\widehat{q}_2 = 0.7386$	$\widehat{v} = 0.6196$	0.2623
High public quality	$\widehat{q}_1 = 0.8258$	$\widehat{q}_2 = 0.5730$	$\widehat{v} = 0.6994$	0.2637

Equilibrium qualities are not first best, and now the deviations from the first best are different from the examples above. If the public firm produces a lower quality than the private firm, both firms produce equilibrium qualities below the first best. If the public firm produces a higher quality, then both firms produce equilibrium qualities above the first best. The equilibrium welfare is higher when the public firm produces high quality.

For $\alpha = 2$ and $\beta = 5$, we have

$$f(v) = 30v(1-v)^4 \qquad F(v) = 15v^2 - 40v^3 + 45v^4 - 24v^5 + 5v^6$$

$$h(v) = \frac{1}{30}\left(\frac{1+4v-5v^2}{v}\right) \qquad k(v) = \frac{15v^2 - 40v^3 + 45v^4 - 24v^5 + 5v^6}{30\,(1-v)^4\, v}$$

and again the hazard and reverse hazard rates are not linear. The first best and equilibria are as follow:

				Welfare
First best	$q_l^* = 0.1818$	$q_h^* = 0.4524$	$v^* = 0.3171$	0.0495
Low public quality	$\widehat{q}_1 = 0.1742$	$\widehat{q}_2 = 0.4270$	$\widehat{v} = 0.3006$	0.0494
High public quality	$\widehat{q}_1 = 0.4995$	$\widehat{q}_2 = 0.2615$	$\widehat{v} = 0.3805$	0.0480

Also in this example, the equilibrium qualities are not first best. If the public firm produces a lower quality than the private firm, both firms produce equilibrium qualities below the first best. If the public firm produces a higher quality, then both firms produce equilibrium qualities above the first best. However, now the equilibrium welfare when the public firm produces low quality is higher than the equilibrium welfare when the public firm produces high quality.

5 Policies and robustness

5.1 Competition and regulatory policies

The analysis in the previous two sections points to various policy implications. The regulation literature has commonly adopted a mechanism-design approach. In our setting, this would take the form of a regulator first committing to the quality and price of the product of a public firm, and then the private firm reacts. Instead, we use a conventional simultaneous-move, quality-price competition model. In fact, the commitment-Stackelberg model, as we will argue, adds few conceptual advantages.

First, Propositions 2 and 5 present conditions for the first best (linear hazard and reverse hazard rates). These propositions have a direct implication for competition policy. Suppose that the market initially consists of private duopolists (as in Section 6). If a regulator would like to improve quality efficiency, taking over a private firm and adopting an objective of social-surplus maximization may be all it takes. Propositions 2 and 5 also indicate whether a public firm should take over a firm producing a low quality or a high quality.

Second, equilibrium qualities are first best in the simultaneous-move games if and only if they are first best in the Stackelberg game (when the public firm can commit to quality or price). The reason is this. Suppose that Stackelberg equilibrium qualities are first best. If public firm chooses the (first-best) low quality, the private firm must choose the (first-best) high quality as a best response. Because the public firm's payoff is social surplus, the (first-best) low quality is a best response against the (first-best) high quality, so the first best is an equilibrium in the simultaneous-move game. Improvement due to commitment is inadequate for the first best.

Proposition 3 implies that the improvement in welfare from a predetermined public quality must come from the public firm choosing a quality closer to the first best. For example, if in an equilibrium, qualities are lower than the first best (as in the reverse truncated exponential distribution case in Example 2), a higher public quality leads to a higher best response by the private firm, so both qualities will become closer to the first best.

5.2 General objective for the public firm and subsidies

So far our focus has been on quality efficiency. The public firm's objective function has been social welfare, so prices are transfers between consumers and firms, so they do not affect social welfare. A more general objective function for a public firm can be a weighted sum of consumer surplus, and profits, also a common assumption in the literature. In this case, we can rewrite Firm 1's objective function as

$$\theta\left\{\int_{\underline{v}}^{\widetilde{v}}[xq_1-p_1]f(x)\mathrm{d}x+\int_{\widetilde{v}}^{\overline{v}}[xq_2-p_2]f(x)\mathrm{d}x\right\}+(1-\theta)\left\{F(\widetilde{v})[p_1-c(q_1)]+[1-F(\widetilde{v})][p_2-c(q_2)]\right\}. \quad (34)$$

Here, consumers are paying for the lower quality q_1 at price p_1, and the higher quality q_2 at price p_2. The weight on consumer surplus is $\theta>\frac{1}{2}$, whereas the weight on profits is $1-\theta$, so profits are unattractive from a social perspective. We can rewrite (34) as

$$\theta\left\{\int_{\underline{v}}^{\widetilde{v}}[xq_1-c(q_1)]f(x)\mathrm{d}x+\int_{\widetilde{v}}^{\overline{v}}[xq_2-c(q_2)]f(x)\mathrm{d}x\right\}-(2\theta-1)\left\{F(\widetilde{v})[p_1-c(q_1)]+[1-F(\widetilde{v})][p_2-c(q_2)]\right\},$$

which always decreases in Firm 1's price. If we impose a balanced-budget constraint, then the public firm must set price p_1 at marginal cost $c(q_1)$ to break even.

In this specification, Lemmas 1 and 4 would not apply. Price differentials no longer equal cost differentials.

In fact, in any price equilibrium, we have $p_2 - p_1 > c(q_2) - c(q_1)$ due to Firm 2's profit-maximizing price-cost margin: $p_2 > c(q_2)$. The incremental price for purchasing the good at a higher quality exceeds the true cost difference, so fewer consumers will use the private firm. The first best cannot be an equilibrium because consumers will never bear the full incremental cost between high and low qualities.

The concern for distribution naturally suggests a subsidy policy. Consider an equilibrium in which Firm 1 chooses a low quality and Firm 2 chooses a higher quality. Each firm's price is given by Lemma 1, so each firm earns a profit. Firm 1's profit can be set aside for distribution. Firm 2's profit can be taxed as a lump sum. The total collection now can be given as a subsidy to consumers who purchase from either the public or the private firms. This subsidy policy is often implemented as a voucher or tax credit. In cases where conditions in Propositions 2 or 5 are satisfied, this would allow the first best to become an equilibrium. From a normative perspective, a government instructing an administrator of a public firm to adopt a goal of social-surplus maximization may allow the implementation of efficient qualities.

5.3 Different cost functions for public and private firms

We now let firms have different cost functions. Let $c_1(q)$ and $c_2(q)$ be Firm 1's and Firm 2's unit cost at product quality q, and these functions are increasing and convex. Often the public firm is assumed to be less efficient, so we can assume $c_1(q) > c_2(q)$ and $c_1'(q) > c_2'(q)$, so both unit and marginal unit costs are higher at the public firm. Our formal model, however, does not require this particular comparative advantage.

The analysis in Sections 3 and 4 remains exactly the same. Simply replace every $c(q_1)$ by $c_1(q_1)$ and every $c(q_2)$ by $c_2(q_2)$. In the price subgame, the equilibrium still has price difference equal to cost difference: $p_2 - p_1 = c_2(q_2) - c_1(q_1)$. The equilibrium qualities continue to satisfy their respective conditions after first-order conditions are simplified.

Propositions 2 and 5 have to be adjusted. This is because the first best in Subsection 2.3 has to be redefined. There are now two ways to assign technology. In one, low quality for low-valuation consumers incurs the cost $c_1(q)$, and high quality incurs the cost $c_2(q)$. In the other, it is the opposite. One of these technology assignments will yield a higher social welfare. However, our abstract model does not allow us to determine which technology should be used for low quality.[8]

Suppose that the first best has the low quality produced by the public firm. Equilibria in Section 4 can never achieve the first best because the low quality is produced by the private firm. Hence, the last statement in Proposition 5 has to be dropped. The same reasoning applies to equilibria in Section 3 and Proposition 2 when the low quality is produced by the private firm in the first best. These qualifications do not seem to pose any conceptual problem. Misallocation is due to a kind of miscoordination on equilibria. Our policy implication in Subsection 5.1 is actually strengthened. If the government takes over a private firm, its decision should be guided by both strategic and technological considerations. It may decide to take over a firm with cost c_1 and produces a low quality because that is what is called for by the first best, and because of the potential for quality efficiency in the mixed oligopoly.

5.4 Consumer outside option and many private firms

Formally, the case of the consumer having an outside option is modeled by a fictitious firm offering a product at zero quality and zero price. The first best may assign null consumption to some consumers whose valuations of quality are below a threshold. A price set by the public firm may affect two margins: whether a consumer should choose between the low-quality good and the high-quality good, as well as whether a consumer should choose between the low-quality good and non-consumption.

In fact, Delbono et al. (1996) show that under a uniform valuation distribution, the first best is not an equilibrium. Efficient allocation requires that all consumers face price differentials that correspond to cost differentials. Hence, if Firm 1 produces a low quality q_1 and Firm 2 produces a high quality q_2, then efficiency requires $p_2 - p_1 = c(q_2) - c(q_1)$. When $p_2 > c(q_2)$ due to Firm 2's market power, $p_1 > c(q_1)$. However, to induce consumers to make efficient nonconsumption decisions, p_1 should be set at $c(q_1)$.

Furthermore, Benassi et al. (2016) show that even the existence of a price equilibrium (under the as-

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[8] As an illustration, let $c_1(q) = (1+s)c(q)$, and $c_2(q) = (1-s)c(q)$. The social welfare from using c_1 to produce the low quality is $\int_{\underline{v}}^{v}[xq_\ell - (1+s)c(q_\ell)]f(x)\mathrm{d}x + \int_{v}^{\overline{v}}[xq_h - (1-s)c(q_h)]f(x)\mathrm{d}x$. At $s = 0$, this is the model in Subsection 2.1. From the Envelope Theorem, the derivative of the maximized welfare with respect to s evaluated at $s = 0$ is the partial derivative of welfare with respect to s: $-c(q_\ell^*)F(v^*) + c(q_h^*)[1 - F(v^*)]$. Properties of q_ℓ^*, q_h^*, and v^* from (3), (4), and (5) do not indicate whether this derivative is positive or negative.

sumption of identical costs among firms and fixed qualities) would require more structure on the valuation distribution (see their Assumption 1 on p88, and Proposition 1 on p89). The usual logconcave distribution assumption (say, in Caplin and Nalebuff (1991)) is inconsistent with the sufficient conditions proposed by Benassi et al.

The case of many private firms is formally very similar. When a public firm has to interact with, say, two private firms, it does not have enough instruments to induce efficient decisions. Suppose that there are three firms, and they produce low, medium, and high qualities. Suppose that the medium quality is produced by a public firm, whereas the other qualities are produced by private firms. Private firms exploit their market power, but the public firm cannot simultaneously use one price to induce two efficient margins, so consumers can choose between medium and low qualities efficiently, and at the same time choose between high and medium qualities efficiently. The lack of tractable analysis seems pervasive in the literature of horizontal and vertical differentiation with multiple firms.

6 Private Duopoly

We now analyze a duopoly model with two private firms under the same extensive form in Subsection 2.4. Firm 1 now maximizes profit, so this is a standard model in which product differentiation is used to relax price competition.

6.1 Subgame-perfect equilibrium prices

Consider a subgame (q_1, q_2) in Stage 2. Without loss of generality, let $q_1 < q_2$. Firm 1's profit is now $F(\widetilde{v})[p_1 - c(q_1)]$, where the demand $\widetilde{v}$ is given by (6). Given Firm 2's price p_2, Firm 1 chooses p_1 to maximize its profit, and the first-order condition is

$$F(\widetilde{v}) - \frac{f(\widetilde{v})[p_1 - c(q_1)]}{q_2 - q_1} = 0.$$

We simplify this first-order condition, and combine the first-order condition of Firm 2's profit maximization (which is derived in the proof of Lemma 1) to obtain the following lemma (whose proof is omitted). (We use the same notation as in the previous sections when Firm 1 is the public firm, but this should not create any confusion.)

Lemma 7 *In subgames* (q_1, q_2) *with* $q_1 < q_2$*, equilibrium prices* $(\widehat{p}_1, \widehat{p}_2)$ *are given by the following:*

$$\widehat{p}_1 - c(q_1) = (q_2 - q_1)\frac{F(\widehat{v})}{f(\widehat{v})} \equiv (q_2 - q_1)k(\widehat{v}), \tag{35}$$

$$\widehat{p}_2 - c(q_2) = (q_2 - q_1)\frac{1 - F(\widehat{v})}{f(\widehat{v})} \equiv (q_2 - q_1)h(\widehat{v}) \tag{36}$$

$$\textit{where} \qquad \widehat{v} = \frac{\widehat{p}_2 - \widehat{p}_1}{q_2 - q_1}.$$

Lemma 7 presents the usual price markups. The key observation is that the first-best allocation of consumers across the two firms is generally not an equilibrium. We substract (35) from (36) to obtain

$$\widehat{v} = \frac{p_2 - p_1}{q_2 - q_1} = \frac{c(q_2) - c(q_1)}{q_2 - q_1} + h(\widehat{v}) - k(\widehat{v}), \tag{37}$$

which says that the price difference between the two firms is different from their cost difference. Compared with either Lemma 1 or Lemma 4, for a given pair of qualities, duopoly prices may be higher or lower than prices when Firm 1 aims to maximize social surplus.

We write equilibrium prices in Stage 2 as $\widehat{p}(q_1, q_2)$ and $\widehat{p}(q_1, q_2)$. The equilibrium marginal consumer $\widehat{v}(q_1, q_2)$ is implicitly defined by (37). Profits of Firm 1 and Firm 2 are, respectively, $F(\widehat{v}(q_1, q_2))[\widehat{p}_1(q_1, q_2) - c(q_1)]$ and $[1 - F(\widehat{v}(q_1, q_2))][\widehat{p}_2(q_1, q_2) - c(q_2)]$. In a subgame-perfect equilibrium, each firm chooses its quality in Stage 1 to maximize its profit, given the rival firm's quality and the continuation equilibrium prices $\widehat{p}_1(q_1, q_2)$ and $\widehat{p}_2(q_1, q_2)$. The following properties of equilibrium prices will be used for the derivation of the equilibrium qualities.

Lemma 8 *From the definitions of* $(\widehat{p}_1, \widehat{p}_2)$ *in (35) and (36), and the marginal consumer* $\widehat{v}(q_1, q_2)$ *implicitly defined by (37) we have* $\widehat{v}$ *increasing in both* q_1 *and* q_2*, and*

$$\frac{\partial \widehat{v}}{\partial q_1} = \frac{\dfrac{c(q_2) - c(q_1)}{q_2 - q_1} - c'(q_1)}{(q_2 - q_1)[1 - h'(\widehat{v}) + k'(\widehat{v})]} > 0 \quad \textit{and} \quad \frac{\partial \widehat{v}}{\partial q_2} = \frac{c'(q_2) - \dfrac{c(q_2) - c(q_1)}{q_2 - q_1}}{(q_2 - q_1)[1 - h'(\widehat{v}) + k'(\widehat{v})]} > 0.$$

Furthermore, Firm 1's equilibrium price increases with Firm 2's quality, but Firm 2's equilibrium price decreases with Firm 1's quality:

$$\frac{\partial \widehat{p}_1}{\partial q_2} = k(\widehat{v}) + (q_2 - q_1)k'(\widehat{v})\frac{\partial \widehat{v}}{\partial q_2} > 0 \quad \textit{and} \quad \frac{\partial \widehat{p}_2}{\partial q_1} = -h(\widehat{v}) + (q_2 - q_1)h'(\widehat{v})\frac{\partial \widehat{v}}{\partial q_1} < 0.$$

Lemma 8 reports classical tendency of more intense price competition when products are more similar. If Firm 1 raises its quality, then the lower quality q_1 gets closer to the higher quality q_2. As a consequence, Firm 2 will reduce its price in Stage 2. Likewise, if Firm 2 raises its quality, then the higher quality q_2 gets farther away from the lower quality q_1, so Firm 1 now raises its price. Our characterization in Lemma 8, however, uses no specific assumptions such as the uniform distribution on quality valuations and quadratic cost functions. Lemma 8 also contrasts with Lemmas 2 and 5. When Firm 1 aims to maximize social surplus, its price responds to quality differences solely to ensure efficient allocation of consumers.

6.2 Subgame-perfect equilibrium qualities

We characterize equilibrium qualities. Profits of Firms 1 and 2 are, respectively, $F(\widehat{v}(q_1,q_2))[\widehat{p}_1(q_1,q_2)-c(q_1)]$ and $[1-F(\widehat{v}(q_1,q_2))][\widehat{p}_2(q_1,q_2)-c(q_2)]$, where $\widehat{p}_1$, $\widehat{p}_2$, and $\widehat{v}$ are subgame-perfect equilibrium prices and marginal consumer in Lemma 7. Equilibrium qualities $\widehat{q}_1$ and $\widehat{q}_2$ are mutual best responses:

$$
\begin{aligned}
\widehat{q}_1 &= \operatorname*{argmax}_{q_1} F(\widehat{v}(q_1,\widehat{q}_2))[\widehat{p}_1(q_1,\widehat{q}_2)-c(q_1)] \quad \text{with} \quad \widehat{v}(q_1,\widehat{q}_2)=\frac{\widehat{p}_2(q_1,\widehat{q}_2)-\widehat{p}_1(q_1,\widehat{q}_2)}{\widehat{q}_2-q_1}\\
\widehat{q}_2 &= \operatorname*{argmax}_{q_2} [1-F(\widehat{v}(\widehat{q}_1,q_2))][\widehat{p}_2(\widehat{q}_1,q_2)-c(q_2)] \quad \text{with} \quad \widehat{v}(\widehat{q}_1,q_2)=\frac{\widehat{p}_2(\widehat{q}_1,q_2)-\widehat{p}_1(\widehat{q}_1,q_2)}{q_2-\widehat{q}_1}.
\end{aligned}
$$

As in the earlier subsections on equilibrium qualities when Firm 1 is a public firm, we substitute $\widehat{v}$ for $\widehat{p}_1$, and rewrite Firm 1's profit function as

$$F(\widehat{v}(q_1,\widehat{q}_2))[\widehat{p}_2(q_1,\widehat{q}_2)-\widehat{v}(q_1,\widehat{q}_2)(\widehat{q}_2-q_1)-c(q_1)].$$

Changing q_1 changes the marginal consumer $\widehat{v}$, Firm 2's price $\widehat{p}_2$, and the surplus for the indifferent consumer $\widehat{v}q_1-c(q_1)$. Now the Envelope Theorem applies, and the effect of q_1 on profit through $\widehat{v}$ is second order. The first-order derivative of Firm 1's profit with respect to q_1 is

$$\frac{\partial \widehat{p}_2(q_1,\widehat{q}_2)}{\partial q_1}+\widehat{v}-c'(q_1), \tag{38}$$

where we have omitted the factor $F(\widehat{v}(q_1,\widehat{q}_2))$ (and the partial derivative with respect to $\widehat{v}$).

Similarly, for Firm 2, we substitute $\widehat{p}_2$ by $\widehat{v}$, and rewrite its profit as

$$[1-F(\widehat{v}(\widehat{q}_1,q_2))][\widehat{p}_1(\widehat{q}_1,q_2)+\widehat{v}(\widehat{q}_1,q_2)(q_2-\widehat{q}_1)-c(q_2)].$$

The effect of q_2 on profit through its effect on $\widehat{v}$ is zero by the Envelope Theorem. The derivative of Firm 2's profit with respect to q_2 is

$$\frac{\partial \widehat{p}_1(\widehat{q}_1, q_2)}{\partial q_2} + \widehat{v} - c'(q_2), \tag{39}$$

where again we have omitted the factor $1 - F$. We now state our main result for the duopoly model.

Proposition 6 *The equilibrium qualities and market share solve the three equations in q_1, q_2, and v:*

$$\begin{aligned}
v &= c'(q_1) + h(v) - h'(v) \left[\frac{\dfrac{c(q_2) - c(q_1)}{q_2 - q_1} - c'(q_1)}{1 - h'(v) + k'(v)} \right] \\
v &= c'(q_2) - k(v) - k'(v) \left[\frac{c'(q_2) - \dfrac{c(q_2) - c(q_1)}{q_2 - q_1}}{1 - h'(v) + k'(v)} \right] \\
v &= \frac{c(q_2) - c(q_1)}{q_2 - q_1} + h(v) - k(v).
\end{aligned}$$

Proposition 6 gives a full characterization of equilibrium qualities. It confirms the product differentiation result: Firm 1 chooses a quality lower than one that is optimal for the indifferent consumer, but Firm 2 does the opposite. From the first two equations in Proposition 6, we have $c'(\widehat{q}_1) < \widehat{v} < c'(q_2)$. Lemma 7 already says that for any given firm qualities, the allocation of consumers across the two firms is not first best. Proposition 6 now says that the equilibrium qualities have very little to do with the first best. In fact, for all the examples we have presented above, equilibrium qualities are not first best.

Excessive product differentiation can be illustrated by the typical uniform-quadratic example. Let $f(v) = 1/10$ and $F(v) = 1/10(v - 10)$, for $v \in [10, 20]$, and $c(q) = \frac{1}{2}q^2$. The first best has $q_l^* = 12\frac{1}{2}$, $q_h^* = 17\frac{1}{2}$, and $v^* = 15$. The equilibrium qualities and the market shares are given by the solution of the three equations in Proposition 6. The equilibrium qualities are $\widehat{q}_1 = 7\frac{1}{2}$ and $\widehat{q}_2 = 22\frac{1}{2}$, and the equilibrium indifferent consumer is $\widehat{v} = 15$.

7 Concluding remarks

In this paper we have studied equilibria in a mixed market in a conventional, two-stage, quality-then-price game. The public firm maximizes social surplus, and the private firm maximizes profit. We have used a general distribution for consumer's valuations and general cost functions for firms. We discuss two classes of equilibria. In one class, the public firm offers low quality and the private offers high quality. In the other class, the opposite is true. It turns out that equilibrium qualities can be first best when hazard and the reverse hazard rates are linear. We have related our results to competition policies, and discussed various robustness issues.

Various directions for further research may be of interest. Clearly, duopoly is a limitation. However, a mixed oligopoly with an arbitrary number of firms is analytically very difficult. In the extant literature, models of product differentiation with many private firms typically need to impose very strong assumptions on either consumer valuation (equivalently location) distribution or production cost (equivalently mismatch disutility). The contribution here relies on our ability to identify the hazard and reverse hazard rates as the determining factors for properties of equilibrium qualities. It may well be that they also turn out to be useful for a richer model. The unit cost being constant with respect to quantity is a common assumption in the literature. We have used the same "constant-returns" approach. Scale effects may turn out to be important even for the mixed duopoly.

We have taken for granted the existence of equilibria. However, firms' payoff functions are discontinuous at the point when they offer the same quality. Proofs of existence in similar duopoly models as in Anderson et al. (1997) or Benassi et al. (2016) have required more assumptions than monotone hazard or reverse hazard rates. Although finding sufficient conditions for existence in the mixed duopoly model is beyond the scope of this paper, further research may turn out to be fruitful.

Appendix

Proof of Lemma 1: Consider $\widehat{p}_2 = \text{argmax}_{p_2}[1 - F(\widetilde{v})][p_2 - c(q_2)]$, where $\widetilde{v} = \dfrac{p_2 - \widehat{p}_1}{q_2 - q_1}$ (see (6)). The first-order derivative of the profit function with respect to p_2 is

$$\begin{aligned} & [1 - F(\widetilde{v})] - f(\widetilde{v})[p_2 - c(q_2)]\frac{1}{q_2 - q_1} \\ = \quad & h(\widetilde{v}) - [p_2 - c(q_2)]\frac{1}{q_2 - q_1}, \end{aligned}$$

where we have used the partial derivative of $\widetilde{v}$ with respect to p_2, namely $1/(q_2 - q_1)$. From the assumption that h is decreasing, the second-order derivative is negative, so the first-order condition is sufficient. Therefore, $\widehat{p}_2$ is given by $\widehat{p}_2 - c(q_2) = (q_2 - q_1)h(\widetilde{v})$.

Next, consider Firm 1 choosing p_1 to maximize (7) where $\widetilde{v} = \dfrac{\widehat{p}_2 - p_1}{q_2 - q_1}$ (see (6)). Because (7) is independent of p_1, we can choose $\widetilde{v}$ to maximize (7) ignoring (6). The optimal value $\widehat{v}$ is given by setting to zero the first-order derivative of (7) with respect to $\widetilde{v}$: $\widehat{v}q_1 - c(q_1) = \widehat{v}q_2 - c(q_2)$. Then we simply choose $\widehat{p}_1$ to satisfy (6) such that $\widehat{v} = \dfrac{\widehat{p}_2 - \widehat{p}_1}{q_2 - q_1} = \dfrac{c(q_2) - c(q_1)}{q_2 - q_1}$. We have shown that $\widehat{p}_1$ and $\widehat{p}_2$ in (9) and (10) are mutual best responses.

Proof of Lemma 2: First, from (10), we obtain $(q_2 - q_1)\mathrm{d}\widehat{v} + \widehat{v}(\mathrm{d}q_2 - \mathrm{d}q_1) = c'(q_2)\mathrm{d}q_2 - c'(q_1)\mathrm{d}q_1$, which, together with the convexity of c, yields

$$\frac{\partial \widehat{v}}{\partial q_1} = \frac{\widehat{v} - c'(q_1)}{q_2 - q_1} = \frac{1}{q_2 - q_1}\left[\frac{c(q_2) - c(q_1)}{q_2 - q_1} - c'(q_1)\right] > 0 \tag{40}$$

$$\frac{\partial \widehat{v}}{\partial q_2} = \frac{c'(q_2) - \widehat{v}}{q_2 - q_1} = \frac{1}{q_2 - q_1}\left[c'(q_2) - \frac{c(q_2) - c(q_1)}{q_2 - q_1}\right] > 0. \tag{41}$$

Next, from (9), we obtain

$$\begin{aligned} \mathrm{d}\widehat{p}_1 - c'(q_1)\mathrm{d}q_1 &= (\mathrm{d}q_2 - \mathrm{d}q_1)h(\widehat{v}) + (q_2 - q_1)h'(\widehat{v})\left(\frac{\partial \widehat{v}}{\partial q_2}\mathrm{d}q_2 - \frac{\partial \widehat{v}}{\partial q_1}\mathrm{d}q_1\right) \\ \mathrm{d}\widehat{p}_2 - c'(q_2)\mathrm{d}q_2 &= (\mathrm{d}q_2 - \mathrm{d}q_1)h(\widehat{v}) + (q_2 - q_1)h'(\widehat{v})\left(\frac{\partial \widehat{v}}{\partial q_2}\mathrm{d}q_2 - \frac{\partial \widehat{v}}{\partial q_1}\mathrm{d}q_1\right). \end{aligned}$$

We then use (40) and (41) to simplify these, and obtain

$$\begin{aligned} \frac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2} &= h(\widehat{v}) + h'(\widehat{v})\left[c'(q_2) - \widehat{v}\right] \\ \frac{\partial \widehat{p}_2(q_1, q_2)}{\partial q_2} &= c'(q_2) + h(\widehat{v}) + h'(\widehat{v})\left[c'(q_2) - \widehat{v}\right], \end{aligned}$$

which are the expressions in the lemma.

Proof of Proposition 1: The first-order derivative of (13) with respect to q_1 is

$$\int_{\underline{v}}^{\widehat{v}(q_1,q_2)} [x - c'(q_1)] f(x) \mathrm{d}x + \{[\widehat{v}(q_1, q_2) q_1 - c(q_1)] - [\widehat{v}(q_1, q_2) q_2 - c(q_2)]\} f(\widehat{v}(q_1, q_2)) \frac{\partial \widehat{v}}{\partial q_1}.$$

By Lemma 1, the term inside the curly brackets is zero. By putting this first-order derivative to zero, we obtain the first equation in the Proposition. Also, because equilibrium prices $\widehat{p}_1(q_1, q_2)$ and $\widehat{p}_2(q_1, q_2)$ must follow Lemma 1, we have

$$\widehat{v}(q_1, q_2) = \frac{c(q_2) - c(q_1)}{q_2 - q_1},$$

which is the last equation in the Proposition.

Next, we use (17) to obtain the first-order derivative of Firm 2's profit with respect to q_2:

$$[1 - F(\widehat{v}(q_1, q_2))] \left[\widehat{v}(q_1, q_2) - c'(q_2) + \frac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2} \right] +$$
$$\{-f(\widehat{v}(q_1, q_2))[\widehat{p}_2(\widehat{q}_1, q_2) - c(q_2)] + [1 - F(\widehat{v}(q_1, q_2))](q_2 - q_1)\} \frac{\partial \widehat{v}(q_1, q_2)}{\partial q_2}.$$

Again, by Lemma 1, the term inside the curly bracket is zero. After setting the first-order derivative to 0, we obtain

$$\widehat{v}(q_1, q_2) - c'(q_2) + \frac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2} = 0.$$

We then use (11) in Lemma 2 to substitute for $\dfrac{\partial \widehat{p}_1(q_1, q_2)}{\partial q_2}$, and write the first-order condition as

$$\widehat{v} - c'(q_2) + h(\widehat{v}) + h'(\widehat{v}) [c'(q_2) - \widehat{v}] = 0,$$

which simplifies to

$$\widehat{v} + \frac{h(\widehat{v})}{1 - h'(\widehat{v})} = c'(q_2),$$

the second equation in the Proposition.

Proof of Lemma 3: By definition, $f(x)h(x) = (1 - F(x))$. We have

$$\begin{aligned}
& \frac{\int_v^{\overline{v}} x f(x)\mathrm{d}x}{1 - F(v)} \\
= & -\frac{\int_v^{\overline{v}} x \mathrm{d}(1 - F(x))}{f(v)h(v)} \\
= & \frac{v(1 - F(v))}{f(v)h(v)} + \frac{\int_v^{\overline{v}} (1 - F(x))\mathrm{d}x}{f(v)h(v)} \\
= & v + \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)},
\end{aligned}$$

where the second equality is due to integration by parts.

Proof of Proposition 2: Suppose that $h(x) = \alpha - \beta x$. We have, $h'(x) = -\beta$, and

$$v + \frac{h(v)}{1 - h'(v)} = v + \frac{\alpha - \beta v}{1 + \beta} = \frac{v + \alpha}{1 + \beta}.$$

Then we compute

$$\begin{aligned}
v + \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} &= v + \frac{\int_v^{\overline{v}} f(x)(\alpha - \beta x)\mathrm{d}x}{f(v)h(v)} \\
&= v + \frac{\alpha[1 - F(v)]}{f(v)h(v)} - \beta \frac{\int_v^{\overline{v}} x f(x)\mathrm{d}x}{f(v)h(v)} \\
&= v + \alpha - \beta \left\{ v + \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} \right\},
\end{aligned}$$

where the expression in the curly brackets comes from the identity (19). Simplifying, we have

$$v + \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} = \frac{v + \alpha}{1 + \beta}.$$

We have proved (20).

The three equations in Proposition 1 are now exactly those that define the first best in (3), (4), and (5). Equilibrium qualities and consumer allocation must be first best.

Proof of Remark 1: When Firm 2 sells to consumers with valuations above v at price p_2, its revenue is $[1 - F(v)]p_2$, where $v = \dfrac{p_2 - p_1}{q_2 - q_1}$. If we express p_2 as a function of v, we have $p_2(v) = p_1 + v(q_2 - q_1)$. The marginal revenue is the derivative of revenue with respect to the firm's quantity, $[1 - F(v)]$:

$$\begin{aligned}
& \frac{\mathrm{d}[1 - F(v)]p_2(v)}{\mathrm{d}[1 - F(v)]} \\
= & \ p_2(v) + [1 - F(v)]\frac{\mathrm{d}p_2(v)}{\mathrm{d}[1 - F(v)]} = p_2(v) + [1 - F(v)]\frac{\mathrm{d}p_2(v)/\mathrm{d}v}{\mathrm{d}[1 - F(v)]/\mathrm{d}v} \\
= & \ p_2(v) - \frac{1 - F(v)}{f(v)}\frac{\mathrm{d}p_2(v)}{\mathrm{d}v} = p_2(v) - h(v)(q_2 - q_1).
\end{aligned}$$

Because $p_2(v)$ is linear in v, marginal revenue is linear in v if and only if the hazard rate $h(v)$ is linear.

Proof of Remark 2: Define $y \equiv 1 - F$, so $y' = -f$. We have $h(x) = \alpha - \beta x$ equivalent to $\frac{y'}{y} = \frac{-1}{\alpha - \beta x}$. First, suppose that $\beta = 0$. We have $\frac{y'}{y} = \frac{-1}{\alpha}$, so $y(v) = A\exp(-\frac{v}{\alpha})$, some A. Therefore, $F(v) = 1 - A\exp(-\frac{v}{\alpha})$. Because we have $F(\underline{v}) = 0$, we must have $A = \exp(\frac{\underline{v}}{\alpha})$. We also have $F(\overline{v}) = 1$, which requires $\overline{v} = \infty$.

Second, suppose that $\beta > 0$. We have $\frac{y'}{y} = \frac{-1}{\alpha - \beta v}$. Solving this differential equation, we have $y\,(v) = A(\alpha - \beta v)^{\frac{1}{\beta}}$, for some constant A. Hence, $F(v) = 1 - A(\alpha - \beta v)^{\frac{1}{\beta}}$, and we obtain the expression for f in the Remark by differentiation. Because $F(\underline{v}) = 0$, we have $A = (\alpha - \beta \underline{v})^{-\frac{1}{\beta}}$. Because $F(\overline{v}) = 1$, we must have $\alpha - \beta\overline{v} = 0$, so that α and β cannot be arbitrary.

Proof of Proposition 3: For any q_2 we consider Firm 1's best response function:

$$\widetilde{q}_1(q_2) = \operatorname*{argmax}_{q_1} \int_{\underline{v}}^{\widehat{v}(q_1,q_2)} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\widehat{v}(q_1,q_2)}^{\overline{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x.$$

First, at $q_2 = q_h^*$, we have $\widetilde{q}_1(q_h^*) = q_\ell^*$. Clearly, if Firm 2 chooses q_h^*, from the definition of the first best, Firm 1's best response is $q_1 = q_\ell^*$ because Firm 1 aims to maximize social surplus. It follows that the first best belongs to the graph of Firm 1's best response function.

Second, we establish that $\widetilde{q}_1(q_2)$ is increasing in q_2. The sign of the derivative of $\widetilde{q}_1(q_2)$ has the same sign of the cross partial derivative of Firm 1's objective function (13) evaluated at $q_1 = \widetilde{q}_1(q_2)$. The derivative of (13) with respect to q_1 is simply

$$\int_{\underline{v}}^{\widehat{v}(q_1,q_2)} [x - c'(q_1)]f(x)dx$$

because the partial derivative with respect to $\widehat{v}$ is zero. The cross partial is then obtained by differentiating the above with respect to q_2, and this gives

$$[\widehat{v}(q_1, q_2) - c'(q_1)]f(\widehat{v})\frac{\partial \widehat{v}(q_1, q_2)}{\partial q_2} > 0$$

where the inequality follows because at $q_1 = \widetilde{q}_1(q_2)$, we have $\widehat{v}(q_1, q_2) > c'(q_1)$ and $\frac{\partial \widehat{v}}{\partial q_2} > 0$ by (41) in the proof of Lemma 2.

Proof of Lemma 6: By definition, $f(x)k(x) = F(x)$. We have

$$\begin{aligned}
& \frac{\int_{\underline{v}}^{v} xf(x)\mathrm{d}x}{F(v)} \\
= & \frac{\int_{\underline{v}}^{v} x\mathrm{d}F(x)}{f(v)k(v)} \\
= & \frac{vF(v)}{f(v)k(v)} - \frac{\int_{\underline{v}}^{v} F(x)\mathrm{d}x}{f(v)k(v)} \\
= & v + \frac{\int_{\underline{v}}^{v} f(x)k(x)\mathrm{d}x}{f(v)h(v)},
\end{aligned}$$

where the second equality is due to integration by parts.

Proof of Proposition 5: Suppose that $k(x) = \gamma + \delta x$. We have $k'(x) = \delta$, and

$$v - \frac{k(v)}{1 + k'(v)} = v - \frac{\gamma + \delta v}{1 + \delta} = \frac{v - \gamma}{1 + \delta}.$$

Then we compute

$$\begin{aligned}
v - \frac{\int_{\underline{v}}^{v} f(x)k(x)\mathrm{d}x}{f(v)k(v)} &= v - \frac{\int_{\underline{v}}^{v} f(x)(\gamma + \delta x)\mathrm{d}x}{f(v)k(v)} \\
&= v - \frac{\gamma F(v)}{f(v)k(v)} - \delta \frac{\int_{\underline{v}}^{v} xf(x)\mathrm{d}x}{f(v)k(v)} \\
&= v - \gamma - \delta \left\{ v - \frac{\int_{\underline{v}}^{v} f(x)k(x)\mathrm{d}x}{f(v)k(v)} \right\},
\end{aligned}$$

where the expression in the curly brackets comes from the identity (32). Simplifying, we have

$$v - \frac{\int_{\underline{v}}^{v} f(x)k(x)\mathrm{d}x}{f(v)k(v)} = \frac{v - \gamma}{1 + \delta}.$$

We have proved (33).

The three equations in Proposition 4 are now exactly those that define the first best in (3), (4), and (5). Equilibrium qualities and consumer allocation must be first best.

Proof of Remark 3: When Firm 2 sells to consumers with valuations below v at price p_2, its revenue is $F(v)p_2$, where $v = \dfrac{p_1 - p_2}{q_1 - q_2}$. If we express p_2 as a function of v, we have $p_2(v) = p_1 - v(q_1 - q_2)$. The marginal revenue is the derivative of revenue with respect to the firm's quantity, $F(v)$:

$$\begin{aligned}
& \frac{\mathrm{d}F(v)p_2(v)}{\mathrm{d}F(v)} \\
= \quad & p_2(v) + F(v)\frac{\mathrm{d}p_2(v)}{\mathrm{d}F(v)} = p_2(v) + F(v)\frac{\mathrm{d}p_2(v)/\mathrm{d}v}{\mathrm{d}F(v)/\mathrm{d}v} \\
= \quad & p_2(v) + \frac{F(v)}{f(v)}\frac{\mathrm{d}p_2(v)}{\mathrm{d}v} = p_2(v) - k(v)(q_1 - q_2).
\end{aligned}$$

Because $p_2(v)$ is linear in v, marginal revenue is linear in v if and only if the reverse hazard rate $k(v)$ is linear.

Proof of Remark 4: Define $y \equiv F$, so $y' = f$. We have $k(x) = \gamma + \delta x$ equivalent to $\dfrac{y'}{y} = \dfrac{1}{\gamma + \delta x}$. First, suppose that $\delta = 0$, then $\dfrac{y'}{y} = \dfrac{1}{\gamma}$, or $\mathrm{d}\ln(y) = \dfrac{\mathrm{d}x}{\gamma}$. Hence, $\ln(y) = \dfrac{x}{\gamma} + B$, some constant B, so $y = \exp(\dfrac{x}{\gamma} + B)$. We require $F(\underline{v}) = \exp(\dfrac{\underline{v}}{\gamma} + B) = 0$, but this is impossible since $\underline{v} > 0$. We conclude that $\delta > 0$.

Second, $\dfrac{y'}{y} = \dfrac{1}{\gamma + \delta x}$, we have $\mathrm{d}\ln(y) = \dfrac{1}{\delta}\dfrac{\mathrm{d}(\gamma + \delta x)}{(\gamma + \delta x)}$. Therefore, $\ln(y) = \dfrac{1}{\delta}\ln(\gamma + \delta x) + B$, some B, or $F = y = A(\gamma + \delta x)^{\frac{1}{\delta}}$, some A. Because F is a distribution function, we require $F(\underline{v}) = 0$ and $F(\overline{v}) = 1$. These requirements are $\gamma + \delta\underline{v} = 0$ and $A(\gamma + \delta\overline{v})^{\frac{1}{\delta}} = 1$. We obtain the expression for f in the Remark by differentiation.

Proof of Remark 5: Suppose we have $F = 1 - (\alpha - \beta v)f$ and $F = (\gamma + \delta v)f$. We use these two equations to solve for f, and obtain $f = [\alpha + \gamma + (\delta - \beta)v]^{-1}$. Then we substitute α by $\beta\overline{v}$ and γ by $-\delta\underline{v}$, and simplify f to the expression of in the Remark. Finally, we obtain F in the Remark by substituting the solution for f in either of the two equations.

Proof of Lemma 8: In subgame (q_1, q_2) the equilibrium indifferent consumer in Stage 2 is given by (37), which is rewritten as

$$[\widehat{v} - h(\widehat{v}) + k(\widehat{v})] = \frac{c(q_2) - c(q_1)}{q_2 - q_1}.$$

Now we use this to differentiate $\widehat{v}$ with respect to the qualities to obtain:

$$\begin{aligned}
\frac{\partial\widehat{v}}{\partial q_1}[1 - h'(\widehat{v}) + k'(\widehat{v})] &= \frac{1}{q_2 - q_1}\left[\frac{c(q_2) - c(q_1)}{q_2 - q_1} - c'(q_1)\right] > 0 \\
\frac{\partial\widehat{v}}{\partial q_2}[1 - h'(\widehat{v}) + k'(\widehat{v})] &= \frac{1}{q_2 - q_1}\left[c'(q_2) - \frac{c(q_2) - c(q_1)}{q_2 - q_1}\right] > 0,
\end{aligned}$$

which simplify to the first two expressions in the lemma, and where the inequalities follow from $h' < 0$, $k' > 0$, and $q_1 < q_2$.

Next, the derivative of $\widehat{p}_1$ in (35) with respect to q_2 and the derivative of $\widehat{p}_2$ in (36) with respect to q_1 in the lemma are obtained by straightforward computation, and we have kept track of $\widehat{v}(q_1, q_2)$ being implicitly defined by (37). Again, the inequalities follow from $h' < 0$, $k' > 0$, and the properties of $\widehat{v}$ derived above.

Proof of Proposition 6: We begin with the derivatives of firms' profits in (38) and (39), and set them to zero to obtain first-order conditions. Equilibrium qualities $\widehat{q}_1$ and $\widehat{q}_2$ are best responses, so must satisfy the first-order conditions simultaneously:

$$\begin{aligned} \frac{\partial \widehat{p}_2(\widehat{q}_1, \widehat{q}_2)}{\partial q_1} + \widehat{v} - c'(\widehat{q}_1) &= 0 && (42) \\ \frac{\partial \widehat{p}_1(\widehat{q}_1, \widehat{q}_2)}{\partial q_2} + \widehat{v} - c'(\widehat{q}_2) &= 0. && (43) \end{aligned}$$

The continuation equilibrium in prices must also satisfy Lemma 7, so (37) must also be satisfied at qualities $\widehat{q}_1$ and $\widehat{q}_2$:

$$\widehat{v} = \frac{c(\widehat{q}_2) - c(\widehat{q}_1)}{\widehat{q}_2 - \widehat{q}_1} + h(\widehat{v}) - k(\widehat{v}),$$

which is the third equation in the Proposition.

Next, we use the expressions for $\dfrac{\partial \widehat{p}_2(\widehat{q}_1, \widehat{q}_2)}{\partial q_1}$ and $\dfrac{\partial \widehat{p}_1(\widehat{q}_1, \widehat{q}_2)}{\partial q_2}$ in Lemma 8. After substitution, the first-order conditions (42) and (43) become

$$\begin{aligned} \widehat{v} &= c'(\widehat{q}_1) + h(\widehat{v}) - (q_2 - q_1) h'(\widehat{v}) \frac{\partial \widehat{v}}{\partial q_1} \\ \widehat{v} &= c'(\widehat{q}_2) - k(\widehat{v}) - (q_2 - q_1) k'(\widehat{v}) \frac{\partial \widehat{v}}{\partial q_2}. \end{aligned}$$

Then we apply the expression for $\dfrac{\partial \widehat{v}}{\partial q_1}$ and $\dfrac{\partial \widehat{v}}{\partial q_2}$ in Lemma 8 to the above, simplify, and obtain the first two equations in the Proposition.

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