Why the Political World is Flat: An Endogenous "Left" and "Right" in Multidimensional Elections

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Abstract

This paper analyzes candidate positioning in multidimensional, common-interest elections. Candidates can polarize in various directions in equilibrium, giving structure to the common concern that issues might be bundled inefficiently. However, the only stable equilibrium efficiently bundles issues that are logically related. Consistently bundling issues in this way can explain why multidimensional policy decisions invariably reduce to the same single dimension.

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1 Introduction

The world of public policy is complex and multifaceted. Elected officials must decide tax policy, foreign policy, health policy, education policy, immigration policy, social policy, and numerous more narrow issues that are themselves complex and multifaceted. To model this properly, the number of dimensions should be enormous. Indeed, every sentence of legislation could be viewed as a separate dimension in which policy could be adjusted. With two sides to each issue on D policy dimensions, there are 2^D policy combinations, plus a continuum of intermediate positions between each pair of extremes.

With so many policy configurations to choose from, politicians seem shockingly one dimensional, taking policy positions that for the most part merely range from liberal to moderate to conservative. Poole and Rosenthal (1985, 1997, 2001) report that a one-dimensional model of policy preference correctly predicts almost 90% of the roll call votes cast in the history of the U.S. House and Senate. They cite similar findings for the European Parliament, the U.N. General Assembly, and several European national parliaments, and Grofman and Brazill (2002) and Shor and McCarty (2011) find the same for the U.S. supreme court and state legislatures. In terms of the notation above, such unidimensionality largely ignores all but *two* of the 2^D orthants of the policy space: one consisting of liberal positions on each of the D issues, and the other consisting of conservative positions on every issue. In the words of Converse (1964, p. 207), "...if a person is opposed to the expansion of social security, he is probably a conservative and is probably opposed as well to any nationalization of private industries, federal aid to education, sharply progressive income taxation, and so forth." Converse (1964) ascribed such unidimensionality only to political elites, but Shor (2014) estimates that a single dimension also explains 80% of the individual policy opinions expressed by ordinary citizens.[1]

Existing election models offer no explanation for such unidimensionality. Most, beginning with Hotelling (1928) and Downs (1957), simply *constrain* policy choices to be one dimensional.[2] Models that allow multiple dimensions, beginning with Plott (1967), often exhibit no pure strategy equilibrium.[3] Probabilistic voting models, beginning with Hinich (1978), have an equilibrium but predict counterfactually that candidates should adopt identical policies and exhibit no ideological differences at all.[4] Polarized equilibria exist in the

[1] Tausanovitch and Warshaw (2014) find that voter preferences correlate on local and national issues, as well.

[2] In one dimension, ideological differences are often attributed to wealth. Since it determines the demand both for redistribution (Romer, 1975; Meltzer and Richard, 1981) and for public goods (Bergstrom and Goodman, 1973), wealth could also correlate preferences across multiple redistributive policies or multiple public goods, thus potentially explaining unidimensionality. As I discuss in McMurray (2017a), however, many policy preferences cannot easily be traced to wealth. Empirically, for example, wealthy voters favor redistribution almost as much as the poor. As another example, defense spending and environmental protection are both textbook examples of public goods but tend to draw voter support from opposite ends of the political spectrum.

[3] More recently, see Duggan and Fey (2005). Mixed-strategy equilibria exist more generally (Duggan and Jackson, 2005), but have unclear empirical relevance in the context of political campaigns (Austen-Smith and Banks, 2005).

[4] More recently, see Xefteris (2017). Bafumi and Herron (2010) and Shor (2011) show empirically that candidates polarize substantially.

"citizen candidate" models of Osborne and Slivinski (1996) and Besley and Coate (1997), but there are many such equilibria, and many non-polarized equilibria as well, making the analysis indeterminate.[5] In particular, this leaves open the possibility that candidates in one election polarize in one direction but candidates in another place or time polarize in a different direction, where the empirical finding is that candidates everywhere polarize in the *same* direction. Indeed, McDonald, Mendes, and Kim (2007) report that the same, single dimension categorizes nearly 90% of the policy positions of eighty political parties in seventeen countries, over twenty-five years.

The models above attribute voter disagreements to exogenous tastes. In McMurray (2017a, 2019) I propose another possibility, which is that voters share a preference for promoting social welfare, but have different *information* about the welfare consequences various policies will have.[6] In that case, the fundamental hope of democracy is to aggregate collective wisdom to determine what is truly best, as in Condorcet's (1785) classic "jury" theorem.[7] This paper extends that model to multiple dimensions but shows that candidates settle endogenously on a single dimension, which does not vary across elections.

The first observation underlying this conclusion is simply that logical links between policy issues naturally correlate voters' opinions. To end an economic recession, for example, it might turn out that fiscal stimulus is effective while monetary stimulus is not, or vice versa, but ex ante it is more likely either that both forms of stimulus are beneficial (because the economy functions more or less as Keynesian models predict) or that both are wasteful (as in more classical models). More broadly, support (or lack thereof) for a host of regulations may stem from an underlying belief or disbelief in market efficiency or trust in government regulators, while support for a variety of redistribution policies jointly depend on a voter's views about the relative importance of luck and effort in creating wealth.

If issues are uncorrelated then, in an otherwise symmetric environment, candidates can bundle policies in any configuration in equilibrium, with each voter supporting the candidate whose policy positions most closely match his own opinion of what is optimal.[8] As in the models above, the problem of indeterminacy is severe in that case. If issues are correlated, however, then voters' and candidates' best response incentives rotate in the direction of correlation, thus breaking symmetry and reducing the number of equilibrium possibilities. The same logic that connects two issues in one election should connect the same two issues in another place or time, so this can explain why policies are bundled so consistently across elections.

Of course, not all issues are as tightly connected as fiscal and monetary stimulus. As

[5] As the latter authors write, "basically *any* pair of candidates who split the voters evenly can be an equilibrium."

[6] As those papers explain, a welfare objective can arise as large elections amplify voter altruism, and is consistent with numerous instances of voters who put social objectives ahead of their own narrow interests. An information model of elections also explains key features of voter behavior, such as why uninformed voters tend to remain ideologically moderate, why confident voters work to persuade their neighbors, and why an extreme candidate still expects to win when on the side of truth.

[7] The jury theorem states that majority opinion in a large electorate favors whichever of two policies is truly superior, as long as individual opinions are correlated with the truth and, conditional on the truth, are mutually independent.

[8] In this paper, masculine pronouns describe voters. Feminine pronouns describe candidates.

Shor (2014) expresses, for example, "it is not clear why environmentalism necessarily hangs together with a desire for more union prerogatives, but [empirically] it does." Logical connections do exist, though, even if they are weak. For example, support for environmental regulations and workers' unions could both stem from a view of businesses as overly ruthless, disregarding employee wellbeing and public health in their pursuit of profit. In fact, this could motivate support for a host of other pro-labor policies. Importantly, the results below do not require a strong connection across issues: *any* non-zero correlation can be enough to orient the equilibrium, so that issues are bundled just as they would be if they were *perfectly* correlated.

A key force driving unidimensionality is that, in an information model, a voter's or candidate's best response anticipates the private information prompting others' behavior. For voters, this means inferring information from the event of a pivotal vote. For candidates, it means learning from the event of winning the election. In the one-dimensional model of McMurray (2019), this leads candidates to polarize, each concluding that if she wins it will be because truth is on her side. The same occurs in multiple dimensions, but the direction of this polarization now depends on the orientation of the voting strategy: when issues are logically unconnected, candidates polarize in exactly the direction of voting, and voters split exactly in the direction of candidates' platforms, so equilibrium polarization can occur in any arbitrary direction. When issues are logically connected, candidates take this into account and adopt more similar positions on each issue, effectively rotating toward the two traditional orthants of the policy space. Correlation rotates citizens' voting response, as well, causing candidates to rotate further. No matter how weak the correlation between issues, this continues until, in equilibrium, platform positions on every issue are comparable. This also makes candidates more ideologically consistent than voters, just as they are empirically.

Observers of politics often express concern that issues are not bundled as sensibly as they could be. The model below lends structure to these concerns, because in addition to the one *major* equilibrium described above, there are *minor* equilibria, where inefficient bundling of issues can be self-perpetuating. With the present formulation of the model, however, minor equilibria are unstable, and therefore unlikely to prevail. In that sense, the model delivers an essentially unique equilibrium prediction. Because the orientation of this unique equilibrium depends on the logical connections between issues and not on institutional details, it should be the same across elections, consistent with empirical evidence.[9]

To emphasize the arbitrariness of issue bundlings when truth variables are uncorrelated, the model below is completely symmetric. Symmetry is also extremely useful for analysis, because with so many truth possibilities and individual opinions and an infinite hierarchy of higher order beliefs, the precise outcome of Bayesian updating is very difficult to characterize, but strategic forces in a known direction produce unambiguous incentives when they break symmetry. Tractability is also achieved by assuming a very special (namely,

[9] As further evidence that institutional features are not essential for the empirical correlation between issues, note that, historically, political factions (e.g. Tories and Whigs in England, Federalists and Anti-Federalists in the U.S.) arose before political parties were ever formalized (see https://en.wikipedia.org/wiki/Political_party). Pan and Xu (2017) find ideological structure similar to that of western democracies, even in authoritarian China. Outside the realm of politics, scientific and other non-political communities similarly divide into competing "schools of thought".

linear) distribution of signals. Finding tractable ways to generalize the baseline model is an important task left largely for future work, but below, a section of extensions suggests that symmetry and linearity are important not for generating, but just for elucidating, the forces that produce unidimensionality: in more general environments, the same forces should operate, so that candidates adopt similar positions on every issue. In perfectly symmetric settings, their policy positions are identical.

2 Literature

There are two papers that study information aggregation in multidimensional common-interest settings, but neither addresses candidate positioning or unidimensionality, which are the focus of this paper. Feddersen and Pesendorfer (1997) show that conflicts of interest impede information aggregation when there are more dimensions than one. Barelli, Bhattacharya, and Siga (2015) identify conditions on the information structure that, in the absence of conflict, are necessary and sufficient for efficient information aggregation in arbitrary dimensions.

Social learning literature offers two accounts of multidimensional beliefs collapsing to a single dimension, but no account as to why the orientation of this direction should be consistent across elections. In Spector (2000), individuals with one of two prior beliefs about a multidimensional common-interest decision take turns reporting private signals to influence the group. Over time, beliefs converge in every direction but the one of prior disagreement, which presumably should differ from election to election.[10] In DeMarzo, Vayanos, and Zwiebel (2003), individuals update their beliefs after circulating their initial signals through a social network but fail to discount repeated information. The direction of slowest consensus can be interpreted as the left and right of politics (see also Louis, Troumpounis, and Tsakas, 2018), but depends on prior beliefs and the network structure, both of which are likely to vary across elections.[11]

The recent private interest election model of Schnakenberg (2016) assumes that voters do not know candidates' policy preferences, and policy platforms are non-binding. In equilibrium, a line divides the policy space in two, and a candidate's platform can only credibly communicate which side of the line her preferred policy bundle lies on. This has the familiar geometry of candidates pitted opposite one another, but since the line dividing them could be oriented in any direction, it offers no reason for different elections to be oriented the same way. With enough symmetry, equilibria with different geometry exist, as well.

[10] With three or more priors, the cross-section of different senders' messages should be fully revealing, as in Battaglini (2002).

[11] In addition to these models, Duggan and Martinelli (2011) and Egorov (2014) show how the orientation of political conflict can be influenced by a monolithic media or by candidate messaging, respectively, but take unidimensionality as an exogenous constraint on communication.

3 The Model

A society consists of candidates A and B, together with N voters, where N is drawn from a Poisson distribution with mean $n \in \mathbb{N}$. Candidates choose platforms x_A and x_B that each will implement if elected, from a space X of D-dimensional policy vectors. Voters each vote for one of the two candidates, and the candidate $w \in \{A, B\}$ who receives the most votes wins the election (breaking a tie, if necessary, by a fair coin toss). That candidate's platform then provides the same policy utility to each voter and candidate.[12] Specifically, utility

$$u(x,z) = -\|x-z\|^2 = -(x-z)\cdot(x-z)$$

decreases quadratically in the Euclidean distance $\|x-z\|$ between the policy outcome $x \in X$ and a policy $z \in X$ that is socially optimal.[13]

To keep the environment as symmetric as possible (for the reasons outlined in Section 1), let X be the unit hyperball. The origin might denote a status quo, for example, where society can deviate from this in any direction, up to some maximal distance, normalized to one.[14] For ease of exposition, let $d = 2$, so that X is the unit disk.[15] A policy vector $x = \binom{x_1}{x_2} \in X$ can then alternatively be written as an ordered pair (x_1, x_2), or using polar coordinates (r_x, θ_x) in terms of its magnitude r_x and polar angle θ_x. Multiplying by $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ then produces a *rotation* $R_\theta x$, which has the same magnitude but polar angle $\theta_x + \theta$, and multiplying by $M_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ produces the *mirror image* through angle θ, which has the same magnitude but polar angle $2\theta - \theta_x$, so that x and $M_\theta x$ are equidistant from a vector with angle θ.

The location of z is unknown, and is modeled as a random variable with domain Z. Section 5 explores the possibility of $Z \subset X$, so that some feasible policy combinations are known not to be optimal, but for now let $Z = X$. Conditional on information Ω (and dropping terms that do not depend on the policy outcome), expected utility

$$E[u(x,z)|\Omega] = -\|x - E(z|\Omega)\|^2 \tag{1}$$

then decreases quadratically in the distance between the policy vector implemented and the updated expectation $E(z|\Omega)$ of the optimum.[16]

[12] Sections 5 and 7 conjecture that similar results would hold for a mix of private and common voter interests, or for candidates whose interests differ from voters', but pure common interests seems a natural starting place for analysis, and also lends tractability and transparency.

[13] As in McMurray (2017b), candidates have an incentive to adjust their platforms after taking office, as they learn more about the location of z, and with common interests, voters have no reason to prevent this. A culture of enforcing platform commitments may be warranted, however, if there is positive probability of candidates holding deviant preferences (as I explore in McMurray, 2019). Binding commitments also make the analysis more directly comparable with existing literature, and avoid the complexities of forecasting candidates' ex post behavior by limiting the number of policy outcomes to one per candidate.

[14] A Cartesian product of intervals, which is perhaps more realistic, produces similar results (see Section 5).

[15] Section 5 explains how the results below extend to arbitrary finite K.

[16] This quadratic specification is convenient but does not seem essential. The important feature of (1) is

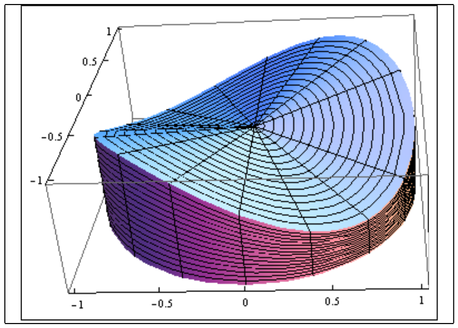

Figure 1: An example density that satisfies Conditions 1 and 2.

To capture the possibility of logical connections across policy issues, let the prior density $f(z;\rho)$ depend on a parameter ρ related to the correlation between z_1 and z_2. In fact, to make the analysis unambiguous, let f satisfy *correlative monotonicity* (Condition 1). For positive ρ, this means that f increases in the direction of the *major diagonal* (i.e. the line defined by $z_1 = z_2$) and decreases in the direction of the *minor diagonal* (i.e. defined by $z_1 = -z_2$), and that this pattern is most pronounced for large ρ. Additionally, assume that f exhibits *dimensional symmetry* (Condition 2), meaning that f is symmetric around the origin and that reorienting one dimension is equivalent to reversing the sign of ρ. Figure 1 illustrates an example

$$f(z;\rho) = \frac{1}{\pi}\left(1 + \rho\frac{z_1 z_2}{\|z\|}\right) = \frac{1}{\pi}\left[1 + \rho r_z \cos(\theta_z)\sin(\theta_z)\right]$$

that satisfies both conditions, where $\rho \in [-1, 1]$ and the correlation coefficient between z_1 and z_2 equals $.4\rho$.

Condition 1 (Correlative monotonicity) *$f(z_1, z_2; \rho)$ is differentiable in z_1, z_2, and ρ. Moreover, $\frac{\partial f(z)}{\partial z_1}$ and $\frac{\partial^2 f(z)}{\partial z_1 \partial \rho}$ have the same signs as z_2 and ρz_2, respectively, and, symmetrically, $\frac{\partial f(z)}{\partial z_2}$ and $\frac{\partial^2 f(z)}{\partial z_2 \partial \rho}$ have the same signs as ρz_1 and z_1 (implying that $\frac{\partial f(z)}{\partial \theta_z}$ has the same sign as $\rho\cos(2\theta_z)$). Also, $\frac{\partial^2 f(z)}{\partial z_1 \partial z_2}$ has the same sign as ρ, $\frac{\partial f(z)}{\partial \rho}$ has the same sign as $z_1 z_2$ (and $\sin(2\theta_z)$), and $\frac{\partial^2 f(z)}{\partial \theta_z \partial \rho}$ has the same sign as $|z_1| - |z_2|$ (and $\cos(2\theta_z)$).*

Condition 2 (Dimensional symmetry) *$f(z_1, z_2) = f(z_2, z_1) = f(-z_1, -z_2)$ and $f(-z_1, z_2) = f(z_1, z_2; -\rho)$. Equivalently, $f(z) = f\left(M_{\frac{\pi}{4}} z\right) = f(R_\pi z)$ and $f\left(M_{\frac{\pi}{2}} z\right) = f(M_0 z) = f\left(R_{\frac{\pi}{2}} z\right) = f(z; -\rho)$.*

When $\rho = 0$, correlative monotonicity implies that f is uniform and thus also satisfies *radial symmetry* (Condition 3), meaning that the optimal policy pair is equally likely to lie in any direction from the origin.

that shifts in the distribution of z shift the desired policy in the same direction. With a linear loss function, for example, a voter would favor the median realizations of z_1 and z_2 (conditional on Ω) instead of the mean, with similar implications for behavior.

Condition 3 (Radial symmetry) *$f(R_\theta z) = f(M_\theta z) = f(z)$ for any $\theta \in \mathbb{R}$ and for any $z \in Z$.*

Opinions about what is optimal are determined by pairs $s_i = (s_{i1}, s_{i2})$ of informative private signals, drawn independently (conditional on z) from the set $S = Z$ of possibly optimal policy pairs. Intuitively, s_{i1} should be informative of z_1 and s_{i2} should be informative of z_2. To ensure further that posterior beliefs are monotonic and symmetric in s_i, even after conditioning on the event of a pivotal vote, let the conditional density $g(s|z)$ of private signals satisfy the more restrictive assumption of *linear informativeness* (Condition 4), meaning that $g(s|z)$ slopes linearly upward in the direction of z. This implies that $g(s|z)$ also satisfies *rotational symmetry* and *error symmetry* (Conditions 5 and 6): rotating z rotates the entire distribution of signals by the same amount, and a signal s is equally likely to deviate from z in a clockwise or counter-clockwise direction.[17]

Condition 4 (Linear informativeness) *$g(s|z) = g_0 + g_1(s \cdot z)$ for some $g_0, g_1 > 0$.*

Condition 5 (Rotational symmetry) *$g(R_\theta s|R_\theta z) = g(s|z)$ for any θ.*

Condition 6 (Error symmetry) *$g(M_{\theta_z} s|z) = g(s|z)$ and $g(s|M_{\theta_s} z) = g(s|z)$.*

Condition 2 implies that $E(z) = 0$, so the linear informativeness of g implies that s_i is uniform, with density $g(s) = g_0$. The posterior expectation of z is then linear in s.

$$\begin{aligned} E(z_1|s) &= \int_Z z_1 \frac{g_0 + g_1(s_1 z_1 + s_2 z_2)}{g_0} f(z)\, dz \\ &= \frac{g_1}{g_0} V(z_1)(s_1 + \rho s_2) \\ E(z_2|s) &= \frac{g_1}{g_0} V(z_2)(\rho s_1 + s_2) \end{aligned}$$

The distribution of signals is continuous, so despite their common objective, voters develop a myriad of different opinions about which policy combination is optimal. Naturally, $E(z_1|s)$ increases in s_1 and $E(z_2|s)$ increases in s_2; if ρ is positive then $E(z_1|s)$ also increases in s_2 and $E(z_2|s)$ also increases in s_1.[18]

Given their career choice in policy making and their privileged access to policy advice, it is reasonable that candidates should observe higher quality signals of z than the typical voter. If voters and candidates all receive private information, however, then each of the

[17] Linearity is much stronger than the combination of Conditions 5 and 6 and, intuitively, seems much stronger than necessary for monotonic posteriors (which are important for guaranteeing monotonic best response incentives). Unfortunately, a tractable condition that is weaker but still sufficient has proven elusive. An alternative approach is to assume that voters are naive, and fail to condition on the event of a pivotal vote. In that case, results similar to those below can be obtained under the much weaker assumption that s_i and z are affiliated (see Section 5).

[18] The assumption that signals on one issue are informative of another issue is consistent with evidence from Brunner, Ross, and Washington (2011) that economic conditions have a causal impact on both economic and non-economic vote choices.

$N+2$ members of society should try to infer the private information of the $N+1$ others, from their equilibrium behavior. Since others make similar inferences, characterizing equilibrium requires analyzing beliefs about others' beliefs about others' beliefs, and so on, which seem hopelessly intractable. To avoid these complexities, candidate signals are not modeled here, and voter behavior constitutes candidates' *only* source of information. However, voting behavior turns out to be surprisingly informative: when the number of voter signals is already large, even very precise candidate signals would be superfluous, as Section 5 explains.

Even without candidate signals, a large number of voter signals makes tractability an important challenge. However, the direction of complicated strategic forces can be isolated by keeping other model features as symmetric as possible. In addition to the symmetry assumptions above, the analysis below restricts attention to equilibria in which voter strategies exhibit natural symmetry. Together, these induce candidates to adopt symmetric platforms in response.

Even focusing on symmetric strategies, of course, establishing equilibrium requires ruling out asymmetric deviations. The natural timing for the game is for voters to vote after candidate platforms are announced, but asymmetric candidate platforms then prompts asymmetric voting, which unfortunately makes the analysis intractable. Instead, therefore, voters and candidates are assumed to move simultaneously, so deviations from symmetric platforms can be evaluated while preserving voter symmetry. This actually doesn't matter to voters, who in equilibrium best-respond to candidate platforms either way. When the number of voters is large, it doesn't matter for candidates, either, as Section 5 explains below.

With simultaneous voting, the appropriate solution concept is Bayesian Nash equilibria (BNE). Given Poisson population uncertainty, such equilibria are necessarily symmetric, in that voters with identical signals vote identically (Myerson, 1998). Such equilibria can be characterized by platforms x_A and x_B together with a single voting strategy, which is a measurable function $v : S \to \{A, B\}$ (from the set V of such functions) that specifies a vote choice for every signal vector $s \in S$ and. Given the simultaneous timing of the game, v does not vary with (x_A, x_B).

4 Equilibrium Analysis

4.1 Voters

The analysis of voter responses to $x_A, x_B \in X$ closely parallels the one-dimensional treatment of McMurray (2017a). A voter prefers the platform closest to z, so the hyperplane (or line, for $D = 2$) midway between x_A and x_B partitions Z into sets Z_A and Z_B of states where x_A and x_B are superior, respectively. With quadratic utility, x_A or x_B is superior when the expectation of z lies in Z_A or Z_B, respectively. This expectation of course depends on a voter's private signal; since a vote only influences a voter's own utility when it is *pivotal* (event P), meaning that it makes or breakes a tie, a voter's best response conditions on this event, as well, as Lemma 1 now states.

Lemma 1 *The voting strategy v^{br} is a best response to $(v, x_A, x_B) \in V \times X^2$ if and only if $v^{br}(s) \in \arg\min_{j\in\{A,B\}} \|x_j - E(z|P,s)\|$ for all $s \in S$.*

Since signals are informative of the truth, $E(z|s)$ is naturally monotonic in s. By itself, this does not guarantee that $E(z|P,s)$ is monotonic, given the intricate relationship between z and P, but Condition 4 ensures that $E(z|P,s)$ indeed is monotonic. In fact, it is linear in s, implying that the best response to any voting strategy is also *linear*, as defined in Definition 1. That is, the hyperplane in Z translates into a hyperplane (or line, for $D = 2$) partitioning S into sets S_A and S_B, such that a voter with $s \in S_j$ prefers to vote for candidate j. Since the best response to a linear strategy is a linear strategy, and continuity is straightforward to verify, a standard fixed point argument over the compact set of hyperplanes in S guarantees equilibrium existence.

Definition 1 *$v_{h,c} \in V$ is linear if h is a unit vector with polar angle $\theta_h \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $v(s) = \begin{cases} A & \text{if } h \cdot s < c \\ B & \text{if } h \cdot s > c \end{cases}$.*[19] *$v_{h,0} \equiv v_h$ is a half-space strategy. A BNE (v_h^*, x_A^*, x_B^*) is a half-space equilibrium if v_h^* is a half-space strategy.*

If the hyperplane defining a linear strategy passes through the origin then S_A and S_B are symmetric halves of S. Definition 1 defines this as a *half-space strategy*, without loss of generality placing S_A on the left of the origin and S_B on the right. An equilibrium with such a voting strategy is a *half-space equilibrium*, and the analysis below restricts attention to this case. This preserves tractability because voting is monotonic and symmetric, and so produces electoral outcomes and best response voting that are monotonic and symmetric, as Lemma 2 now states. Below, it is often convenient to use the normal vector h and its orthogonal rotation $h' = R_{\frac{\pi}{2}} h$ as basis vectors for the policy space. Events A and B are shorthand for $w = A$ and $w = B$, respectively.

Lemma 2 *If $v_h \in V$ is a half-space strategy then, for any $z \in Z$,*

1. *(Outcome monotonicity) $\nabla_z \Pr(B|z) \cdot h > 0$ and $\nabla_z \Pr(B|z) \cdot h' = 0$.*
2. *(Pivot monotonicity) $\nabla_z \Pr(P|z) \cdot h$ and $z \cdot h$ have opposite signs and $\nabla_z \Pr(P|z) \cdot h' = 0$.*
3. *(Outcome symmetry) $\Pr(A|-z) = \Pr(B|z)$ and $\Pr(A) = \Pr(B) = \frac{1}{2}$.*
4. *(Pivot symmetry) $\Pr(P|z) = \Pr(P|-z)$.*
5. *(Half-space response) The unique best response to $(v_h, -x, x)$ is a half-space strategy $v_{h^{br}}$.*

Part 1 of Lemma 2 states that, when voters follow a half-space strategy, electoral outcomes are monotonic: as z moves in the direction of h, signals in that vicinity become more likely, so more citizens vote B and victory becomes more likely. Part 2 states that pivot probabilities decline as z moves in the direction of h. Orthogonal moves do not change the probability of winning or of a pivotal vote. Parts 3 and 4 state that opposite realizations of z generate opposite candidate fortunes but identical pivot probabilities. Part 5 states that, when candidates adopt symmetric platforms (as they will in equilibrium, given the outcome symmetry of Part 3), best-response voting follows another half-space strategy.

[19] The behavior of voters for whom $h \cdot s = c$ exactly is inconsequential, occurring with zero probability.

4.2 Candidates

Like voters, a candidate prefers to implement her expectation of the optimal policy. With no private signals, both candidates' basic inclination would be to adopt policy platforms at 0. Just as a vote only matters if it is pivotal, however, a candidate's platform only matters if she wins office (event j, shorthand for $w = j$). Her best response platform therefore conditions on this event, as Lemma 3 now states.[20] Note that this depends on the strategy that a candidate expects voters to follow, but not on the platform choice of her opponent.

Lemma 3 *For any voting strategy $v \in V$, the unique best response for candidate j is given by $x_j^{br} = E(z|j)$.*

When voters follow a half-space strategy, candidate A tends to win the election in certain states of the world while B wins in opposite states. From the event of winning, therefore, the two candidates infer opposite information and therefore adopt opposite platforms, as Lemma 4 now states, which will each be optimal in the states of the world where they respectively win.

Lemma 4 *If $v_h \in V$ is a half-space strategy then $x_A^{br} = -x_B^{br} \neq 0$.*

In stating that candidates are symmetric, Lemma 3 says nothing about the extent of polarization. As I show in McMurray (2018), however, polarization can be substantial when n is large. Candidate A almost surely wins when $z \in Z_A$, while B almost surely wins when $z \in Z_B$. Anticipating this, candidates adopt positions close to $E(z|z \in Z_A)$ and $E(z|z \in Z_B)$, which are center points of opposite halves of the policy space.

4.3 Equilibrium

If z_1 and z_2 are uncorrelated then $f(z)$ exhibits radial symmetry, as Section 3 notes, meaning that the optimal policy pair is equally likely to lie in any direction from the origin. The consequence of this, as Proposition 1 now states, is that *any* half-space strategy v_h, together with candidates' best response policies, constitutes an equilibrium. In such an equilibrium, candidates simply take policy positions in the directions of $-h$ and h, symmetric around the origin. Voters take the event of a pivotal vote into account, but behave just as they would if they did not: those with s_i closer to x_A vote A and those with s_i closer to x_B vote B.

Proposition 1 *Let $\rho = 0$. For any unit vector h there exists a unique half-space equilibrium (v_h^*, x_A^*, x_B^*), with $x_A^* = -x_B^* \neq 0$.*

[20] The proof of Lemma 3 is a straightforward generalization of Theorem 2 in McMurray (2018), and so is not presented here. Note that this pivotal logic does not require the quadratic specification of utility. With linear utility loss, for example, a candidate would prefer the median realization of z instead of the mean, but her posterior $f(z|w = j)$ would still condition on the event of winning the election.

The logic underlying Proposition 1 is straightforward: when voters follow v_h, electoral victory will be most likely when the optimal policy lies in the general directions of $-h$ and h, and perfect symmetry ensures that candidates form expectations precisely in these directions. A vote is most likely to be pivotal when z is roughly equidistant from $-h$ and h, and therefore roughly equidistant from x_A and x_B. This conveys nothing about which of the two platforms is superior, so a voter votes sincerely, as he would have done if he had not conditioned on the event of a pivotal vote.

Proposition 1 shows how a multidimensional environment reduces to a single dimension in equilibrium, with x_A and x_B endogenously defining "left" and "right" positions on the line between them, and voters dividing according to the projections of their opinions onto this line. So far, this has nothing to do with the specific structure of information, or even with common interests; it follows simply from having two candidates: *any* two positions in a multidimensional space define a line, and if each voter supports the candidate closest to himself (whether to his private interest or to his private estimate of the common interest) then voters will split into two groups, in the direction of that line.[21] However, showing how a *single* election reduces to one dimension does nothing to resolve the puzzle above, which is that issues are bundled together consistently *across* elections, so that different elections reduce to the *same* line. A unique equilibrium would have resolved the puzzle, by identifying a single orientation that must prevail in every election, but Proposition 1 states that equilibrium can be oriented in *any* direction. With perfect symmetry, then, indeterminacy is severe, and there is no apparent reason why different electorates could not bundle issues differently from one another.

Proposition 1 identifies infinitely many equilibria, but qualitatively, two policy dimensions create only two ways to bundle the issues: any $\theta_h \in \left[0, \frac{\pi}{2}\right]$ produces a *major* equilibrium, meaning that one candidate is more liberal on both issues while the other is more conservative on both issues; any $\theta_h \in \left(-\frac{\pi}{2}, 0\right)$ produces a *minor* equilibrium, meaning that each candidate is more liberal on one issue and more conservative on the other. Within these categories, different θ_h merely correspond to different levels of polarization: for $|\theta_h| < \frac{\pi}{4}$, candidates polarize more on issue 1 than issue 2; for $|\theta_h| > \frac{\pi}{4}$, they polarize more on issue 2 than issue 1. In stating that any θ_h can sustain a half-space equilibrium, then, Proposition 1 implies *both* that either bundling of issues is possible in equilibrium, and that either issue can be more polarizing, with a continuum of possible polarization levels.

4.4 Correlation

When $\rho = 0$, the distinction between major and minor equilibria is immaterial. When $\rho > 0$, however, major equilibria bundle issues in the direction of correlation while minor equilibria bundles issues oppositely. In that case, the number of equilibria falls precipitously, as Proposition 2 now states: there is a unique major equilibrium oriented exactly in the direction of the major diagonal, and a unique minor equilibrium oriented exactly in the direction of the minor diagonal.

[21] With more than two candidates, voters should ignore all but the two front runners (Duverger, 1954), which would split the electorate in a similar way.

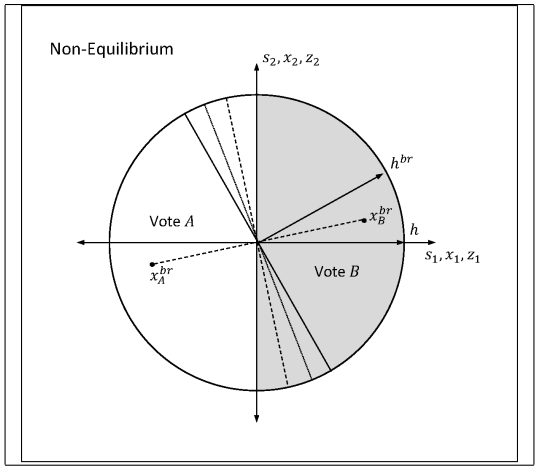

Figure 2: Non-equilibrium half-space strategy.

Proposition 2 *If $\rho > 0$ then there exists one major half-space equilibrium $E^+ = \left(v_{h^+}, x_A^+, x_B^+\right)$ with $\theta_{h^+} = \theta_{x_B^+} = \frac{\pi}{4}$ and one minor half-space equilibrium $E^- = \left(v_{h^-}, x_A^-, x_B^-\right)$ with $\theta_{h^+} = \theta_{x_B^+} = -\frac{\pi}{4}$. No other half-space equilibrium exists.*

To give some intuition for why equilibrium half-space strategies can only be oriented in directions h^+ and h^-, Figure 2 illustrates the case of a half-space strategy with polar angle $\theta_h = 0$. When following this strategy, voters ignore s_2 completely: those with negative s_1 (unshaded region) vote A while those with positive s_1 (shaded region) vote B. Candidate B then infers that, if she wins the election, it will likely be because z_1 is positive. For $\rho = 0$, she would learn nothing about z_2, and would adopt a policy position exactly on the horizontal axis, but for $\rho > 0$ a positive z_1 suggests that z_2 is likely positive as well, so if she wins, candidate B takes positive positions on both issues—that is, adopting a platform with polar angle $\theta_{x_B^{br}} > 0$. Candidate A behaves symmetrically.

If candidates respond to v_h with platforms x_A^{br} and x_B^{br}, as illustrated in Figure 2, then the dashed line between them separates Z_A from Z_B: voters with expectations $E\left(z|P,s\right)$ in the southwest and northeast regions prefer to vote A and B, respectively. Rotated counter-clockwise from the dashed line in Figure 2 is a dotted line. Voters whose expectations $E\left(z|s\right)$ lie southwest of the dotted line (call this region $\tilde{Z}_A$) on the basis of private information alone form updated expectations $E\left(z|P,s\right)$ in Z_A and prefer to vote A; those with expectations northeast of the dotted line (region $\tilde{Z}_B$) update to expectations in Z_B, and prefer to vote B. These lines differ because, when a voter's peers vote on the basis of s_1 alone, they are most likely to tie (making his own vote pivotal) when z_1 is close to zero. Thus, for any s, $E\left(z|P,s\right)$ lies closer to the vertical axis than $E\left(z|s\right)$ does. In particular, if $E\left(z|s\right)$ lies exactly on the dotted line then $E\left(z|P,s\right)$ lies exactly on the dashed line, leaving a voter indifferent between x_A and x_B.[22]

[22] A voter whose expectation $E\left(z|s\right)$ is northeast of the dotted line but southwest of the dashed line has a slightly negative signal of z_1 but a strongly positive signal of z_2. Since the two candidates are polarized largely only in the horizontal dimension, his basic inclination would be to vote for candidate A. If his vote is pivotal, however, it is likely that $z_1 \approx 0$. After conditinoning on event P, therefore, he puts relatively

Corresponding to the dotted and dashed lines in Z is a solid line, also depicted in Figure 2, that partitions the space of signals into S_A and S_B. Voters with signal realizations southwest of this line form expectations $E(z|s)$ and $E(z|P,s)$ southwest of the dotted and dashed lines, respectively, and so prefer to vote A; voters with signals northeast of the line instead prefer to vote B. In other words, if his peers follow v_h and candidates adopt best-response platforms x_A^{br} and x_B^{br}, then a voter's best response is the half-space strategy oriented in the direction of h^{br}, where $\theta_{h^{br}} > \theta_{x_B^{br}} > \theta_h$. Since v_h is not the best response to itself (and to the platforms that are candidates' best responses to v_h), it cannot be sustained in equilibrium.

The logic above is easiest to see for the case of $\theta_h = 0$, but holds more generally. As long as z_1 and z_2 are correlated, information that voters communicate about either dimension informs candidates about both dimensions. Half-space strategies with polar angles $\theta_h \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ communicate more about issue 1 than about issue 2; those with θ_h below $-\frac{\pi}{4}$ or above $\frac{\pi}{4}$ communicate more about issue 2 than about issue 1. Either way, taking the correlation across issues into account lessens the distinction between issues, so candidates' beliefs and platforms are less disproportionate, and closer to the main diagonal than h is. h^{br} must be closer still to the main diagonal, so that the voting behaviors assigned to each signal match candidates' positions even after pivotal considerations push voters' beliefs back in the direction of the original voting strategy.

If $\theta_h = \pm\frac{\pi}{4}$ then candidates infer equal information about the two issues, even after taking ρ into account, and adopt policy platforms exactly on the diagonal. Since this aligns perfectly with v_h, pivotality tells a voter nothing about which candidate is superior. h^{br} then coincides with h, thus constituting an equilibrium. Z_j and $\tilde{Z}_j$ coincide, meaning that the voters who favor A or B after the pivotal voting calculus are the same ones who did so before taking P into account.

4.5 Polarization and Welfare

In the major equilibrium, votes for candidate B tend to reflect positive realizations of s_1, suggesting that z_1 is likely positive. They also tend to reflect positive s_2, suggesting that z_2 is likely positive, as well. Given the positive correlation between issues, these conclusions reinforce each other, so $E(z_1|B)$ and $E(z_2|B)$ are both more extreme than they would be (for the same voting strategy) if the truth variables were uncorrelated. By contrast, in the minor equilibrium, votes for candidate B tend to reflect positive realizations of s_1 but negative realizations of s_2, suggesting that $z_1 > 0$ and $z_2 < 0$. Since the issues are positively correlated, however, these inferences undermine each other, so $E(z_1|B)$ and $E(z_2|B)$ are less extreme than they would be if the truth variables were uncorrelated. Proposition 3 states this formally, and notes further that the degree of polarization increases in ρ.

Proposition 3 *$\left\|x_j^+\right\|$ and $\left\|x_j^-\right\|$ increase and decrease with ρ, respectively, and are equal if and only if $\rho = 0$.*

higher weight on his signal of z_2 than before, and votes for candidate B instead.

Private interest models highlight the utilitarian value of moderate policies, which compromise between the competing interests at either extreme to minimize the total disutility that voters suffer from policies far from their ideal points. From that perspective, Proposition 3 might seem to indicate that the minor equilibrium promotes greater social welfare. In common interest settings, however, a voter benefits from a policy that is close to the true optimum, not from a policy that is close to his current opinion, so centrist policies need not hold the same utilitarian appeal.

In any case, the result that there are two equilibria with differing levels of polarization raises the question of what is best for society. Defining social welfare $W(v, x_A, x_B)$ is uncontroversial here, unlike many settings, because voters and candidates share the same objective function, which can be written as follows.

$$W(v, x_A, x_B) = E_{w,z}[u(x_w, z)] = \int_Z \left[\sum_{j=A,B} u(x_j, z) \Pr(j|z) \right] f(z)\, dz \tag{2}$$

Proposition 4 now states that, in fact, (2) is higher in the major equilibrium, even though policy outcomes are more extreme. Like polarization, the welfare difference between equilibria increases in ρ.

Proposition 4 *$W(E^+;\rho)$ and $W(E^-;\rho)$ increase and decrease with ρ, respectively, and are equal if and only if $\rho = 0$.*

A simple intuition for the result that the major equilibrium is superior to the minor equilibrium is that v_{h^+} and v_{h^-} specify the same voter behavior, but in different states of the world. When z happens to be in quadrant 1 or 3, v_{h^+} does well at identifying the right quadrant but v_{h^-} does not; when z is in quadrant 2 or 4, v_{h^-} does well at identifying the right quadrant but v_{h^+} does not. Since quadrants 1 and 3 occur more frequently, v_{h^+} is the more informative voting strategy. In fact, it seems reasonable to conjecture that no other combination of voter and candidate behavior generates higher welfare than E^+.[23]

The existence of an inferior equilibrium implies that an inferior bundling of political issues could be self-perpetuating: even if it were known that issues had somehow come to be bundled together inefficiently, voters and candidates would go along with the inefficient bundling. If issues are only loosely correlated then little welfare is lost, but if ρ is large then the loss is more severe.

Proposition 2 reduces the number of equilibria from infinity to two, but since the two surviving equilibria entail opposite bundlings of the policy issues, it still gives no explanation as to why a major equilibrium should not prevail in one election while a minor equilibrium prevails in another. Proposition 4 is useful in that regard, in that a Pareto superior equilibrium survives the payoff dominance refinement of Harsanyi and Selten (1988). An even stronger reason to favor the major equilibrium is that the minor equilibrium is *unstable*, as

[23] In common interest games such as this, behavior that is socially optimal is also individually optimal, and therefore constitutes an equilibrium (McLennan, 1998), so no half-space strategy other than v_{h^+} can maximize welfare, by Propositions 2 and 4. This does not rule out equilibria with asymmetric voting, but asymmetry seems unlikely to improve welfare.

the proof of Proposition 2 makes clear: rotating the voting strategy slightly away from v_{h^-} leads candidates to adopt platforms further from the minor diagonal, and voters respond by rotating further still, with a chain of best responses converging to the major diagonal. In contrast, rotating slightly away from v_{h^+} generates best response platforms that rotate back again, prompting $v_{h^{br}}$ even closer to the major diagonal. Both for its efficiency and its stability, then, the major equilibrium emerges as the unique behavioral prediction of the model. Importantly, ρ need not be large: *any* positive correlation, no matter how small, orients the equilibrium so that issues are bundled just as they would be if ρ were equal to one.

5 Extensions

5.1 Higher Dimensions

The model above takes the crucial first step of accommodating more than one dimension, but the eventual goal is to model a large number D of political issues. A thorough treatment of higher dimensions is beyond the scope of this paper, but this section discusses how, as long as symmetry is preserved, the two-dimensional analysis extends in a natural way to arbitrary D. To see this, let X be a D-dimensional unit hyperball with optimal issue positions $z_1, z_2, ..., z_D$ and suppose that the pairwise correlations between any two of these variables are the same, and proportional to $\rho \geq 0$. Assume further that $-z$ or permutations of z leave $f(z)$ unchanged (analogous to Condition 2) and that $\frac{\partial f(z)}{\partial z_d}$, $\frac{\partial^2 f(z)}{\partial z_d \partial \rho}$, and $\frac{\partial^2 f(z)}{\partial z_d \partial z_{d'}}$ have the same signs as $\rho \prod_{d' \neq d} z_{d'}$, $\prod_{d' \neq d} z_{d'}$, and ρ, respectively (as in Condition 1). Then let $g(s|z)$ satisfy linear informativeness, as already formulated in Condition 4.[24]

Formally extending the results of Section 4 would require the cumbersome notation of hyperspherical coordinates, but it should be clear from the analysis above that all of the results above have multidimensional analogs, based on identical reasoning. As in Lemma 1, each voter still favors the policy platform closest to his expectation $E(z|P, s)$ of the optimal policy, conditional on the event of a pivotal vote. Half-space strategies can still be defined by a single normal vector, now defining the hyperplane that partitions S, and such strategies still imply the symmetry properties of Lemma 2. A candidate's optimal platform choice is still her expectation $E(z|j)$ of the optimal policy, conditional on winning, and with a half-space voting strategy, this still implies that candidates will adopt substantially polarized platforms, opposite one another.

When $\rho = 0$, correlative monotonicity still implies that $f(z)$ is uniform. Together with the rotational symmetry of $g(s|z)$ and the symmetry of half-space voting, this guarantees that candidates' expectations $E(z|A)$ and $E(z|B)$ lie in the exact directions of $-h$ and h, respectively. If they adopt these expectations as platforms, a pivotal vote conveys nothing to voters about the magnitude of z in the direction of x_A and x_B, so $E(z|P, s)$ and $E(z|s)$

[24]Examples of densities that satisfies these conditions are $\frac{1}{V_K}\left(1 + \rho \prod_{k=1}^{K} z_k\right)$, where V_K denotes the hypervolume of a K-dimensional unit hyperball (e.g. $f(z_1, z_2, z_3) = \frac{3}{4\pi}(1 + \rho z_1 z_2 z_3)$, in three dimensions), and $g(s|z) = \frac{1}{V_K}(1 + s \cdot z)$.

lie in the same direction, and voters with $s \cdot h < 0$ prefer to vote B, while those with $s \cdot h > 0$ prefer to vote A. In other words, v_h constitutes its own best response, for *any* normal vector h.

When $\rho > 0$, the number of equilibria reduces dramatically, as before. Voters cannot simply ignore all issues but the first, for example, for the same reason illustrated in Figure 2: if voting reflected s_1 alone, candidate B would infer upon winning that z_1 is likely positive, implying that z_2 through z_D are likely positive, as well, and would respond with positive positions on every issue. Relative to h, this reflects a rotation toward the major diagonal (i.e., where $z_d = z_{d'}$ for all d, d'). A voter's best response would then be a half-space strategy rotated even further toward the major diagonal.

Clearly, a major equilibrium still exists, with h^+, x_A^+, and x_B^+ all oriented along the major diagonal. In that equilibrium, a voter votes A if his average signal is negative and votes B if his average signal is positive. When candidate B wins the election, her updated expectations of z_d are then all (equally) positive. Given the positive correlation across issues, the inference that $z_d > 0$ reinforces the inference about the other issue dimensions, so that, as in Proposition 3, she adopts a more extreme position than she would have adopted if the issues had been uncorrelated. Candidate A takes an opposite position, and with the h^+, x_A^+, and x_B^+ exactly on the major diagonal, a pivotal vote conveys no information about which candidate is superior, thus sustaining the same voting strategy in response.

For three dimensions, Figure 3 illustrates the candidate platforms that best respond to the non-equilibrium half-space strategy described above, along with major and minor equilibrium platforms. In two dimensions there is only one minor equilibrium, but $D = 3$ produces four of one type of minor equilibrium and three of another. The first possibility is that candidates polarize in opposite directions on two issues but not at all on the third. Maintaining the assumption that $x_{A1} \leq x_{B1}$, there are four such equilibria, because candidates can converge on any issue, and converging on issue 1 gives two ways to polarize on issues 2 and 3. In the first minor equilibrium of Figure 3, for example, $x_{A3} = x_{B3} = 0$. In response, voters ignore s_3 completely, voting A if $s_1 < s_2$ and voting B if $s_1 > s_2$. Upon winning, candidate B then expects z_1 to be positive and z_2 to be negative. These inferences undermine one another, so she is less extreme than she would otherwise be. The inferences that $z_1 > 0$ and $z_2 < 0$ have opposite implications for z_3, which cancel out in equilibrium so that $E(z_3|j) = 0$.

In the second type of minor equilibrium, one candidate takes a negative position on two issues and a positive position on one, while the other does the opposite. There are three such equilibria, orienting any of the three issues opposite the other two. In the final example of Figure 3, for example, $x_{A1} < x_{B1}$ and $x_{A2} < x_{B2}$ but $x_{A3} > x_{B3}$. Upon winning, candidate B then infers that $z_1 > 0$ and $z_2 > 0$ but that $z_3 < 0$. The inference about z_1 and z_2 undermines the inference about z_3, so she takes a position on issue 3 that is less extreme than it would be if issues were uncorrelated. The inference that $z_1 > 0$ is similarly undermined by the inference that $z_3 < 0$, but is bolstered by the inference that $z_2 > 0$, so she adopts a more extreme position on issue 1 than on issue 3. Issues 1 and 2 are symmetric, so she takes equally extreme positions.

In higher dimensions, the number of minor equilibria grows quickly. In four dimensions, for example, there are sixteen minor equilibria: three with both candidates adopting leftist positions on two issues and rightist positions on two issues; four with one candidate taking

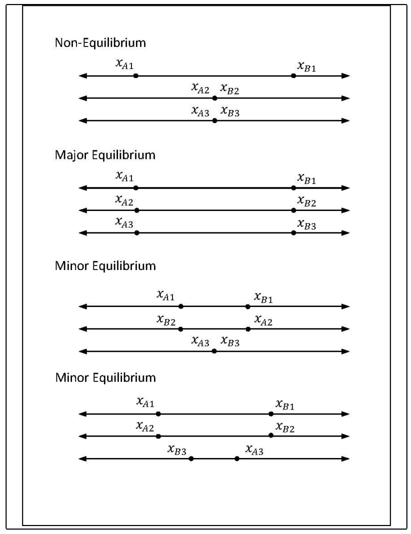

Figure 3: Equilibrium (and non-equilibrium) candidate positions in three dimensions with symmetric prior distribution.

leftist positions on three issues and a rightist position on the fourth issue or vice versa; and nine with each candidate taking a leftist position on one issue, a rightist position on one issue, and centrist positions on the remaining two issues.[25] For arbitrary D, there are enough minor equilibria for a candidate to take leftist or rightist positions on *any* strict subset of the issues. In other words, *any* bundling of issues could persist in equilibrium, with polarization depending on how many issues are bundled in each direction.

For large D, coordinating on any one minor equilibrium seems difficult. In contrast, there is only ever a single major equilibrium, so coordination is easy. This equilibrium also Pareto dominates the others, and so survives the payoff dominance refinement (Harsanyi and Selten, 1988). Moreover, the minor equilibria are all unstable. For example, perturbing the first minor equilibrium of Figure 3 so that candidates polarize slightly more on issue 1 than issue 2 would lead voters to place greater weight on s_1 than s_2, thus conveying more information about z_1 than about z_2, so that candidates polarize even more on issue 1 and even less on issue 2 in response. Information about z_3 would then no longer cancel out, so $E(z_3|A) < 0 < E(z_3|B)$ and therefore $x_A^{br} < 0 < x_B^{br}$. As candidates polarize more on issues 1 and 3 and less on issue 2, $v_{h^{br}}$ rotates further. As A and B votes increasingly convey information that z_1 and z_3 are positive, platforms on issue 2 become less and less polarized, until they are not polarized at all, and then polarize in the opposite direction, consistent with issues 1 and 3. As before, this chain of best responses converges to the major equilibrium.

[25] This accounting of equilibria retains the convention above that the candidate who is weakly to the right on issue 1 is labeled as candidate B. Dropping this convention, there are two minor equilibria in two dimensions, twelve in three dimensions, and twenty-six in four dimensions.

5.2 Asymmetric Issue Importance

To keep the pivotal voting calculus tractable, Section 3 assumes that X and Z are symmetric in every direction, and so is f when $\rho = 0$ (Condition 3); even when $\rho > 0$, f is symmetric along both diagonals (Condition 2); g exhibits both rotational symmetry and error symmetry (Conditions 5 and 6); utility u weights directions and issues symmetrically; half-space equilibria ensure symmetric platforms; and simultaneous timing preserves symmetric voting even when candidates deviate asymmetrically. Perfect symmetry is worrisome, because it is a knife-edge condition and because various asymmetries seem entirely plausible. If reducing the number of symmetric directions from infinity to two reduces the number of equilibria similarly, relaxing symmetry further might eliminate remaining equilibria.

This section makes progress on this question by relaxing symmetry in a way that preserves one form of symmetry, while still preserving the half-space structure of equilibrium voting that makes analysis tractable. Consider the following generalized utility function,

$$u(x, z) = -(1+\lambda)(x_1 - z_1)^2 - (1-\lambda)(x_2 - z_2)^2$$

where the model of Section 3 imposes $\lambda = 0$, but $\lambda \in (0, 1)$ allows the possibility that issue 1 is more important to voters than issue 2.[26] Dropping terms that don't depend on the policy outcome, expected utility then generalizes from (1) to the following.

$$E_z[u(x, z) | \Omega] = -(1+\lambda)[x_1 - E(z_1|\Omega)]^2 - (1-\lambda)[x_2 - E(z_2|\Omega)]^2 \tag{3}$$

Proposition 5 now characterizes equilibrium for $\rho = 0$ and $\lambda > 0$. Like correlation across issues, asymmetric issue importance eliminates all of the infinitely many equilibria identified in Proposition 1, except two. One of these focuses entirely on issue 1 ($\theta_{h^+} = 0$) and the other focuses entirely on issue 2 ($\theta_{h^-} = -\frac{\pi}{2}$). Clearly, the first of these provides higher welfare, since issue 1 is more important.

Proposition 5 *If $\rho = 0$ and $\lambda > 0$ then there exists one half-space equilibrium with $\theta_{h^*} = x_B^* = 0$ and another with $\theta_{h^*} = x_B^* = \frac{\pi}{2}$. No other half-space equilibrium exists.*

The logic of Proposition 5 is that candidate incentives do not depend on λ: each simply prefers to implement her expectations of z_1 and z_2, regardless of which is more important. If voters followed a half-space strategy with $\theta_h = \frac{\pi}{4}$, for example, candidates would still respond with platforms on the major diagonal. However, λ does affect voter responses. For example, a voter with s_i on the minor diagonal was previously indifferent between candidates on the major diagonal, as each was better on one issue. Now, since issue 1 is more important, such a voter responds with a half-space strategy oriented clockwise from h^+. In general, voters with the new utility function value policy vector x just as voters with the old utility function valued $\tilde{x} = \begin{pmatrix} 1+\lambda \\ 1-\lambda \end{pmatrix} \cdot x$. As in the case of $\rho > 0$, this drives a wedge between v_h and the best response to candidates' best responses to v_h, except when voters and candidates exactly align with one of the major axes, so that x_j and $\tilde{x}_j$ lie in the same direction.

[26] Or, isomorphically, that that deviations from the status quo are easier in one direction than the other, so that the effective policy space is an ellipse, not a circle.

When λ and ρ are both zero, the model above is symmetric in every direction, and there are half-space equilibria oriented in every direction. When either parameter is positive, the model is symmetric only in two directions, and only two half-space equilibria remain. If λ and ρ are both positive then the model is no longer symmetric in any direction, which raises the question of whether equilibrium exists at all. The answer, as Proposition 6 now states, is that a major equilibrium and a minor equilibrium remain. These are no longer oriented exactly along the major and minor diagonals, but are in the same vicinity.

Proposition 6 *If $\lambda > 0$ and $\rho > 0$ then there exists a major equilibrium $\left(v_{h^+}, x_A^+, x_B^+\right)$ with $\theta_{h^+} \in \left(0, \frac{\pi}{4}\right)$ and a minor equilibrium $\left(v_{h^-}, x_A^-, x_B^-\right)$ with $\theta_{h^-} \in \left(-\frac{\pi}{4}, -\frac{\pi}{2}\right)$.*

The proof of Proposition 6 notes that if a voter's peers follow a half-space strategy with $\theta_h = 0$ then, because $\rho > 0$, candidates respond with platforms rotated counter-clockwise, to $\theta_{x_B^{br}} > 0$, rotating the best voting voting strategy counter-clockwise from h. If $\theta_h = \frac{\pi}{4}$ then $\theta_{x_B^{br}} = \frac{\pi}{4}$, as well, but since $\lambda > 0$, a voter's best response is rotated clockwise from h. By continuity, there is a polar angle $\theta_{h^+} \in \left(0, \frac{\pi}{4}\right)$ that prompts the same polar angle in response. By similar arguments, half-space strategies with polar angles $\theta_h = -\frac{\pi}{4}$ and $\theta_h = -\frac{\pi}{2}$ generate best-response half space strategies with larger and smaller polar angles, so continuity implies the existence of an equilibrium oriented toward $\theta_{h^-} \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$.

Proposition 6 makes clear that Proposition 2 is robust even though Proposition 6 is not: in spite of asymmetry, one major equilibrium and one minor equilibrium still exist. It would be straightforward to adapt the proof of Proposition 4 to show that the major equilibrium Pareto dominates, both because it bundles issues more efficiently and because it focuses more squarely on the more important of the two issues. The minor equilibrium offers little differentiation on the more important issue, focusing more heavily on what is less important, in addition to bundling issues inefficiently. The major equilibrium is also stable, as a half-space strategy close to h^+ generates a best-response vector even closer to h^+, while the minor equilibrium is unstable, as a half-space strategy close to h^- generates a sequence of best responses that converges to h^+.

The equilibria of Proposition 6 are no longer oriented exactly along the major and minor diagonals. However, $x_{Ad} < x_{Bd}$ for both d still, so this represents the same bundling of issues as before. That they are off the diagonals simply means that polarization differs across issues. In other words, λ determines which of the two issues is more polarized, but ρ still determines how the issues are bundled in equilibrium.

5.3 General Asymmetry and Nonlinearity

Relaxing all of the symmetry and linearity of Section 3, equilibrium exists as long as X is non-empty and compact and S is measurable. To see this, note that in common interest environments, socially optimal behavior constitutes an equilibrium (McLennan, 1998). Fixing (x_A, x_B), the correspondence of welfare-maximizing voting strategies $v_n^{**}(x_A, x_B)$ is non-empty, convex, compact, and upper hemicontinuous in (x_A, x_B) for any n, by the maximum theorem.[27] Lemma 3 still gives candidates' unique best responses $x_{A,n}^{br}(v) = E(z|A; n, v)$

[27] Measurable S implies that Lebesgue integration, and therefore welfare, are continuous on the set V of measurable (mixed) voting strategies $v : S \to [0, 1]$, which is compact under the product topology (by

and $x^{br}_{B,n}(v) = E(z|B; n, v)$ to any voting strategy, so v^{**}_n can be reinterpreted as a correspondence from the compact set V of voting strategies into itself (i.e., the optimal response to candidates' best responses to any voting strategy). For any n, Kakutani's theorem guarantees a fixed point v^*_n, and $\left(v^*_n, x^{br}_{A,n}(v^*_n), x^{br}_{B,n}(v^*_n)\right)$ therefore constitutes an equilibrum.

Intuitively, it seems that voting in the optimal equilibrium—or, for that matter, best-response voting in *any* equilibrium—should be monotonic, meaning that voters with signals closest to x_j vote j (making j's victory most likely when z is in the same vicinity), even if the boundary in S between A voters and B voters is nonlinear or no longer passes through the origin. Unfortunately, the intricate relationship between z, P, s, and w makes these conjectures difficult to verify.

As long as utility is monotonic in $\|x - z\|$, A and B should optimally win in the sets Z_A and Z_B of states closer to x_A and to x_B. As long as Z_A and Z_B generate distinguishable distributions of signals, Barelli, Bhattacharya, and Siga (2018) show that a sequence of voting strategies $(v_n)_n$ exists that satisfies full information equivalence in the limit.[28] Since $v^*_n(x_A, x_B)$ produces (weakly) greater welfare than v_n, $(v^*_n(x_A, x_B))_n$ then satisfies FIE as well. Moreover, since $V \times X^2$ is compact, a subsequence of $\left(v^*_n, x^{br}_{A,n}(v^*_n), x^{br}_{B,n}(v^*_n)\right)_n$ converges to some $\left(v^*_\infty, x^*_{A,\infty}, x^*_{B,\infty}\right)$, and the continuity of $x^{br}_{j,n}(v)$ guarantees that $x^*_{j,\infty} = E\left(z|z \in Z_j\left(x^*_{A,\infty}, x^*_{B,\infty}\right)\right)$. In other words, candidates behave in large elections as if they were playing a simpler game, where they choose platforms and then the platform closer to z is implemented.

One general consequence of this is substantial polarization in large elections, as platforms correspond to expectations $E(z|Z_A)$ and $E(z|Z_B)$ over opposite sides of a partition. Another general consequence is (approximate) endogenous symmetry within the policy space, in that the unconditional mean $E(z)$ tends to be located centrally within the policy space, and is a weighted average of the two platforms.

Without assuming specific functional forms, it seems inherently difficult to characterize the role of ρ explicitly. However the rest of the model is specified, though, it seems intuitively that correlation across issues should operate as above. In particular, starting from any

Tychanoff's theorem) since X is compact. Convexity of $v^{**}(x_A, x_B)$ follows because welfare is monotonic (and therefore quasiconcave) in v.

[28] More precisely, there exists $(v_n)_n$ satisfying FIE, if and only if a hyperplane can partition the space $\Delta(S)$ of signal distributions such that $\{g(s|z) \in \Delta(S) : z \in Z_A\}$ and $\{g(s|z) \in \Delta(S) : z \in Z_B\}$ lie on opposite sides of the partition. Above, Conditions 2 and 4 guarantee that, interpreting the set of functions on S as a vector space, $H(s) = (x_B - x_A) \cdot s$ is normal to such a hyperplane. To see this, note that $g(s|z = \bar{x})$ must lie on the hyperplane of indifference, and when s is uniform, the difference

$$\begin{aligned}
\langle g(s|z) - g(s|\bar{x}), H(s)\rangle &= \int_S [(z - \bar{x}) \cdot s]\,[(x_B - x_A) \cdot s]\, ds \\
&= (z - \bar{x}) \cdot \begin{pmatrix} (x_{B1} - x_{A1}) \int_S s_1^2 ds + (x_{B1} - x_{A1}) \int_S s_1 s_2 ds \\ (x_{B1} - x_{A1}) \int_S s_1 s_2 ds + (x_{B1} - x_{A1}) \int_S s_2^2 ds \end{pmatrix} \\
&= (z - \bar{x}) \cdot (x_B - x_A) \int_S s_1^2 ds
\end{aligned}$$

is proportional to the utility difference $u(x_B, z) - u(x_A, z) = -(x_B - z) \cdot (x_B - z) + (x_A - z) \cdot (x_A - z) = 2(x_B - x_A) \cdot (z - \bar{x})$.

equilibrium where $\rho = 0$ and $x_A = E(z|Z_A)$ and $x_B = E(z|Z_B)$, increasing ρ amounts to raising the density of (z_1, z_2) pairs with the same sign and lowering the density of pairs with opposite signs, which will tend to rotate $E(z|Z_A)$ and $E(z|Z_B)$ in the direction of the major diagonal. In response, candidates should rotate in this direction, and Z_A and Z_B should rotate with them, so that $E(z|Z_A)$ and $E(z|Z_B)$ rotate even further. Without perfect symmetry, of course, equilibrium platforms will not lie exactly on the major diagonal (see Section 5.2), but should generally lie in the same quadrant (or orthant, in higher dimensions), thus generating the same bundling of issues. Formally, the maximum theorem implies that $\left(v_n^*, x_{A,n}^{br}(v_n^*), x_{B,n}^{br}(v_n^*)\right)$ is upper hemicontinuous in u, X, Z, S, f, and g, implying that slight deviations from the case of perfect symmetry produce equilibria that deviate only slightly from the equilibrium characterized above.

Starting from the case of $\rho = 0$, platforms can rotate toward the major diagonal both in clockwise or counter-clockwise directions. In that light, the logic above also suggests the possibily general existence of a minor equilibrium, delicately balanced between rotating in either direction. As before, such an equilibrium should be unstable, in the sense that rotating slightly away rotates best responses even further, in contrast with the major equilibrium, where rotations in either direction should trigger best responses that rotate back again.

To illustrate this numerically, consider $X = [-1,1]^2$ and the density $f(z_1, z_2) = \frac{1}{8}(1 + \rho z_1 z_2) + \frac{1}{8}(1 - z_1)$, which is not symmetric in any direction when $\rho > 0$. For $\rho = 0$, there are four policy pairs satisfying $x_j = E(z|j)$. These are oriented vertically $\left(\binom{-.17}{-.50}, \binom{-.17}{.50}\right)$, horizontally $\left(\binom{-.58}{0}, \binom{.39}{0}\right)$, southwest/northeast $\left(\binom{-.50}{-.27}, \binom{.21}{.31}\right)$, and southeast/northwest $\left(\binom{-.50}{.27}, \binom{.21}{-.31}\right)$. Increasing ρ (which is proportional to the correlation coefficient) slightly to .1, these rotate so that three $\left(\binom{-.19}{-.50}, \binom{-.15}{.50}\right)$, $\left(\binom{-.57}{-.03}, \binom{.39}{.04}\right)$, and $\left(\binom{-.51}{-.27}, \binom{.22}{.31}\right)$ are southwest/northeast but one $\left(\binom{-.50}{.28}, \binom{.20}{-.30}\right)$ is still southeast/northwest. The latter seems to be the most fragile, and produces the lowest welfare.

5.4 Timing, Office Motivation, and Candidate Information

Section 3 assumes that candidates and voters move simultaneously but, in reality, candidates announce policy positions before voting takes place. Timing does not matter to voters, who best respond to candidates' platforms either way, but knowing that her policy choice will change how voters vote does matter to a candidate. The analysis of the previous section is useful in that regard: in large elections (as long as FIE is satisfied), equilibrium behavior is equivalent to the simpler game in which candidates move first and then voters, observing candidates' positions, determine which platform is superior.[29] In that sense, equilibrium behavior in simultaneous and sequential games are asymptotically equivalent.

In Section 3, candidates care only about the policy outcome. In reality, candidates might also value winning office. With simultaneous timing, candidates have no way to influence voters, but since timing is immaterial in large elections, insights from one-dimensional sequential game in McMurray (2018) should apply. As that paper shows, polarization is lower when candidates are office motivated, but can still be substantial, because truth motivated

[29] Equilibrium exists in a sequential game, for reasons similar to those above.

voters only sometimes reward moderation.[30] Formally extending to multiple dimensions would be difficult for the reasons described above, but electoral concerns should give a candidate no reason to rotate her policy position, at least in a symmetric environment like the above, as doing so would attract votes in some states of the world but sacrifice votes in opposite states, which are equally likely.

As noted above, another unrealistic assumption of Section 3 is that candidates have no private information about z. With sequential timing, however, this is actually immaterial, as I explain in McMurray (2018). As long as her signal is informative of the truth, a candidate's platform must be monotonic in her signal; with sequential timing, this means that voters can infer candidates' private signals, effectively updating the common prior. Voters then adjust their behavior accordingy, and in equilibrium elect the candidate whose platform is superior *after* taking this information into account. A candidate's optimal platform is still her expectation $E(z|j)$ conditional on winning, but the event $w = j$ now incorporates both her own information and her opponent's.[31] In the limit as n grows large, this still converges to $\mathbf{1}_{u(x_j,z)>u(x_{-j},z)}$, so $E(z|j)$ still converges to $E(z|Z_j)$, as before.[32]

6 Applications

As Sections 1 and 2 explain, private interest literature cannot adequately explain why political attitudes are so unidimensional. As Shor (2014) expresses, for example, "it is not clear why environmentalism necessarily hangs together with a desire for more union prerogatives, but it does." In a common interest setting, such correlation arises naturally from the logical connections between issues: for example, environmentalism and union support might both reflect a view of businesses as ruthless, willing to pursue profit at the expense of employees or the environment. In fact, such a view could also engender support for minimum wage laws and a host of other pro-labor policies. Such logical connections may seem too weak to justify such consistent issue bundling, but *any* non-zero ρ is sufficient to orient the equilibrium, so that issues are bundled just as they would be if correlation were perfect. As issue importance fluctuates, canddiates polarize most highly on the prominent issue of the day, but the underlying bundling of issues remains largely the same.[33]

Converse (1964) and Shor (2014) find that political candidates are more ideologically consistent than voters. This, too, is consistent with the analysis above. With two issues,

[30] Moderating her platform always attracts votes, but this only matters to a candidate for moderate realizations of z: when z is extreme in her (or her opponent's) favor, a candidate will win (or lose) whether she moderates or not.

[31] Modeling this explicitly would involve higher order beliefs that seem hopelessly intractable. For example, the informational content of voting behavior now mixes original private information, responses to a candidate's own information (which she finds redundant), responses to her opponent's original information (which she also hopes to infer), responses to her opponent's guess of her own information, and so on.

[32] Similar reasoning would apply with subjective prior beliefs about z.

[33] Policy realignments do seem to take place occasionally. One possibility is that this reflects learning about ρ. With new insights about the relationship between issues, for example, a correlation that had long been presumed positive might prove to be negative. In that case, what had seemed to be a stable, major equilibrium would suddenly be revealed as a minor equilibrium, giving way to a (rather sudden) rebundling of issues.

for example, voters hold opinions in every quadrant, but candidates only ever take positions in quadrants 1 and 3. For large D, candidates adopt consistent positions on every issue, but voters form opinions in every orthant, and the fraction of voters who favor the party line on every issue tends to zero.

The results above also shed light on modern political arguments. The U.S. Libertarian party, for example, is liberal on social issues such as immigration, abortion and marriage, but conservative on economic issues such as taxes and regulation. Its website emphasizes logical consistency, arguing that while Democrats and Republicans each favor personal or economic liberty, Libertarians favor both.[34] The model above formalizes this as a claim that the optimal social and economic policies should be correlated, implying that society is stuck in an inferior equilibrium. The analysis above affirms this as a possibility, although it also concludes that such an equilibrium is unlikely to prevail. One possibility is simply that correlation goes the other way, and the present bundling of issues is efficient: like their names suggest, for example, conservative policies might be logically unified by a commitment to preserve social and economic traditions, while liberal or progressive policies seek to modernize on both fronts. On the other hand, enriching the present model might vindicate the Libertarian narrative: like z, for example, ρ may be imperfectly observed, with different voters seeing it as positive or negative. The point here is not to settle any philosophical debate, but to show that the model clarifies current public debate.

[35]

7 Conclusion

Multidimensional election models are plagued by convergence or equilibrium non-existence or multiplicity. Empirical unidimensionality makes this problem seem less urgent, but remains its own mystery. This paper has pointed out that, in a common interest setting, logical connections across issues are a natural source of correlation, and breaks symmetry to reduce the potential of multiple equilibria. The possibility remains of an inferior bundling of issues, mirroring prevalent public concerns, but the only stable equilibrium efficiently bundles related issues together. Decisions that are inherently multidimensional thus endogenously reduce to a single, “left-right” axis, and a fixed logical structure explains why the direction of disagreement remains so consistent over space and time.

That a common interest paradigm sheds light where private interest literature has not highlights the utility of this general approach to elections. Literally identical preferences are improbable, however, so generalizing this is an important direction for future work.[36] As long as preferences share a substantial common element, the results above seem likely to be

[34] See www.lp.org/platform.

[35] Policy realignments do seem to take place occasionally. One possibility is that this reflects learning about ρ. With new insights about the relationship between issues, for example, a correlation that had long been presumed positive might prove to be negative. In that case, what had seemed to be a stable, major equilibrium would suddenly be revealed as a minor equilibrium, giving way to a (rather sudden) rebundling of issues.

[36] Even if private interests are important empirically, a pure common interest model is useful as a theoretical benchmark against which imperfectly aligned interests can be compared.

robust. If a voter's ideal policy were some weighted average of the policy $\hat{x}_i$ that maximizes his narrow self-interest and the social optimum z, for instance, he should still formulate an expectation $E(z|P, s_i)$ of z based on his private signal, together with whatever he can infer from the event of a pivotal vote. As in the model above, correlation between z_1 and z_2 should lead voters to develop correlated expectations. As long as private interests are not too dominant, voting should then reveal information about voters' private signals, which a truth motivated candidate can use in much the same way as before.[37] Candidates who mix private and public interest should still condition their beliefs about z on the event of winning, and if voter strategies reveal information in a particular direction then $E(z|j)$ should tend to rotate from there, toward the major diagonal, as before. Sufficiently selfish candidates (with particular policy preferences) might divide the electorate southeast/northwest, but those who sufficiently value the public good should divide it southwest/northeast.[38]

Within the pure common interest paradigm, future work should enrich the information structure above. In the current framework, for example, pairs of voters could reach a consensus, simply by sharing and combining their private signals. In large groups, public opinion should be so reliable that voters abandon minority opinions. Empirically, individuals routinely disagree with one another, and maintain unpopular opinions. In McMurray (2018) I discuss informational limitations that might explain these features, and the same discussion applies here.[39] In higher dimensions, it would be useful to explore correlation structures that lack the symmetry assumed in Section 5.1. With three dimensions, for example, it is possible for z_1 and z_2 to correlated positively with each other, but negatively with z_3. A dynamic model of how opinions update between elections would be useful, as well: Krasa and Polborn (2014) present evidence, for example, that the correlation of voter attitudes across political issues has increased over time.

Condorcet's (1785) jury theorem ensures that voters elect the candidate with the better platform. In one dimension, this means that the policy outcome x is generally in the same direction as z. In higher dimensions, x is only guaranteed to be in the right half-space; as the number of dimensions grows large, the probability of being in the same orthant as z shrinks to zero, and the expected number of issues on which x and z differ grows without bound. This is an inherent limitation of a binary decision in a highly complex policy environment. Note that additional candidates cannot easily solve this problem, as they reduce the best candidate's ability to win a plurality of votes. If two candidates close to z split voters' support, for example, someone far inferior may win. Precisely to avoid such situations, strategic voters might also ignore all but two candidates, as Duverger's (1954) law predicts.

In the one-dimensional model of McMurray (2017b), the winning candidate infers a policy

[37] If voting is only partly informative then candidates will polarize less, for a given n, but may be just as polarized in the limit, as they approach perfect information.

[38] Even candidates who are inherently very selfish may put substantial weight on the public interest, to secure a favorable legacy.

[39] Conflicts of interest are a more obvious barrier to consensus, but similar disagreements persist on purely speculative questions, where interests should be irrelevant. Another relevant observation is that, in learning from other voters' opinions, an individual must discount any information that is already shared. Identifying which pieces of information another voter shares would require extensive communication, and disagreements could persist in the meantime.

mandate from the size of her vote share, which shapes her policy choices and can even steer her precisely to z. Candidates who lose the election then still add value, by giving voters a way to signal more or less extreme opinions. With multiple dimensions, losing candidates may play an even more important role, allowing voters to signal opinions orthogonal to the winner's platform. Whether Democrat and Republican platforms reflect a major or a minor equilibrium, for example, votes for Libertarian or Green candidates could nudge a major party to increase freedom or environmental protections. With far more issues than parties, however, the ability to precisely identify z remains limited, underscoring the importance of informal political activities such as petitions, rallies, public opinion surveys, and letters to legislators, which communicate policy-specific opinions that a coarse voting mechanism cannot.[40]

A Appendix

Proof of Lemma 1. Given $z \in Z$ and $v \in V$, a voter votes for candidate $j \in \{A, B\}$ with probability

$$\phi\left(j|z\right) = \int_S 1_{v(s)=j} g\left(s|z\right) ds \tag{4}$$

where $1_{v(s)=j}$ equals one if $v\left(s\right) = j$ and zero otherwise. The numbers N_A and N_B of A and B votes are then independent Poisson random variables with means $n\phi\left(A|z\right)$ and $n\phi\left(B|z\right)$, with the following probability of vote totals $\left(N_A, N_B\right) = \left(a, b\right)$.

$$\psi\left(a, b|z\right) = \frac{e^{-n}}{a!b!}\left[n\phi\left(A|z\right)\right]^a \left[n\phi\left(B|z\right)\right]^b \tag{5}$$

From within the game, an individual reinterprets N_A and N_B as the numbers of votes cast by his peers (Myerson, 1998); by voting himself, he can add one to either total. A vote for j will be *pivotal* (event P_j) with probability $\Pr\left(P_j|z\right) = \frac{1}{2}\Pr\left(N_j = N_{-j}\right) + \frac{1}{2}\Pr\left(N_j = N_j - 1\right)$, because candidates tie but j loses the tie-breaking coin toss or j wins the coin toss but loses the election by exactly one vote. The total probability of a vote for either candidate being pivotal (event P) is then given by $\Pr\left(P|z\right) = \Pr\left(P_A|z\right) + \Pr\left(P_B|z\right)$.

In terms of $\Pr\left(P_j\right)$ and $\Pr\left(P\right)$, the difference in expected benefit between voting B and voting A is given by the following,

$$\begin{aligned}
\Delta E_{w,z}\left[u\left(x_w, z\right)|s\right] &= \int_Z \left[u\left(x_B, z\right) - u\left(x_A, z\right)\right] \Pr\left(P_B|z\right) f\left(z|s\right) dz \\
&\quad - \int_Z \left[u\left(x_A, z\right) - u\left(x_B, z\right)\right] \Pr\left(P_A|z\right) f\left(z|s\right) dz \\
&= \int_Z \left[u\left(x_B, z\right) - u\left(x_A, z\right)\right] \Pr\left(P|z\right) f\left(z|s\right) dz \\
&= \Pr\left(P|s\right) E_z\left[u\left(x_B, z\right) - u\left(x_A, z\right)|P, s\right]
\end{aligned}$$

[40]Making a similar observation, Besley and Coate (2008) advocate un-bundling complex legislation, allowing separate dimensions to be decided separately.

$$= \Pr(P|s)\left(-\|x_B - E(z|P,s)\|^2 + \|x_A - E(z|P,s)\|^2\right) \qquad (6)$$

where the third equality follows because $f(z|s) = \frac{f(z)g(s|z)}{\int_Z g(s|z)f(z)dz}$, $f(z|P,s) = \frac{\Pr(P|z)f(z)g(s|z)}{\int_Z \Pr(P|z)g(s|z)f(z)dz}$, and $\Pr(P|s) = \frac{\int_Z \Pr(P|z)g(s|z)f(z)dz}{\int_Z g(s|z)f(z)dz}$, so $\Pr(P|z)f(z|s) = \Pr(P|s)f(z|P,s)$, and the final equality follows from (1). This benefit is positive if and only if x_B is closer to $E(z|P,s)$ than x_A is. ■

Proof of Lemma 2. 1. With a half-space strategy, (4) reduces to $\phi(A|z) = \int_{\{s\in S:s\cdot h<0\}} g(s|z)\,ds$ and $\phi(B|z) = \int_{\{s\in S:s\cdot h>0\}} g(s|z)\,ds$. These decrease and increase with z in the direction of h: for example, $\nabla_z\phi(B|z)\cdot h = g_1\int_{\{s\in S:s\cdot h>0\}}(s\cdot h)\,ds > 0$. In the orthogonal direction, both are constant: using basis vectors h and h', for example, $\nabla_z\phi(B|z)\cdot h' = g_1\int_{\{s\in S:s_1>0\}} s_2 ds = g_1\int_{s_1>0}\int_{-1}^{1} s_2 ds_2 ds_1 = 0$.

Conditional on k total votes (and on z), the number of B votes follows a binomial distribution, with probability parameter $\phi(B|z)$. $\Pr(B|k,z)$ is the probability that this number exceeds $\frac{k}{2}$, which increases with $\phi(B|z)$. Summing over all k, this implies that $\Pr(B|z)$ increases in $\phi(B|z)$, too. $\Pr(B|z)$ does not otherwise depend on z, so $\nabla_z\phi(B|z)\cdot h > 0$ implies that $\nabla_z\Pr(B|z)\cdot h = \frac{\partial\Pr(B|z)}{\partial\phi(B|z)}\nabla_z\phi(B|z)\cdot h > 0$ and, similarly, $\nabla_z\phi(B|z)\cdot h' = 0$ implies that $\nabla_z\Pr(B|z)\cdot h' = 0$.

2. $\Pr(P|z)$ increases in $\phi(B|z)$ only if $\phi(B|z) < \frac{1}{2}$. To see this, rewrite $\Pr(P)$ in terms of $\phi(B|z)$ using (5) and $\Pr(P_j)$, as follows.

$$\begin{aligned}
\Pr(P|z) &= \sum_{k=m}^{\infty}\left[\psi(k,k|z) + \frac{1}{2}\psi(k+1,k|z) + \frac{1}{2}\psi(k,k+1|z)\right] \\
&= \sum_{k=m}^{\infty}\frac{n^{2k}e^{-n}}{k!k!}\left[\phi(A|z)\,\phi(B|z)\right]^k\left[1 + \frac{n\phi(B|z)}{2(k+1)} + \frac{n\phi(A|z)}{2(k+1)}\right] \\
&= \sum_{k=m}^{\infty}\frac{n^{2k}e^{-n}}{k!k!}\left\{\left[1-\phi(B|z)\right]\phi(B|z)\right\}^k\left[1 + \frac{n}{2(k+1)}\right] \qquad (7)
\end{aligned}$$

This expression increases in $[1-\phi(B|z)]\,\phi(B|z)$, which increases in $\phi(B|z)$ if and only if $\phi(B|z) < \frac{1}{2}$. By Part 1 of this lemma, $\phi(B|z=0) = \frac{1}{2}$, and since $\nabla_z\phi(B|z)\cdot h' = 0$, this implies further that $\phi(B|z) = \frac{1}{2}$ whenever $z\cdot h = 0$. Since $\nabla_z\phi(B|z)\cdot h > 0$, this implies that $\Pr(P|z)$ increases in $\phi(B|z)$ when $z\cdot h < 0$ and decreases in $\phi(B|z)$ when $z\cdot h > 0$. Accordingly, $\nabla_z\Pr(P|z)\cdot h = \frac{\partial\Pr(P|z)}{\partial\phi(B|z)}\nabla_z\phi(B|z)\cdot h$ and $z\cdot h$ have opposite signs, while $\nabla_z\Pr(P|z)\cdot h' = \frac{\partial\Pr(P|z)}{\partial\phi(B|z)}\nabla_z\phi(B|z)\cdot h' = 0$.

3. If v_h is a half-space strategy then $\phi(A|-z) = \int_{\{s\in S:s\cdot h<0\}} g(-s|z)\,ds = \int_{\{s\in S:s\cdot h>0\}} g(s|z)\,ds = \phi(B|z)$, where the first equality utilizes the rotational symmetry of g (Condition 5) and the second reflects a change of variables. From (5) it is then clear that $\psi(a,b|-z) = \psi(b,a|z)$ for any $a,b\in Z_+$, implying that $\Pr(A|-z) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\psi(k+m,k|-z) + \frac{1}{2}\Pr(N_A = N_B)$ equals $\Pr(B|z) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\psi(k,k+m|z) + \frac{1}{2}\Pr(N_A = N_B)$ and, integrating over z, that $\Pr(A) = \Pr(B) = \frac{1}{2}$.

4. $\phi(A|-z) = \phi(B|z)$ implies that $\Pr(P|-z) = \Pr(P|z)$, as is clear from (7).

5. If $x_A = -x_B$ then the difference (6) in utility between voting B and voting A can be written as $\left[-(x_{B1}-z_1)^2 - (x_{B2}-z_2)^2\right] - \left[-(-x_{B1}-z_1)^2 - (-x_{B2}-z_2)^2\right] = 4(x_{B1}z_1 + x_{B2}z_2)$ in state z. Averaging across states, this equals

$$\begin{aligned}\Delta E_{w,z}[u(x_w)|s] &= \frac{4}{g_0}\int_Z (x_{B1}z_1 + x_{B2}z_2)\Pr(P|z)[g_0 + g_1(s\cdot z)] f(z)\,dz \\ &= \frac{4g_1}{g_0}\int_Z (x_B\cdot z)\Pr(P|z)(s\cdot z) f(z)\,dz\end{aligned}$$

where the final equality follows from Condition 2. This expression is linear in s and equals zero for $s=0$, implying that $\Delta E_{w,z}[u(x_w)|s]$ is positive and negative on opposite sides of a line in S that passes through the origin. In other words, the best response is another half-space strategy, as claimed. ■

Proof of Lemma 4. Lemma 3 states that $x_j^{br} = E(z|j)$ for $j = A, B$, and symmetry is a straightforward consequence of Condition 2, together with Part 3 of Lemma 2, since $x_{Ak}^{br} = \int_Z z_k \frac{\Pr(A|z)f(z)}{\Pr(A)}dz = \int_Z (-z_k)\frac{\Pr(A|-z)f(-z)}{\Pr(A)}dz$ equals $-\int_Z z_k \frac{\Pr(B|z)f(z)}{\Pr(B)}dz = -x_{Bk}^{br}$. Using h and h' as basis vectors, $E(z|B)\cdot h$ can be written simply as $E(z_1|B)$. Part 1 of Lemma 2 implies that $\Pr(B|z_1,z_2)$ increases in z_1 and is constant with respect to z_2, so that this reduces to the following, and can seen to be positive.

$$\begin{aligned}E(z_1|B) &= \int_Z z_1\frac{\Pr(B|z_1,z_2)f(z_1,z_2)}{\Pr(B)}dz_2dz_1 \\ &= 2\int_{Z_{1,2}}[z_1\Pr(B|z_1,z_2)f(z_1,z_2) - z_1\Pr(B|-z_1,z_2)f - z_1,z_2 \\ &\quad + z_1\Pr(B|z_1,-z_2)f(z_1,-z_2) - z_1\Pr(B|-z_1,-z_2)f(-z_1,-z_2)]dz_2dz_1 \\ &= 2\int_{Z_{1,2}} z_1[\Pr(B|z_1,z_2) - \Pr(B|-z_1,z_2)][f(z_1,z_2) + f(z_1,-z_2)]dz_2dz_1 \quad (8)\end{aligned}$$

Here, $Z_{1,2}$ denotes the union of the first and second octants (i.e., the first quadrant) and the second equality holds because $\Pr(B) = \frac{1}{2}$, by Lemma 2. ■

Proof of Proposition 1. By Lemmas 3 and 4, candidates' best responses to v_h are symmetric expectations $x_A^{br} = E(z|A) = -E(z|B) = -x_B^{br}$. When $\rho = 0$, $x_B^{br}\cdot h' = 0$, meaning that $\theta_{x_B^{br}} = \theta_h$. To see this, write x_B^{br} using h and h' as basis vectors, so that $x_B^{br}\cdot h$ and $x_B^{br}\cdot h'$ reduce simply to $E(z_1|B)$ and $E(z_2|B)$, respectively. The first of these is given by (8) in the proof of Lemma 4, and the second reduces in a similar way to the following.

$$E(z_2|B) = 2\int_{Z_{1,2}} z_2[\Pr(B|z_1,z_2) - \Pr(B|-z_1,z_2)][f(z_1,z_2) - f(z_1,-z_2)]dz_2dz_1 \quad (9)$$

When $\rho = 0$, Condition 3 implies that $f(z_1,-z_2) = f(z_1,z_2)$, so (9) equals zero. Thus, x_B^{br}—and, by symmetry, x_A^{br}—are orthogonal to h'. As shown in the proof of Lemma 4, however, $x_B^{br}\cdot h > 0$. Thus, x_A^{br} and x_B^{br} lie exactly in the directions of $-h$ and h, respectively.

By Lemma 2, a voter's best response to v_h and symmetric candidate platforms $(-x, x)$ is another half-space strategy, $v_{h^{br}}$. By Proposition 1, a voter prefers to vote B if and only if $E(z|P,s) \cdot x_B > 0$, and since $\theta_{x_B} = \theta_h$ in equilibrium, this is equivalent to the condition that $E(z|P,s) \cdot h > 0$ (where P depends implicitly on v_h). With basis vectors h and h', this dot product reduces simply to $E(z_1|P,s)$, which is proportional to the following.

$$\begin{aligned}
&\int_{Z_{1,2}} [z_1 \Pr(P|z_1,z_2)\, g(s|z_1,z_2)\, f(z_1,z_2) - z_1 \Pr(P|-z_1,z_2)\, g(s|-z_1,z_2)\, f(-z_1,z_2) \\
&+ z_1 \Pr(P|z_1,-z_2)\, g(s|z_1,-z_2)\, f(z_1,-z_2) - z_1 \Pr(P|-z_1,-z_2)\, g(s|-z_1,-z_2)\, f(-z_1,-z_2)] dz_2 dz_1 \\
= &\int_{Z_{1,2}} z_1 \Pr(P|z_1,z_2)\, [[g(s|z_1,z_2) - g(s|-z_1,-z_2)]\, f(z_1,z_2) \\
&+ [g(s|z_1,-z_2) - g(s|-z_1,z_2)]\, f(z_1,-z_2)] dz_2 dz_1 \qquad (10)
\end{aligned}$$

Equality follows from Part 2 of Lemma 2 since (with the rotated basis) $\Pr(P|z_1,z_2)$ is constant with respect to z_2, and depends on the magnitude of z_1 but not the sign. Condition 3 implies that $f(-z_1,z_2) = f(z_1,z_2)$ and Condition 4 implies that $g(s|z_1,z_2) - g(s|-z_1,z_2)$ has the same sign as $\binom{s_1}{s_2} \cdot \binom{z_1}{z_2} - \binom{s_1}{s_2} \cdot \binom{z_1}{z_2} = 2s_1 z_1$. Since z_1 is positive on $Z_{1,2}$, this has the same sign as s_1, or to $s \cdot h$ for the original Euclidean basis vectors. In other words, v_h is the best response to itself, and together with the best-response candidate platforms constitutes a half-space equilibrium. ■

Lemma A1 *If $\theta_h \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta_z \in \left(\theta_h, \theta_h + \frac{\pi}{2}\right)$ then $f(z) - f(M_{\theta_h} z)$ has the same sign as $\rho\left(\frac{\pi}{4} - |\theta_h|\right)$.*

Proof. If $|\theta_h| \leq \frac{\pi}{4}$ then $\theta_z - \theta_h \in \left(0, \frac{\pi}{2}\right)$ implies that $\max\left\{\theta_{M_{\theta_h} z}, \theta_{M_{-\frac{\pi}{4}} M_{\theta_h} z}\right\} < \min\left\{\theta_z, \theta_{M_{\frac{\pi}{4}} z}\right\}$. By Condition 2, both vectors on the left-hand side of this inequality have density $f(M_{\theta_h} z)$ and both vectors on the right have density $f(z)$. Moreover, both sides of the inquality lie in the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, so by Condition 1, $f(z) - f(M_{\theta_h z})$ has the same sign as ρ, as claimed. Similar reasoning applies for other values of θ_h: $\theta_z - \theta_h \in \left(0, \frac{\pi}{2}\right)$ implies that $-\frac{\pi}{4} < \max\left\{\theta_{R_{-\pi} z}, \theta_{M_{-\frac{\pi}{4}} R_{-\pi} z}\right\} < \min\left\{\theta_{M_{\theta_h} z}, \theta_{M_{\frac{\pi}{4}} M_{\theta_h} z}\right\} < \frac{\pi}{4}$ if $\theta_h > \frac{\pi}{4}$ and implies that $-\frac{\pi}{4} < \max\left\{\theta_z, \theta_{M_{-\frac{\pi}{4}} z}\right\} < \min\left\{\theta_{R_\pi M_{\theta_h} z}, \theta_{M_{\frac{\pi}{4}} R_\pi M_{\theta_h} z}\right\} < \frac{\pi}{4}$ if $\theta_h < -\frac{\pi}{4}$. Either way, both vectors on the left have density $f(z)$ and both vectors on the right have density $f(M_{\theta_h z})$, so Condition 1 implies that $f(z) - f(M_{\theta_h z})$ and ρ have opposite signs. ■

Proof of Proposition 2. If his peers vote according to v_h then, according to Lemma 1, a voter's best response is to vote j if and only if $E(z|P,s) \cdot x_j > 0$. For v_h to be its own best response, voters with signals satisfying $s \cdot h > 0$ should have a best response to vote B while the rest vote A. A signal orthogonal to h should make a voter indifferent. According to Lemma 3, candidate j's best response to v_h is the policy $x_j^{br} = E(z|j)$ that is optimal in expectation, conditional on winning. Taking these conditions together, a half-space equilibrium requires that $E(z|P,s) \cdot E(z|j)$ have the same sign as $s \cdot h$. The logic of this proof is to show that this is possible if and only if $\theta_h \in \left\{-\frac{1}{4}, \frac{1}{4}\right\}$. In particular, other values of θ_h produce $E(z|P,s) \cdot E(z|j) \neq 0$ for s orthogonal to h.

The cleanest way to compare $E(z|P,s)$ and $E(z|j)$ with each other is to compare both vectors with h and h'. This is accomplished most simply by using h and h' as basis vectors, so that $x_j^{br} \cdot h$ and $x_j^{br} \cdot h'$ reduce to $E(z_1|B)$ and $E(z_2|B)$, respectively, which are given by (8) and (9). Similarly, $E(z|P,s) \cdot h$ and $E(z|P,s) \cdot h'$ reduce to $E(z_1|P,s)$ and $E(z_2|P,s)$. For generic s, the first of these is given in the proof of Proposition 1 as proportional to (10). Analogously, the second is as follows.

$$\begin{aligned} E(z_2|P,s) \quad \propto \quad & \int_{Z_{1,2}} z_2 \Pr(P|z_1,z_2)\left[\left[g(s|z_1,z_2) - g(s|-z_1,-z_2)\right] f(z_1,z_2)\right. \\ & \left. + \left[g(s|-z_1,z_2) - g(s|z_1,-z_2)\right] f(z_1,-z_2)\right] dz_2 dz_1 \end{aligned} \tag{11}$$

However, a signal that is orthogonal to h has $s_1 = 0$; in that case, Condition 6 implies that $g(s|-z_1,z_2) = g(s|z_1,z_2)$ for any z_1, so (10) and (11) reduce to the following.

$$\begin{aligned} E(z_1|P,s=h') \quad \propto \quad & \int_{Z_{1,2}} z_1 \Pr(P|z_1,z_2)\left[g(s|z_1,z_2) - g(s|z_1,-z_2)\right] \\ & \times \left[f(z_1,z_2) - f(z_1,-z_2)\right] dz_2 dz_1 \end{aligned} \tag{12}$$

$$\begin{aligned} E(z_2|P,s=h') \quad \propto \quad & \int_{Z_{1,2}} z_2 \Pr(P|z_1,z_2)\left[g(s|z_1,z_2) - g(s|z_1,-z_2)\right] \\ & \times \left[f(z_1,z_2) + f(z_1,-z_2)\right] dz_2 dz_1 \end{aligned} \tag{13}$$

Together, $s = h'$ and $z \in Z_{1,2}$ also imply that $g(s|z_1,z_2) > g(s|z_1,-z_2)$.

If $|\theta_h| < \frac{\pi}{4}$ then, given $\rho > 0$, Lemma A1 implies that $f(z) > f(M_{\theta_h} z)$. In terms of basis vectors h and h', this means that $f(z_1,z_2) > f(z_1,-z_2)$, implying that (8), (9), (12), and (13) are all positive. That $x_B^{br} \cdot h > 0$ and $x_B^{br} \cdot h' > 0$ implies that x_B^{br} has polar angle strictly between those of h and h'.[41] That $E(z|P,s) \cdot h > 0$ and $E(z|P,s) \cdot h' > 0$, together with the result that x_B^{br} lies between h and h', imply in turn that $E(z|P,s) \cdot x_B^{br} > 0$, as well.[42] In short, $|\theta_h| < \frac{\pi}{4}$ is not compatible with equilibrium: when his peers follow v_h and candidates respond optimally, equilibrium would require that a voter with orthogonal signal $s = h'$ be indifferent between voting A and voting B, but instead such a voter prefers to vote B.

If $|\theta_h| > \frac{\pi}{4}$ then Lemma A1 implies that $f(z) < f(M_{\theta_h} z)$, which in terms of basis vectors h and h' means that $f(z_1,z_2) > f(z_1,-z_2)$. In that case, (8) is still positive but (9) is now negative, meaning that $x_B^{br} \cdot h' < 0 < x_B^{br} \cdot h$, so that x_B^{br} has polar angle strictly between those of $-h'$ and h. Similarly, for $s = h'$ (13) is still positive but (12) is negative, meaning that $E(z|P,s) \cdot x_B^{br} < 0$.[43] When his peers follow v_h and candidates respond optimally, therefore, a voter with signal $s = h'$ prefers to vote A. Thus, v_h is incompatible with equilibrium.

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[41] Formally, $x_B^{br} \cdot h = r_{x_B^{br}} \cos\left(\theta_{x_B^{br}} - \theta_h\right) > 0$ and $x_B^{br} \cdot h' = r_{x_B^{br}} \cos\left(\theta_{x_B^{br}} - \theta_{h'}\right) > 0$ together imply that $\theta_{x_B^{br}} - \theta_h < \frac{\pi}{4}$ and $\theta_{h'} - \theta_{x_B^{br}} < \frac{\pi}{4}$.

[42] Formally, $x_B^{br} = r_{x_B^{br}}\left[\alpha h + (1-\alpha) h'\right]$ for some $\alpha \in (0,1)$ implies that $E(z|\mathcal{P},s) \cdot x_B^{br} = r_{x_B^{br}} \alpha E(z|\mathcal{P},s) \cdot h + r_{x_B^{br}} (1-\alpha) E(z|\mathcal{P},s) \cdot h' > 0$.

[43] Formally, $x_B^{br} = r_{x_B^{br}}\left[\alpha h + (1-\alpha)(-h')\right]$ for some $\alpha \in (0,1)$ and $E(z|\mathcal{P},s) \cdot h < 0 < E(z|\mathcal{P},s) \cdot h'$ together imply that $E(z|\mathcal{P},s) \cdot x_B^{br} = r_{x_B^{br}} \alpha E(z|\mathcal{P},s) \cdot h - r_{x_B^{br}} (1-\alpha) E(z|\mathcal{P},s) \cdot h' < 0$.

If $|\theta_h| = \frac{\pi}{4}$ then Lemma A1 implies that $f(z) = f(M_{\theta_h} z)$, which in terms of basis vectors h and h' means that $f(z_1, z_2) = f(z_1, -z_2)$. In that case, (8) and (12) are still positive but (9) and (13) are zero. That $x_B^{br} \cdot h > 0$ and $x_B^{br} \cdot h' = 0$ implies that x_B^{br} is colinear with h. That $E(z|P,s) \cdot h > 0$ and $E(z|P,s) \cdot h' = 0$ when $s = h'$ then implies that the best response to v_h (and $x_A^{br}(v_h)$ and $x_B^{br}(v_h)$) is v_h, so $(v_h, x_A^{br}, x_B^{br})$ constitutes a half-space equilibrium. ■

Proof of Proposition 3. Since x_B^+ lies in the direction of h^+ and x_B^- lies in the direction of h^-, their magnitudes can be written as the projection of x_B^+ on h^+ and the projection of x_B^- on h^-, respectively. Generically, (8) gives the projection of x_B^{br} on h, in terms of the basis vectors h and h'. That candidate platforms in the major and minor equilibria are symmetric implies that $\Pr(B|-z) = \Pr(A|z)$, while $|\theta_h| = \frac{\pi}{4}$ implies that $f(M_{\theta_h} z) = f(z)$ by Lemma A1. Thus, (8) reduces to $\|x_B\| = 8\int_{Z_{1,2}} z_1 \left[\Pr(B|z_1,z_2) - \frac{1}{2}\right] f(z_1,z_2)\, dz$, and can be differentiated as follows.

$$\frac{\partial \|x_B\|}{\partial \rho} = 8\int_{Z_{1,2}} z_1 \left[\Pr(B|z_1,z_2) - \frac{1}{2}\right] \frac{\partial}{\partial \rho} f(z_1,z_2)\, dz$$

A policy vector z under these rotated basis vectors corresponds to a rotated vector $R_{\theta_h} z$ under the original basis vectors. In the cases of the major and minor equilibria, this corresponds to $R_{\frac{\pi}{4}} z = \frac{1}{\sqrt{2}}\binom{z_1 - z_2}{z_1 + z_2}$ and $R_{-\frac{\pi}{4}} z = \frac{1}{\sqrt{2}}\binom{z_1 + z_2}{z_2 - z_1}$, respectively. Reversing z_1 and z_2 therefore corresponds to reversing the sign of either the first or the second component. Either way, because of the dimensional symmetry of f (Condition 2), this is equivalent to reversing the sign of ρ. Thus, $\frac{\partial \|x_B\|}{\partial \rho}$ reduces further, as follows,

$$\begin{aligned}
\frac{\partial \|x_B\|}{\partial \rho} &= 8\int_{Z_1} \{ z_1 \left[\Pr(B|z_1,z_2) - \frac{1}{2}\right] \frac{\partial}{\partial \rho} f(z_1,z_2) \\
&\quad + z_2 \left[\Pr(B|z_2,z_1) - \frac{1}{2}\right] \frac{\partial}{\partial \rho} f(z_2,z_1) \} dz \\
&= 8\int_{Z_1} \{ z_1 \left[\Pr(B|z_1,z_2) - \frac{1}{2}\right] \\
&\quad - z_2 \left[\Pr(B|z_2,z_1) - \frac{1}{2}\right] \} \frac{\partial}{\partial \rho} f(z_1,z_2)\, dz
\end{aligned}$$

where $z_1 > z_2 > 0$ for policy pairs in Z_1, implying that $\Pr(B|z_1,z_2) > \Pr(B|z_2,z_1)$, and therefore that the bracketed difference is positive. With the original basis vectors, Condition 1 states that $\frac{\partial}{\partial \rho} f(z)$ has the same sign as $z_1 z_2$; with the rotated basis vectors, this means that $\frac{\partial}{\partial \rho} f(z)$ has the same sign as $(z_1 - z_2)(z_1 + z_2)$ for the major equilibrium and $(z_1 + z_2)(z_2 - z_1)$ for the minor equilibrium. Since $z_1 > z_2 > 0$ for all policy pairs in Z_1, this implies that $\frac{\partial \|x_B^+\|}{\partial \rho}$ is positive and $\frac{\partial \|x_B^-\|}{\partial \rho}$ is negative, as claimed. ■

Proof of Proposition 4. If voter and candidate strategies $(v, x_A, x_B) = (v_+, (-x,-x), (x,x))$

are oriented along the major diagonal (for some $x > 0$) then (2) can be rewritten as follows

$$\begin{aligned} W_+(x) &= \int_Z \left\{\left[-(-x-z_1)^2 - (-x-z_2)^2\right]\Pr(A|z) + \left[-(x-z_1)^2 - (x-z_2)^2\right]\Pr(B|z)\right\} f(z)\,dz \\ &= 2\int_Z \left[-(x-z_1)^2 - (x-z_2)^2\right]\Pr(B|z) f(z)\,dz \end{aligned}$$

using a change of variables and noting that $\Pr(A|z) = \Pr(B|-z)$ by Lemma 2. This function is concave in x, achieving a maximum at the major equilibrium policy position $x_+ = E(z_1|B) = E(z_2|B)$. Implicitly, $\Pr(B|z)$ and $f(z)$ in this expression depend on v_+ and ρ, respectively.

Instead orienting voter and candidate strategies $(v, x_A, x_B) = (v_-, (-x, x), (x, -x))$ along the minor diagonal (for some $x > 0$) generates the same welfare as reversing the sign of ρ,

$$\begin{aligned} W_-(x;\rho) &= \int_Z \{[u((-x,x),(z_1,z_2))\Pr(A|z_1,z_2;v_-) \\ &\quad + u((x,-x),(z_1,z_2))\Pr(B|z_1,z_2;v_-)]\} f(z_1,z_2;\rho)\,dz \\ &= \int_Z \{[u((x,x),(-z_1,z_2))\Pr(B|-z_1,z_2;v_+) \\ &\quad + u((x,x),(z_1,-z_2))\Pr(B|z_1,-z_2;v_+)]\} f(z_1,z_2;\rho)\,dz \\ &= \int_Z \{u((x,x),(z_1,z_2))\Pr(B|z_1,z_2;v_+) \\ &\quad + u((x,x),(z_1,z_2))\Pr(B|z_1,z_2;v_+)\} f(z_1,z_2;-\rho)\,dz \\ &= W_+(x;-\rho) \end{aligned}$$

where the second equality holds because $u((-x,x),(z_1,z_2)) = -(-x-z_1)^2 - (x-z_2)^2$ and $u((x,x),(-z_1,z_2)) = -(x+(-z_1))^2 - (x-z_2)^2$ are equivalent algebraically (as are $u((x,-x),(z_1,z_2))$ and $u((x,x),(z_1,-z_2))$), because $\Pr(A|z) = \Pr(B|-z)$, and because expected vote shares $\phi(B|z_1,z_2;v_-) = \int_{s\cdot\binom{x}{-x}>0} g(s|z_1,z_2)\,ds$ and $\phi(B|z_1,-z_2;v_+) = \int_{s\cdot\binom{x}{x}>0} g(s|z_1,-z_2)\,ds$ are equivalent (as can be seen by a change of variables in the second dimension), so $\Pr(B|z_1,z_2;v_-) = \Pr(B|z_1,-z_2;v_+)$.

An alternative way of rewriting (2) is as follows.

$$\begin{aligned} W_+(x) &= \int_Z [u(x_A,z)\Pr(A|z) + u(x_B,z)\Pr(B|z)] f(z)\,dz \\ &= 2\int_{Z_{1,2,3,8}} [u(x_A,z)\Pr(A|z) + u(x_B,z)\Pr(B|z)] f(z)\,dz \\ &= 4\int_{Z_{1,8}} [u(x_A,z)\Pr(A|z) + u(x_B,z)\Pr(B|z)] f(z)\,dz \\ &= 4\int_{Z_1} (\{\left[-(-x-z_1)^2 - (-x-z_2)^2\right]\Pr(A|z_1,z_2) \\ &\quad + \left[-(x-z_1)^2 - (x-z_2)^2\right]\Pr(B|z_1,z_2)\} f(z_1,z_2) \\ &\quad + \{\left[-(-x-z_1)^2 - (-x+z_2)^2\right]\Pr(A|z_1,-z_2) \end{aligned}$$

$$+\left[-\left(x-z_1\right)^2-\left(x+z_2\right)^2\right]\Pr\left(B|z_1,-z_2\right)\}f\left(z_1,-z_2\right))dz \qquad (14)$$

The second equality here holds because $x_A=-x_B$, so $u\left(x_A,z\right)\Pr\left(A|z\right)=u\left(x_B,-z\right)\Pr\left(B|-z\right)$ and $u\left(x_B,z\right)\Pr\left(B|z\right)=u\left(x_A,-z\right)\Pr\left(A|-z\right)$, implying that each realization of z in orthants 1, 2, 3, or 8 generates identical welfare to the opposite state, $-z$, from orthant 4, 5, 6, or 7. The third equality holds because $u\left(\left(-x,-x\right),\left(z_2,z_1\right)\right)$ and $u\left(\left(-x,-x\right),\left(z_1,z_2\right)\right)$ are algebraically equivalent (as are $u\left(\left(-x,-x\right),\left(z_2,z_1\right)\right)$ and $u\left(\left(x,x\right),\left(z_1,z_2\right)\right)$) and because the expected vote shares $\phi\left(B|z_2,z_1\right)=\int_{s\cdot\binom{x}{x}>0}g\left(s|z_2,z_1\right)ds$ and $\phi\left(B|z_1,z_2\right)=\int_{s\cdot\binom{x}{x}>0}g\left(s|z_1,z_2\right)ds$ in states $\left(z_2,z_1\right)$ and $\left(z_1,z_2\right)$ are the same, so $\Pr\left(j|z_2,z_1\right)=\Pr\left(j|z_1,z_2\right)$ for $j=A,B$. The final equality associates with each policy pair $\left(z_1,z_2\right)$ in Z_1 a corresponding policy pair $\left(z_1,-z_2\right)$ in Z_8.

$f\left(-z_1,z_2;\rho\right)=f\left(z_1,z_2;-\rho\right)$ by Condition 2, so differentiating (14) with respect to ρ and replacing $\Pr\left(A|z_1,z_2\right)=1-\Pr\left(B|z_1,z_2\right)$ yields the following.

$$\begin{aligned}
\frac{\partial W_+\left(x;\rho\right)}{\partial\rho} &= 4\int_{Z_1}\{\left[-\left(-x-z_1\right)^2-\left(-x-z_2\right)^2\right]\left[1-\Pr\left(B|z_1,z_2\right)\right] \\
&\quad +\left[-\left(x-z_1\right)^2-\left(x-z_2\right)^2\right]\Pr\left(B|z_1,z_2\right) \\
&\quad -\left[-\left(-x-z_1\right)^2-\left(-x+z_2\right)^2\right]\left[1-\Pr\left(B|z_1,-z_2\right)\right] \\
&\quad -\left[-\left(x-z_1\right)^2-\left(x+z_2\right)^2\right]\Pr\left(B|z_1,-z_2\right)\}\frac{\partial}{\partial\rho}f\left(z_1,z_2\right)dz \\
&= 4\int_{Z_1}\{-4xz_2+\left(4xz_1+4xz_2\right)\Pr\left(B|z_1,z_2\right) \\
&\quad +\left(-4xz_1+4xz_2\right)\Pr\left(B|z_1,-z_2\right)\}\frac{\partial}{\partial\rho}f\left(z_1,z_2\right)dz \\
&= 16x\int_{Z_1}\{z_1\left[\Pr\left(B|z_1,z_2\right)-\Pr\left(B|z_1,-z_2\right)\right] \\
&\quad +z_2\left[\Pr\left(B|z_1,z_2\right)-\frac{1}{2}+\Pr\left(B|z_1,-z_2\right)-\frac{1}{2}\right]\}\frac{\partial}{\partial\rho}f\left(z_1,z_2\right)dz
\end{aligned}$$

$\left(z_1,z_2\right)\in Z_1$ (equivalently, $z_1>z_2>0$) implies: that $\Pr\left(B|z_1,z_2\right)>\frac{1}{2}$ and $\Pr\left(B|-z_1,z_2\right)>\frac{1}{2}$, since $\binom{z_1}{z_2}=\frac{z_2-z_1}{2}\binom{-1}{1}+\frac{z_1+z_2}{2}\binom{1}{1}$ and $\binom{z_1}{-z_2}=-\frac{z_1+z_2}{2}\binom{-1}{1}+\frac{z_1-z_2}{2}\binom{1}{1}$, so Parts 1 and 3 of Lemma 2 imply that $\Pr\left(B|0,0\right)=\frac{1}{2}$ and that $\nabla_z\Pr\left(B|z\right)\cdot\binom{z_1}{z_2}=\frac{z_1+z_2}{2}\nabla_z\Pr\left(B|z\right)\cdot h_+>0$ and $\nabla_z\Pr\left(B|z\right)\cdot\binom{z_1}{-z_2}=\frac{z_1-z_2}{2}\nabla_z\Pr\left(B|z\right)\cdot h_+>0$; that $\Pr\left(B|z_1,z_2\right)>\Pr\left(B|z_1,-z_2\right)$, since $\binom{z_1}{z_2}-\binom{z_1}{-z_2}=\frac{z_1+2z_2}{2}\binom{-1}{1}+z_2\binom{1}{1}$, so $\nabla_z\Pr\left(B|z\right)\cdot\left[\binom{z_1}{z_2}-\binom{z_1}{-z_2}\right]=z_2\nabla_z\Pr\left(B|z\right)\cdot h_+>0$; and that $\frac{\partial}{\partial\rho}f\left(z_1,z_2\right)>0$. Thus, $W_+\left(x;\rho\right)$ increases in ρ (strictly, as long as $x>0$). If $\left(x_+^\rho,x_+^\rho\right)$ and $\left(x_-^\rho,-x_-^\rho\right)$ denote the major and minor equilibrium platforms of candidate B, therefore, then for $\rho'>\rho$, $W_+\left(x_+^{\rho'};\rho'\right)\geq W_+\left(x_-^\rho;\rho'\right)\geq W_+\left(x_-^\rho;\rho\right)$. That is, major equilibrium welfare increases in ρ. Minor equilibrium welfare decreases in ρ, since $W_-\left(x_-^{\rho'};\rho'\right)=W_+\left(x_-^{\rho'};-\rho'\right)<W_+\left(x_-^\rho;-\rho\right)<W_+\left(x_-^{-\rho};-\rho\right)=W_-\left(x_-^{-\rho};\rho\right)<W_-\left(x_-^\rho;\rho\right)$, implying that $W_+\left(x_+^\rho;\rho\right)=W_-\left(x_-^\rho;\rho\right)=W_+\left(x;-\rho\right)$ if and only if $\rho=0$, as claimed. ∎

Proof of Proposition 5. Regardless of λ, Lemmas 3 and 2 remain valid: best response platforms are given by $x_j^{br} = E(z|j)$ and half-space voting v_h produces expected vote shares that increase in the direction of h, symmetric platforms $x_A^{br} = -x_B^{br}$, and a half-space best-response strategy $v_{h^{br}}$. The logic for the last of these claims mirrors the proof of Lemma 2: the difference in expected utility between voting B and voting A now becomes the following,

$$\begin{aligned}\Delta E_{w,z}\left[u\left(x_w\right)|s\right] &= \int_Z \{\left[-\left(1+\lambda\right)\left(x_{B1}-z_1\right)^2-\left(1-\lambda\right)\left(x_{B2}-z_2\right)^2\right] \\ &\quad -\left[-\left(1+\lambda\right)\left(-x_{B1}-z_1\right)^2-\left(1-\lambda\right)\left(-x_{B2}-z_2\right)^2\right]\}\Pr\left(P|z\right)f\left(z|s\right)dz \\ &= 4\int_Z \left[\left(1+\lambda\right)x_{B1}z_1+\left(1-\lambda\right)x_{B2}z_2\right]\left[1+\frac{g_1}{g_0}\left(s_1z_1+s_2z_2\right)\right]f\left(z\right)dz\end{aligned}$$

which is still linear in s and still zero for $(s_1, s_2) = (0, 0)$.

From (3), the benefit of voting B has the same sign as $(1+\lambda)\,x_{B1}E\left(z_1|P,s\right)+(1-\lambda)\,x_{B2}E\left(z_2|P,s\right) = E\left(z|P,s\right)\cdot\binom{(1+\lambda)x_{B1}}{(1-\lambda)x_{B2}}$. The proof of Proposition 1 shows that if $\rho = 0$ then a voter with $\theta_s = \theta_h + \frac{\pi}{2}$ is indifferent between voting A and B. That is, $E\left(z|P,s\right)\cdot\binom{x_{B1}}{x_{B2}} = 0$. Such a voter is no longer indifferent when $\lambda > 0$, unless $x_{B1} = 0$ or $x_{B2} = 0$. That proof shows further that x_B^{br} has the same polar angle as h, however, so equilibrium requires $h_1 = 0$ or $h_2 = 0$, meaning that $\theta_h \in \left\{0, \frac{\pi}{2}\right\}$. ■

Lemma A2 *If voting follows v_h then: if $\theta_h \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ then $\theta_{x_B^{br}} \in \left[\theta_h, \frac{\pi}{4}\right]$; if $\theta_h \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$ then $\theta_{x_B^{br}} \in \left[-\frac{3\pi}{4}, \theta_h\right]$; if $\theta_h \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ then $\theta_{x_B^{br}} \in \left[\frac{\pi}{4}, \theta_h\right]$.*

Proof. Write the difference between platforms as $x_{B1}^{br} - x_{B2}^{br} = E\left(z_1|B\right) - E\left(z_2|B\right) = \int_Z \left(z_1 - z_2\right)\frac{\Pr(B|z)}{\Pr(B)}f\left(z\right)dz$. Noting that $\Pr\left(B\right) = \frac{1}{2}$ and expressing all eight octants in terms of the first octant Z_1, this reduces to the following,

$$\begin{aligned}&2\int_{Z_1}\left[\left(z_1-z_2\right)\Pr\left(B|z_1,z_2\right)f\left(z_1,z_2\right)+\left(-z_1-z_2\right)\Pr\left(B|-z_1,z_2\right)f\left(-z_1,z_2\right)\right.\\&+\left(z_1+z_2\right)\Pr\left(B|z_1,-z_2\right)f\left(z_1,-z_2\right)+\left(-z_1+z_2\right)\Pr\left(B|-z_1,-z_2\right)f\left(-z_1,-z_2\right)\\&+\left(z_2-z_1\right)\Pr\left(B|z_2,z_1\right)f\left(z_2,z_1\right)+\left(-z_2-z_1\right)\Pr\left(B|-z_2,z_1\right)f\left(-z_2,z_1\right)\\&\left.+\left(z_2+z_1\right)\Pr\left(B|z_2,-z_1\right)f\left(z_2,-z_1\right)+\left(-z_2+z_1\right)\Pr\left(B|-z_2,-z_1\right)f\left(-z_2,-z_1\right)\right]dz\\=\;&4\int_{Z_1}\{\left(z_1-z_2\right)\left[\Pr\left(B|z_1,z_2\right)-\Pr\left(B|z_2,z_1\right)\right]f\left(z_1,z_2\right)\\&+\left(z_1+z_2\right)\left[\Pr\left(B|z_1,-z_2\right)-\Pr\left(B|-z_2,z_1\right)\right]f\left(z_1,z_2;-\rho\right)\}dz\end{aligned}$$

where equality follows from Condition 2 and Lemma 2. Invoking Lemma 2 a second time, the two differences in brackets have the same signs as $h\cdot(z_1,z_2) - h\cdot(z_2,z_1) = (h_1-h_2)(z_1-z_2)$ and $h\cdot(z_1,-z_2) - h\cdot(-z_2,z_1) = (h_1-h_2)(z_1+z_2)$, respectively, which means that the entire expression has the same sign as $h_1 - h_2$ (since $z_1 - z_2$ and $z_1 + z_2$ are both positive in Z_1). In other words, $x_B^{br}\cdot\binom{1}{-1}$ and $x_B^{br}\cdot\binom{1}{1}$ have the same signs as $h\cdot\binom{1}{-1}$ and $h\cdot\binom{1}{1}$, respectively, so θ_h and $\theta_{x_B^{br}}$ both belong to the same quadrant: either $\left[-\frac{3\pi}{4},-\frac{\pi}{4}\right]$, $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$, or $\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$. However, the proof of Proposition 2 shows that x_B^{br} differs from h in the direction of the major diagonal. That is, $\theta_{x_B^{br}} < \theta_h$ if and only if $\frac{\pi}{4} < |\theta_h| < \frac{3\pi}{4}$. ■

Proof of Proposition 6. As in the proof of Proposition 5, best response platforms are given by $x_j^{br} = E(z|j)$ and v_h produces expected vote shares that increase in the direction of h, symmetric best-response platforms $x_A^{br} = -x_B^{br}$, and a half-space best-response strategy $v_{h^{br}}$. Also, the benefit of voting B instead of voting A has the same sign as $E(z|P,s)\cdot\tilde{x}_B$, where $\tilde{x}_B = \begin{pmatrix} (1+\lambda)x_{B1} \\ (1-\lambda)x_{B2} \end{pmatrix}$.

The proof of Proposition 2 shows that $\theta_{x_B^{br}} \in \left(0,\frac{\pi}{4}\right)$, which implies that $x_{B1}^{br} > x_{B2}^{br} > 0$, and therefore that $(1+\lambda)x_{B1}^{br} > (1-\lambda)x_{B2}^{br} > 0$, or $\tilde{x}_{B1}^{br} > \tilde{x}_{B2}^{br} > 0$. Thus, $\theta_{\tilde{x}_B^{br}} \in \left(0,\frac{\pi}{4}\right)$. The proof of Proposition 1 shows that if $\theta_{x_B} = 0$ and $\theta_s = \frac{\pi}{2}$ then $E(z|P,s)\cdot x_B = 0$. For the same signal realization, then, $\theta_{\tilde{x}_B^{br}} > 0$ implies that $E(z|P,s)\cdot\tilde{x}_B^{br} > 0$. In other words, a voter whose signal is orthogonal to h prefers voting B, and a voter who is indifferent between voting A and B has a signal with polar angle $\theta_s > \frac{\pi}{2}$. Thus, the half-space voting strategy that best responds to $\tilde{x}_B^{br}(v_h)$ has a normal vector h^{br} with $\theta_{h^{br}} > 0$.

If $\theta_h = \frac{\pi}{4}$ then the proof of Proposition 2 shows that $\theta_{x_B^{br}} = \frac{\pi}{4}$, which implies that $x_{B1}^{br} = x_{B2}^{br} > 0$, and therefore that $(1+\lambda)x_{B1}^{br} > (1-\lambda)x_{B2}^{br} > 0$, or $\tilde{x}_{B1}^{br} > \tilde{x}_{B2}^{br} > 0$. Thus, again $\theta_{\tilde{x}_B^{br}} \in \left(0,\frac{\pi}{4}\right)$. The proof of Proposition 1 shows that if $\theta_{x_B} = \frac{\pi}{4}$ and $\theta_s = \frac{3\pi}{4}$ then $E(z|P,s)\cdot x_B = 0$. For the same signal realization, then, $\theta_{\tilde{x}_B^{br}} < \frac{\pi}{4}$ implies that $E(z|P,s)\cdot\tilde{x}_B^{br} < 0$. In other words, a voter whose signal is orthogonal to h prefers voting A, and a voter who is indifferent between voting A and B has a signal with $\theta_s < \frac{3\pi}{4}$. Thus, the half-space voting strategy that best responds to $\tilde{x}_B^{br}$ has a normal vector h^{br} with $\theta_{h^{br}} < \frac{\pi}{4}$. Since $\theta_{h^{br}(h)}$ is a continuous function of θ_h, the results that $\theta_{h^{br}(h)} > \theta_h$ for $\theta_h = 0$ and $\theta_{h^{br}(h)} < \theta_h$ for $\theta_h = \frac{\pi}{4}$ together imply (by the intermediate value theorem) the existence of $\theta_{h^+} \in \left(0,\frac{\pi}{4}\right)$ such that $\theta_{h^{br}(h^+)} = \theta_{h^+}$, implying that v_{h^+} is a best response to $\left(v_{h^+}, x_A^+, x_B^+\right)$, and the latter constitutes a half-space equilibrium.

If $\theta_h = -\frac{\pi}{4}$ then the proof of Proposition 2 shows that $\theta_{x_B^{br}} = -\frac{\pi}{4}$, which implies that $x_{B1}^{br} > 0 > x_{B2}^{br}$ and $\left|x_{B1}^{br}\right| = \left|x_{B2}^{br}\right|$, and therefore that $(1+\lambda)x_{B1}^{br} > 0 > (1-\lambda)x_{B2}^{br}$ and $\left|(1+\lambda)x_{B1}^{br}\right| = \left|(1-\lambda)x_{B2}^{br}\right|$, or $\tilde{x}_{B1}^{br} > 0 > \tilde{x}_{B2}^{br}$ with $\left|\tilde{x}_{B1}^{br}\right| > \left|\tilde{x}_{B2}^{br}\right|$. Thus, $\theta_{\tilde{x}_B^{br}} \in \left(-\frac{\pi}{4},0\right)$. For the same voting strategy, the proof of Proposition 1 shows that if $\theta_{x_B} = -\frac{\pi}{4}$ and a citizen's signal realization has polar angle $\theta_s = \frac{\pi}{4}$ orthogonal to h then $E(z|P,s)\cdot x_B = 0$. For the same signal realization, then, $\theta_{\tilde{x}_{br}} > -\frac{\pi}{4}$ implies that $E(z|P,s)\cdot\tilde{x}_B^{br} > 0$. In other words, a voter whose signal is orthogonal to h prefers voting B, and one who is indifferent between voting A and B has a signal with $\theta_s > \frac{\pi}{4}$. Thus, the half-space voting strategy that best responds to $\tilde{x}_B^{br}$ has a normal vector h^{br} with $\theta_{h^{br}} > -\frac{\pi}{4}$.

If $\theta_h = -\frac{\pi}{2}$ then the proof of Proposition 2 shows that $\theta_{x_B^{br}} \in \left(-\frac{3\pi}{4},-\frac{\pi}{2}\right)$, which implies that $x_{B1}^{br} < 0$ and $x_{B2}^{br} < 0$, and therefore that $(1+\lambda)x_{B1}^{br} < 0$ and $(1-\lambda)x_{B2}^{br} < 0$, or $\tilde{x}_{B1}^{br} < 0$ and $\tilde{x}_{B2}^{br} < 0$. Thus, $\theta_{\tilde{x}_B^{br}} < -\frac{\pi}{2}$. For the same voting strategy, the proof of Proposition 1 shows that if $\theta_{x_B} = -\frac{\pi}{4}$ and a citizen's signal realization has polar angle $\theta_s = 0$ orthogonal to h then $E(z|P,s)\cdot x = 0$. For the same signal realization, then, $\theta_{\tilde{x}_{br}} < -\frac{\pi}{4}$ implies that

$E(z|P,s) \cdot \tilde{x}_B^{br} < 0$. In other words, a voter whose signal is orthogonal to h prefers voting A, and one who is indifferent between voting A and B has a signal with $\theta_s < 0$. Thus, the half-space voting strategy that best responds to $\tilde{x}_B^{br}(v_h)$ has a normal vector h^{br} with $\theta_{h^{br}} < -\frac{\pi}{2}$. Since $\theta_{h^{br}(h)}$ is a continuous function of θ_h, the results that $\theta_{h^{br}(h)} > \theta_h$ for $\theta_h = 0$ and $\theta_{h^{br}(h)} < \theta_h$ for $\theta_h = \frac{\pi}{4}$ together imply (by the intermediate value theorem) the existence of $\theta_{h^-} \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ such that $\theta_{h^{br}(h^-)} = \theta_{h^-}$, implying that v_{h^-} is a best response to $\left(v_{h^-}, x_A^-, x_B^-\right)$, and the latter constitutes a half-space equilibrium. ■

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