Debt Maturity Management

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Abstract

This paper studies how a borrower issues long- and short-term debt in response to shocks to the enterprise value. Our theory highlights the tradeoff between commitment and hedging. Short-term debt protects creditors from future dilution and forces the borrower to reduce leverage after negative shocks. Long-term debt postpones default and allows the borrower time to recover after a downturn, thereby providing hedging in the upturn. Borrowers issue both types of debt when they are far from default in the upturn. By contrast, distressed borrowers exclusively issue short-term debt. Our model generates novel implications for the dynamic adjustment of debt maturities.

Keywords: capital structure; debt maturity; risk management; dynamic tradeoff theory.

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1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in several aspects, and an important one is its maturity. Borrowing can be short, as in the case of trade credit, or long as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing following shocks to their enterprise value?

The academic literature falls behind in providing a useful framework to study these questions, despite their obvious importance. For example, the Leland model (Leland, 1994) and the vast follow-up literature typically assume that (1) all debt has the same (expected) maturity and (2) the borrower either commits to the total leverage or may only increase leverage after retiring all existing debt and paying some exogenous issuance cost.[1] Although these assumptions simplify the analysis, they are inconsistent with the ample empirical evidence that borrowers often issue a mix of long- and short-term debt and that adjusting the outstanding debt's maturity profile can take some time to accomplish.

This paper introduces a simple and tractable framework to address these questions. Our theory highlights the tradeoff between commitment and hedging in borrowing long and short. Long-term debt has a staggered structure: it matures in the future, and the borrower cannot commit to not issuing more debt before the existing debt is due. Due to this lack of commitment, creditors of long-term debt are exposed to dilution. Therefore, a borrower financed exclusively by long-term debt never voluntarily reduces leverage following negative shocks. By contrast, short-term debt does not suffer from dilution because it matures before the borrower gets the chance to borrow again. Following negative shocks, short-term debt forces the borrower to reduce the leverage. Meanwhile, long-term debt has important hedging benefits: if a downturn arrives, long-term debt postpones default and allows the borrower some time to recover. These merits of long-term debt offer hedging benefits to the borrower in the upturn.

More specifically, a risk-neutral borrower has assets that generate an income flow that follows a regime-switching geometric Brownian motion (GBM) process. The expected growth rate of the income is high in an upturn but low in a downturn. There is also a disaster shock in the downturn, after which the borrower defaults immediately.[2] A transition from the upturn to the downturn can be interpreted as a downside risk. Creditors are competitive, risk-neutral, and have a lower

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss in the subsection on related literature.

[2] In the appendix, we show that immediate default after the disaster shock can be the borrower's endogenous optimal choice.

cost of capital than the borrower. The difference in the cost of capital and the tax-shield benefits motivate the borrower to issue debt. Two types of debt are available. The short-term debt matures instantaneously (i.e., has zero maturity) and needs to be continuously rolled over. Long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing either type of debt at any time to adjust the maturity profile of the outstanding debt. This feature distinguishes us from the existing literature.

The flexibility to issue more debt before the current long-term one matures exposes long-term creditors to dilution (Fama and Miller, 1972; Black and Scholes, 1973; Admati et al., 2018; DeMarzo and He, 2021). Specifically, the borrower always has incentives to borrow more after the existing long-term debt has been issued because the additional borrowing dilutes the existing long-term claims. In equilibrium, creditors anticipate future dilution, and the price of long-term debt adjusts downwards to the level at which the borrower cannot capture any benefit.

By contrast, short-term debt protects its creditors from being diluted and resolves the commitment problem because short-term creditors' debt matures before the borrower can issue it again. As a result, short-term creditors need to be compensated only by the probability of default within a short period but not by the cost of being diluted in the future. Following negative shocks, short-term debt forces the borrower to reduce the leverage, even though she does not commit to doing so. This result contrasts the literature on the leverage-ratchet effect (Admati et al., 2018), which suggests that a borrower financed exclusively by long-term debt never voluntarily reduces leverage, even after negative shocks.

Given the advantage of short-term debt in resolving the commitment problem, it is natural for the borrower to issue it. Indeed, our results show that long-term debt is never issued in the downturn when there is no additional downside risk that can be potentially hedged. Instead, the borrower fully levers up by borrowing short. In the upturn, the potential arrival of the downturn introduces an interesting tradeoff in issuing short-term debt. The borrower chooses between issuing a lower amount of riskless short-term debt (i.e., fully repaid even after a downturn) and a higher amount of risky short-term debt (i.e., immediately default once the downturn comes). We show the borrower issues risky (riskless) short-term debt in the upturn if she is very close to (far from) the default boundary. The reason is that when the borrower is close to the default boundary, both the amount of riskless short-term borrowing and the cost of default – the option value of continuing to operate in the downturn – become very low. Therefore, it is optimal to borrow risky short-term debt, which leads to an immediate default after the downturn arrives. In this case, long-term debt is exposed to the same downside risk as short-term debt and suffers from dilution. Therefore, long-term debt offers no hedging benefit, and there is no reason for the borrower to issue it.

Results are different in the upturn if the borrower is far from the default boundary. In this case, the borrower issues riskless short-term debt, and creditors do not anticipate an immediate default if the downturn arrives. In addition, the borrower also issues some long-term debt. There are two reasons behind this result. First, long-term debt could postpone default after the downturn arrives. In other words, by offering the borrower an option to default later instead of immediately after the downturn, long-term debt provides better hedging against the downside risk and is valued in the upturn. We show this result analytically by studying a limiting model in which the volatility of the GBM process converges to zero. If the cash flow in the downturn declines faster than the debt matures so that the borrower defaults deterministically, she will issue long-term debt in the upturn to postpone default in the downturn. The second reason is related. By delaying default, long-term debt also allows the borrower time to recover after the downturn, thereby increasing the chances of avoiding default. This second channel is only present if the volatility of the cash flow process is non-zero. Note that the borrower values the benefit of long-term debt in hedging the downside risk, even though she is risk-neutral. The reason is that the cost in default introduces constraints in the financing, which makes the borrower behave as if she were risk-averse.

By constructing a dynamic model, our model emphasizes that empirical studies of debt maturity need to differentiate stock (outstanding debt) versus flow (newly issued debt). For example, our results in the upturn imply that in states where the borrower has a lot of outstanding long-term debt and is therefore very close to the default boundary, the newly issued debt is exclusively short-term. Almeida et al. (2011) present findings during the financial crisis that if a large amount of existing long-term debt is due very soon, this can push the borrower very close to default and reduces real activities. By contrast, Brunnermeier (2009) and Krishnamurthy (2010) also document that during the crisis when a borrower is close to default, the newly-issued debt was primarily short-term.

Moreover, our model implies that defaults can be classified into two types. First, they can be driven by the gradual deterioration of the borrower's cash flows relative to the accumulation of long-term debt. Second, defaults can occur suddenly after a large negative shock and an excessive amount of short-term debt borrowed prior to the shock. Furthermore, our model implies that cross-sectionally, borrowers more exposed to large downside risks should use more long-term debt in their capital structure. These are implications for future empirical tests.

Our model is consistent with the evidence that market leverage is counter-cyclical (Adrian and Shin, 2014), whereas debt maturity structure is pro-cyclical (Xu, 2018; Chen et al., 2021). Moreover, our model implies that within a regime, cash flows and debt maturity negatively comove with each other. However, if we compare across regimes, the regime with higher expected growth rates in cash flows has on average longer debt maturity. These implications, once again, can be tested by future empirical studies.

Related literature

Our paper builds on the literature of dynamic corporate finance pioneered by Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019).[3] Important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the related problem when the borrower can only issue zero-maturity debt (see also Bolton et al. (2021), who further model costly equity issuance). In these papers, the borrower can only issue one type of debt, so the tradeoff between borrowing long and short is not explicitly studied. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed, and the borrower can choose between two types of exponentially-maturing debt. Our paper differs in two aspects. First, we allow for flexibility in adjusting total leverage. Second, we model short-term debt as debt that matures instantaneously. The different approaches in modeling short-term debt render the mechanisms of the two papers drastically different. Whereas we emphasize the tradeoff between commitment and hedging, their paper focuses on rollover losses and dilution. Brunnermeier and Yogo (2009) also study debt maturity in the context of liquidity risk, and they show long-term debt is optimal if the firm is close to default (or close to debt restructuring as in their paper). Our results are the opposite: the borrower will issue exclusively short if she is close to default. The difference is driven by the assumption that the borrower can issue debt at any time without commitment in our model. By contrast, the borrower in Brunnermeier and Yogo (2009) can only issue new debt after the current debt is repaid and effectively has commitment.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. One advantage of a fully-dynamic setup is that it allows us to make empirical predictions regarding the stock (existing debt) and the flow (new issuance) of debt maturity. The insight that short-term debt resolves the lack of commitment is also present in another related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runnable feature of short-term debt. In our paper, the reason that short-term debt resolves commitment is fundamentally different: the short rate would increase drastically if the borrower issued more debt.[4] This feature resembles the leasing solution (Bulow, 1982) to the durable-goods monopoly problem. Relatedly, Gertner and Scharfstein (1991) show that conditional on financial distress, short-term debt has a higher market value and increases leverage, leading to more ex-post debt overhang (also see Diamond and He (2014)).

[3] Malenko and Tsoy (2020) study the role of reputation.

[4] Also see Hu and Varas (2021) on this feature of short-term debt in the context of financial intermediaries.

The insight that long-term debt can be diluted has been recognized by Fama and Miller (1972) and Black and Scholes (1973) and has been more recently formalized by Admati et al. (2018). Brunnermeier and Oehmke (2013) show equity and short-term debt can dilute long-term debt's recovery value in bankruptcy. Our paper rules out this mechanism by assuming zero recovery value in the benchmark model. Instead, we focus on dilution outside the bankruptcy, which comes exclusively from the borrower's lack of commitment to issuance and default.

The hedging benefits of long-term debt are also related to the literature on fiscal policy and sovereign debt. For example, Angeletos (2002) shows that the ex-post variations in the market value of public debt hedge the government against bad fiscal conditions. Aguiar et al. (2019) show that in the absence of hedging motives, the borrower never actively issues any long-term debt due to the lack of commitment. By contrast, we show that with hedging, the borrower issues a combination of long- and short-term debt (also see Niepelt (2014)). Bigio et al. (2021) study debt maturity management under liquidity cost but without dilution. In their model, the borrower's choice depends on the bond demand curve, micro-founded via search (Duffie et al., 2005). The mechanisms of the two papers are complementary.

A similar trade-off is studied by Arellano and Ramanarayanan (2012), who calibrate a quantitative model of sovereign borrowing with two maturities. Our paper complements their analysis in several dimensions. First, we develop a tractable model with a transparent characterization of the equilibrium. Unlike Arellano and Ramanarayanan (2012), we can fully characterize the optimal debt policy and highlight the fundamental economic forces underlying the maturity choices. Second, our framework identifies the type of risk – downside risk – that the borrower wants to hedge using long-term debt. Specifically, we emphasize that by offering the borrower an option to default later instead of immediately after a downturn, long-term debt provides better hedging against downside risk, which is valued in good times. By contrast, Arellano and Ramanarayanan (2012) emphasizes that long-term debt offers a hedge against future fluctuations in spreads. However, because credit spreads are endogenous, it is unclear what the underlying risks driving the fluctuations are and hence the hedging benefit. Third, we show that long- and short-term debt offer different kinds of flexibility following shocks. Short-term debt forces the borrower to reduce leverage in response to negative shocks. If there are only small shocks, short-term debt is chosen such that leverage is never excessive. When there are large shocks, the option to postpone default embedded in long-term debt is valuable. Finally, we follow the standard corporate finance tradition by casting the model in a risk-neutral setting. *A priori*, the risk-neutral borrower in our model does not have a reason to value the merit of hedging by long-term debt. The cost of default makes the borrower behave as if she is risk-averse, as emphasized by the finance literature on risk management (Bolton et al., 2011; Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To our knowledge, no previous work has established the link between maturity management and risk management in a corporate finance setting.

2 The Model

2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower, often interpreted as a firm. The relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate ρ, which exceeds r, the discount rate of creditors. In other words, creditors have a lower cost of capital compared to the borrower.

The borrower's asset generates earnings before interest and taxes (EBIT) at a rate X_t, which evolves according to:

$$\frac{dX_t}{X_{t-}} = \mu_{\theta_t} dt + \sigma dB_t - \mathbb{1}_{\{\theta_t = L\}} dN_t, \tag{1}$$

where B_t is a standard Brownian motion, N_t is a Poisson process with arrival rate η, and $\theta_t \in \{H, L\}$ represents the regime with $\theta_0 = H$. After an independent Poisson event with intensity λ, θ_t switches to L and stays unchanged. The drift μ_{θ_t} differs across the two regimes with $\mu_L < \mu_H$, so that the high state H is associated with a higher expected growth rate in the borrower's cash flow. Below, we refer to the high state as the *upturn* and the low state as the *downturn*. In addition, we allow the possibility of a *disaster* shock in the downturn that arrives at a rate η, upon which the cash flow X_t permanently drops to zero. In Internet Appendix B.5, we show that this disaster shock is equivalent to a third state in which the drift falls below μ_L. We establish conditions such that immediately after the third state arrives, the borrower finds it optimal to default.

2.2 Debt Maturity Structure

The borrower would like to issue debt for two reasons: creditors' lower cost of capital and tax shields. Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the outstanding debt maturity structure. In particular, we do not restrict the borrower to commit to a particular issuance path but instead, let the issuance decisions be made at each instant.

All *short-term debt* matures instantaneously and therefore needs to be continuously rolled over.[5] We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt \downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time t and let y_{t-} be the associated short rate. With tax shields, the borrower makes a total interest payment of $(1-\pi)y_{t-}D_{t-}$ to short-term creditors, where π is the corporate tax rate. *Long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing bonds with coupon rate r and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. The coupon payments of long-term debt are tax deductible, so the borrower makes coupon payments $r(1-\pi)F_t dt$, where F_t is the aggregate face value of long-term debt outstanding at time t.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume the bankruptcy cost is 100%. In other words, creditors cannot recover any value once the borrower defaults.

2.3 Valuation

Let τ_b be the endogenous time at which the borrower chooses to default. We define p_t as the price per unit of the face value, which for $t < \tau_b$ is

$$p_t = \mathbb{E}_t \left[\int_t^{\tau_\xi \wedge \tau_b} e^{-r(s-t)} r ds + e^{-r(\tau_\xi - t)} \mathbb{1}_{\{\tau_b > \tau_\xi\}} \right], \tag{2}$$

where τ_ξ is the (stochastic) maturing date. The two components in the previous expression correspond to the coupon and final payments. The short rate y_{t-} depends on the borrower's equilibrium default decisions:

$$y_{t-} = r + \lim_{dt \downarrow 0} \frac{\Pr_{t-dt}\left(\tau_b \leq t | \tau_b > t - dt\right)}{dt}, \tag{3}$$

where the second term on the right-hand side is the hazard rate of default. According to (3), y_{t-} compensates the creditors for the probability of default occurring between $t-dt$ and t. For example, if in the upturn, short-term creditors expect default only upon a transition to the downturn, then $y_{t-} = r + \lambda$. Similarly, if in the downturn, default on short-term debt only happens upon the disaster shock hits, then $y_{t-} = r + \eta$.

To simplify notation, we denote after-tax rates with a hat. In particular, let $\hat{r} = (1-\pi)r$,

[5] The fact that short-term debt matures instantaneously implies that the borrower does not have the chance to issue new debt before the existing short-term debt matures.

$\hat{\eta} = (1-\pi)\eta$, $\hat{\lambda} = (1-\pi)\lambda$, and $\hat{y}_{t-} = (1-\pi)y_{t-}$. The net income of the borrower is

$$\mathrm{NI}_t \equiv \underbrace{X_t}_{\text{EBIT}} - \underbrace{\pi(X_t - rF_t - y_{t-}D_{t-})}_{\text{tax payment}} - \underbrace{(rF_t + y_{t-}D_{t-})}_{\text{interest payments}} = (1-\pi)X_t - \hat{r}F_t - \hat{y}_{t-}D_{t-}. \tag{4}$$

The net cash flow corresponds to the net income plus the net proceeds from debt issuance, with the latter given by $(p_t dG_t - \xi F_t dt) + dD_t$. Note the notations dG_t and dD_t allow for both atomistic and flow issuance, and the price of long-term debt p_t could also depend on the issuance amount dG_t.

Define V_t as the continuation value of the borrower, which we sometimes refer to as the equity value at time t. The borrower chooses the endogenous time of default and the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate function as given. Once again, let us emphasize that all these decisions, default and issuance, are made without commitment:

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \geq t\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-\rho(s-t)} \left(\mathrm{NI}_s ds + (p_s dG_s - \xi F_s ds) + dD_s \right) \right]. \tag{5}$$

To guarantee the valuation remains finite, we assume both $\hat{r} + \hat{\lambda} > \mu_H$ and $\hat{r} + \hat{\eta} > \mu_L$ hold. These assumptions follow from the literature. Specifically, it requires in both states that the creditor's effective discount rate is above the expected growth rate of the cash flow.

2.4 Smooth Equilibrium

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state θ_t, the cash-flow level X_t, and the amount of outstanding debt $\{D_{t-}, F_t\}$. The equilibrium requires the following: (1) creditors break even; that is, p_t follows equation (2) and y_{t-} follows equation (3); and (2) the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t \geq 0$. Finally, an MPE is *smooth* if no jump occurs in long-term debt issuance, in which case we write $dG_t = g_t F_t dt$. In a smooth equilibrium, the aggregate face value of long-term debt evolves according to

$$dF_t = (g_t - \xi) F_t dt. \tag{6}$$

3 Equilibrium

3.1 An Equivalent Maximization Problem

Let us define J_t as the joint (maximized) continuation value of the borrower and short-term creditors if default does not occur at time t. The following result motivates us to work with J_t for the remainder of this paper.

Proposition 1. *The joint continuation value satisfies $J_t = J_{\theta_t}(X_t, F_t)$. Specifically,*

$$J_H(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ (1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s + (\rho+\lambda-\hat{y}_{s-})D_{s-} + \lambda \max\{J_L(X_s, F_s) - D_{s-}, 0\} \Big\} ds \Bigg] \tag{7}$$

$$J_L(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \Big\{ (1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s + (\rho+\eta-\hat{y}_{s-})D_{s-} \Big\} ds \Bigg],$$

where the maximization is subject to the issuance constraint $D_s \le J_{\theta_s}(X_s, F_s)$. The equity value is $V_t = V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_{\theta_t}(X_t, F_t) - D_{t-}, 0\}$.

The terms in (7) are related to those in (5). Here, $(\rho+\lambda-\hat{y}_s)D_{s-}$ reflects the gains from issuing short-term debt, where $\rho+\lambda-\hat{y}_s$ is the difference between the borrower's discount rate and the required rate of return by short-term creditors. Finally, the last term in (7) stands for the event of regime-shifting, upon which the borrower would rather default and renege on the payments if the amount of outstanding short-term debt exceeds the maximum joint value if default does not immediately occur; that is, if $D_{s-} > J_L(X_s, F_s)$. In this case, both parties receive a payoff of zero. The terms in $J_L(X, F)$ can be interpreted similarly.

Proposition 1 generates a very interesting economic insight: even though the borrower makes decisions on debt issuance, these decisions are made to maximize the borrower's and short-term creditors' joint value. This is because any issuance decisions will be immediately reflected in the credit risk faced by short-term creditors, affecting the proceeds from issuing short-term debt. Note that the payoff to existing long-term creditors is ignored in the maximization problem because their debts have been issued in the past, and the borrower has incentives to dilute them.[6] This result relates to Aguiar et al. (2019) in the context of sovereign debt, where the equilibrium issuance decisions can be characterized by the solution to a planner's problem that ignores the payoff to existing long-term creditors. Meanwhile, the max operator in (7) shows that the borrower and short-term creditors may still have conflicts on the immediate default decisions.

[6] The payoff to new long-term creditors is at dt order in the smooth equilibrium.

3.2 Value Function and Short-Term Debt Issuance

Proposition 1 implies the state variable D_{t-} only enters the problem by affecting whether the borrower chooses to default immediately at time t. Following this result, we can suppress the problem's dependence on D_{t-} and treat it as a decision variable. A smooth MPE is characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F,D)$, $D_\theta(X,F)$, and $g_\theta(X,F)$. By exploiting the homogeneity of the problem, we can further reduce the problem's dimension and write $J_\theta(X,F) = Xj_\theta(f)$, $D_\theta(X,F) = Xd_\theta(f)$, where $f = F/X$. Moreover, g_θ, p_θ, and y_θ are homogeneous of degree zero, so $g_\theta(X,F) = g_\theta(f)$, $p_\theta(X,F) = p_\theta(f)$, and $y_\theta(X,F,D) = y_\theta(f,d)$. For the rest of this paper, we sometimes refer to $j_\theta(f)$ as the scaled value function.

Given that the low state is absorbing, it is convenient to solve for the equilibrium starting from there and working up to the high state.

Low state $\theta_t = L$. Default driven by the Brownian shock can be anticipated by short-term creditors. Given so, short-term debt can only default after the disaster shock. Therefore, the short rate is $y_L(f,d) \equiv r + \eta$ and $\hat{y}_L = (1-\pi)(r+\eta)$ for $d \le j_L(f)$. By considering the change in the value function in (7) over a small interval, we can derive the following equation:

$$\underbrace{(\rho+\eta)J_L(X,F)}_{\text{required return}} = \max_{D_L\in[0,J_L(X,F)],\, g_L} \underbrace{(1-\pi)X - (\hat{r}+\xi)F}_{\text{cash flow net long payments}} + \underbrace{(\rho+\eta-\hat{y}_L)D_L}_{\text{gains from borrowing short}}$$
$$+ \underbrace{p_L(X,F)\,g_L F}_{\text{proceeds from issuing long}} + \underbrace{\frac{\partial J_L(X,F)}{\partial F}(g_L-\xi)F}_{\text{evolution of } dF} + \underbrace{\frac{\partial J_L(X,F)}{\partial X}X\mu_L + \frac{1}{2}\frac{\partial^2 J_L(X,F)}{\partial X^2}X^2\sigma^2}_{\text{evolution of } dX}. \quad (8)$$

Note that the choice of short-term debt D_L is capped by the level of value function J_L, and the choice of D_L also affects J_L.

The net benefits of issuing long-term debt become clear once we examine all the terms that involve g_L on the right-hand side. Whereas $p_L(X,F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, $\frac{\partial J_L(X,F)}{\partial F}$ is the drop in the borrower's continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, the marginal proceeds must be fully offset by the drop in continuation value so that the borrower is indifferent; that is,

$$p_L(X,F) + \frac{\partial J_L(X,F)}{\partial F} = 0. \quad (9)$$

As in DeMarzo and He (2021), Equation (9) provides an indifference condition that allows solving the value function $J_L(X,F)$ as if $g_L(f) = 0$, corresponding to the case that the borrower never issues any further long-term debt.

Substituting $J_\theta(X,F) = X j_\theta(f)$ in equation (8) (together with the indifference condition (9)) we get the following HJB for the scaled value function $j_L(f)$:

$$(\rho+\eta-\mu_L)\, j_L(f) = \max_{d_L\in[0,j_L(f)]} (1-\pi) - (\hat{r}+\xi)\, f + (\rho+\eta-\hat{y}_L)\, d_L \\ -(\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f). \quad (10)$$

Given that the coefficient in front of d_L satisfies $\rho+\eta-\hat{y}_L = \rho+\eta-(\hat{r}+\hat{\eta}) > 0$, it is always optimal for the borrower to issue as much short-term debt as possible, which leads to $d_L(f) = j_L(f)$. Intuitively, the borrower benefits from issuing short-term debt due to lower costs and tax shields. The rest of the problem becomes standard. The borrower defaults if and only if $f \uparrow f_L^b$, where f_L^b satisfies the value matching condition $j_L(f_L^b) = 0$ and the smooth pasting condition $j_L'(f_L^b) = 0$.

High state $\theta_t = H$. The smooth equilibrium leads to an indifference condition in long-term debt issuance that relates to (9):

$$p_H(X,F) + \frac{\partial J_H(X,F)}{\partial F} = 0. \quad (11)$$

In the upturn, the borrower faces the Brownian shock and a downside risk whereby the state may transit from high to low. Suppose default does not occur immediately upon the downturn's arrival. In that case, the borrower and short-term creditors receive a maximum value of $j_L(f)$, among which d_H must be repaid to short-term creditors. The borrower will default immediately upon the state transition if and only if $d_H > j_L(f)$. Expecting so, short-term creditors demand a short rate

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \le j_L(f) \\ r+\lambda & \text{if } d_H > j_L(f). \end{cases} \quad (12)$$

Following a similar analysis to the one in the low state, we arrive at the HJB for the scaled value function $j_H(f)$:

$$(\rho+\lambda-\mu_H)\, j_H(f) = \max_{d_H\in[0,j_H(f)]} (1-\pi) - (\hat{r}+\xi)\, f + (\rho+\lambda-\hat{y}_H)\, d_H + \lambda \max\{j_L(f)-d_H, 0\} \\ -(\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \quad (13)$$

The optimal issuance of short-term debt follows from (13): the borrower borrows either $j_L(f)$ at rate r or $j_H(f)$ at $r+\lambda$. The riskless short-term debt $j_L(f)$ brings a flow benefit of $(\rho+\lambda-\hat{r})j_L(f)$, whereas the risky short-term debt brings $(\rho+\pi\lambda-\hat{r})j_H(f)$. A comparison of the two shows that risky short-term debt is preferred if and only if

$$(\rho+\pi\lambda-\hat{r})j_H(f) > (\rho+\lambda-\hat{r})j_L(f) \Rightarrow \underbrace{(\rho-\hat{r})\left(j_H(f)-j_L(f)\right)+\pi\lambda j_H(f)}_{\text{benefit from additional leverage}} > \underbrace{\lambda j_L(f)}_{\text{additional bankruptcy cost}}.$$

We conjecture, and later verify, that the optimal policy is characterized by a threshold

$$f_\dagger = \min\left\{f \geq 0 : (\rho-\hat{r})\left(j_H(f)-j_L(f)\right)+\pi\lambda j_H(f) \geq \lambda j_L(f)\right\}. \tag{14}$$

When $f < f_\dagger$, the borrower issues the maximum amount of risk-free short-term debt $d_H(f) = j_L(f)$. When $f > f_\dagger$, the borrower issues the maximum amount of risky short-term debt $d_H(f) = j_H(f)$. At $f = f_\dagger$, the borrower is indifferent between the two. As usual, the value function satisfies the value matching and smooth pasting conditions $j_H(f_H^b) = 0$ and $j'_H(f_H^b) = 0$ at the default boundary $f = f_H^b$. The next proposition provides the optimal short-term policy together with the joint continuation value $j_\theta(f)$.

Proposition 2 (Short-term debt Issuance)**.** *There is $\bar{\lambda}$ – defined in equation* (A.16) *– such that $f_\dagger > 0$ if and only if $\lambda > \bar{\lambda}$, where $f_\dagger$ is defined in* (14).

- *In state $\theta = L$:*
  - *The borrower issues short-term debt $d_L(f) = j_L(f)$ and pays a short rate $y_L(f, d_L(f)) = r + \eta$.*
  - *The joint continuation value is*

$$j_L(f) = \underbrace{\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L} - \frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}f}_{\textit{no default value}} + \underbrace{\frac{1}{\gamma-1}\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}\left(\frac{f}{f_L^b}\right)^\gamma}_{\textit{default option}}, \tag{15}$$

    where $\gamma > 1$ is provided in equation (A.1)*, and the default boundary f_L^b in* (A.2).

- *In state $\theta = H$:*
  - *The borrower issues short-term debt*

$$d_H(f) = \begin{cases} j_L(f) & \text{if } f < f_\dagger \\ j_H(f) & \text{if } f \geq f_\dagger \end{cases}$$

    and pays a short rate given by (12).

– *The joint continuation value is*

$$j_H(f) = \begin{cases} u_0(f) + \big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^\phi & f \in [0, f_\dagger) \\ u_1(f) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)h_0(f) - u_1(f_H^b)h_1(f) & f \in \left[f_\dagger, f_H^b\right], \end{cases} \tag{16}$$

where $\phi > 1$ *is provided in* (A.9)*, and*

$$u_0(f) = \underbrace{\frac{1-\pi}{\rho+\lambda-\mu_H}\left(1 + \frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}-\mu_L}\right) - \frac{\hat{r}+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}+\xi}\right)f}_{\text{no default value}} + \underbrace{\delta\frac{1}{\gamma-1}\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}\left(\frac{f}{f_L^b}\right)^\gamma}_{\text{default option in low state}} \tag{17}$$

$$u_1(f) = \underbrace{\frac{1-\pi}{\hat{r}+\hat{\lambda}-\mu_H} - \frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi}f}_{\text{no default value}}. \tag{18}$$

The discount factors δ*,* $h_0(\cdot)$*, and* $h_1(\cdot)$ *are defined in equations* (A.11) *and* (A.13)*. The boundaries* $f_\dagger$ *and* f_H^b *are determined using the boundary conditions* $j_H(f_H^b) = j_H'(f_H^b) = 0$ *and equation* (14)*.*

The condition $\lambda \leq \bar{\lambda}$ corresponds to $(\rho - \hat{r})(j_H(0) - j_L(0)) + \pi\lambda j_H(0) \geq \lambda j_L(0)$, which implies that even for a borrower without any outstanding long-term debt, the benefit from additional leverage exceeds the additional bankruptcy cost. Therefore, she borrows risky short-term debt. In this case, $f_\dagger = 0$, and the results are very similar across the two states. For the remainder of this paper, we focus on the more general case of $f_\dagger > 0$, which requires $\lambda > \bar{\lambda}$. In this case, a borrower without any outstanding long-term debt will borrow riskless short-term debt, so $d_H(0) = j_L(0)$. Meanwhile, the maximum riskless short-term borrowing decreases as f increases and eventually gets to zero as f approaches f_L^b. Therefore, the borrower chooses risky short-term debt $d_H(f) = j_H(f)$ for f sufficiently high. We show in Lemma 3 of the appendix that a unique threshold $f_\dagger \in \left(0, f_L^b\right)$ exists such that $(\rho - \hat{r})(j_H(f) - j_L(f)) + \pi\lambda j_H(f) \geq \lambda j_L(f)$ if and only if $f \geq f_\dagger$. Figure 1 offers a graphical illustration. Intuitively, the additional bankruptcy cost $\lambda j_L(f)$ associated with risky short-term borrowing is high when f is low but becomes very low when f is high. In contrast, the difference between the amount of risky and riskless short-term borrowing declines much slower as f grows.

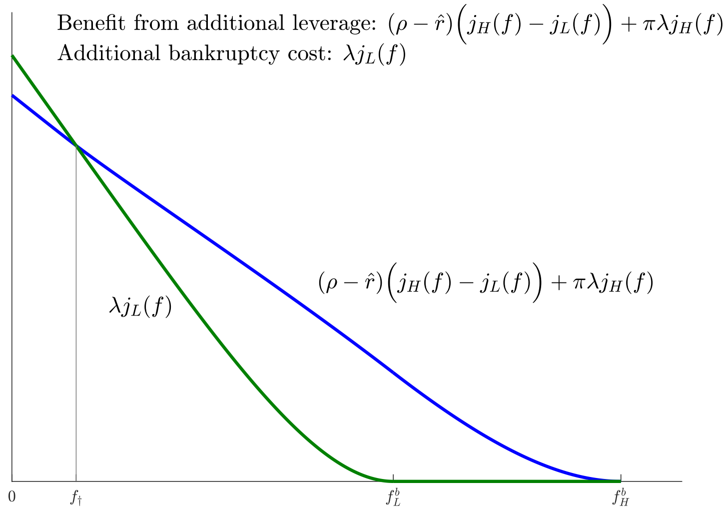


Figure 1: Cost and benefit of riskless short-term debt

3.3 Price and Issuance Policy of Long-Term Debt

We have shown that in equilibrium, the borrower is indifferent between issuing long-term debt and not. However, the result does not imply she never borrows long on the equilibrium path. In this subsection, we derive the issuance policy of long-term debt.

It follows from Itô's lemma that, before the disaster $dN_t = 1$ in the low state, f_t evolves according to[7]

$$\frac{df_t}{f_t} = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right) dt - \sigma dB_t. \tag{19}$$

Let us start with the downturn $\theta_t = L$, where equation (9) (or equivalently, $p_L(f) = -j'_L(f)$) is the necessary condition for the borrower to be indifferent between issuing long-term debt and not. The price satisfies the following HJB equation:

$$(r+\xi+\eta)\, p_L\,(f) = \underbrace{r+\xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p'_L(f) + \frac{1}{2}\sigma^2 f^2 p''_L(f)}_{\text{expected change in bond price}}. \tag{20}$$

To derive the issuance function g_L, we plug $d_L = j_L(f)$ into (10), differentiate the resulting equation once, and add (20) on both sides. Turning to the upturn $\theta_t = H$, the equity holder's indifference in

[7] We omit the disaster shock dN_t when $\theta_t = L$. Upon the disaster shock $dN_t = 1$, X_t gets absorbed at 0, so f_t jumps to ∞. The borrower then defaults immediately, and the price of the long-term debt jumps to zero.

long-term debt issuance becomes $p_H(f) = -j'_H(f)$, and $p_H(f)$ satisfies the following HJB equation:

$$(r + \xi + \lambda)\, p_H(f) = r + \xi + \mathbb{1}_{\{f < f_\dagger\}} \cdot \lambda p_L(f) + \big(g_H(f) - \xi - \mu_H + \sigma^2\big)\, f p'_H(f) + \frac{1}{2}\sigma^2 f^2 p''_H(f). \quad (21)$$

Compared with (20), (21) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \leq f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The derivation of the issuance policy $g_H(f)$ follows the same steps as the one in the low state.

Proposition 3 (Long-term debt issuance)**.** *The equilibrium price and issuance of long-term debt follow:*

1. *Downturn $\theta = L$: $\forall f \in [0, f_L^b)$, the price of long-term debt is $p_L(f) = -j'_L(f)$, and the issuance policy is*

$$g_L(f) = \frac{\pi\big(r - (r + \eta)\, p_L(f)\big)}{-f p'_L(f)}.$$

2. *Upturn $\theta = H$: $\forall f \in [0, f_H^b)$, the price of long-term debt is $p_H(f) = -j'_H(f)$.*
   - *For $f \in [0, f_\dagger)$*

$$g_H(f) = \frac{\pi r\,(1 - p_L(f))}{-f p'_H(f)} + \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-f p'_H(f)}.$$

   - *For $f \in [f_\dagger, f_H^b)$*

$$g_H(f) = \frac{\pi\big(r - (r + \lambda) p_H(f)\big)}{-f p'_H(f)}.$$

Proposition 3 shows that both in the low state and in the high state when $f \geq f_\dagger$, the borrower issues long-term debt only to take advantage of the tax shields. By contrast, in the high state when $f < f_\dagger$, the borrower has an additional reason to issue long-term debt. Note that when there is no tax shield, that is, when $\pi = 0$, the issuance policy becomes

$$g_\theta(f) = \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-f p'_H(f)} \mathbb{1}_{\{f < f_\dagger,\, \theta = H\}}, \quad (22)$$

so that long-term debt is only issued in the high state when $f < f_\dagger$. In (22), it becomes clear that without the tax shields, the amount of long-term debt issuance is proportional to the difference in the long-term debt's price in the two states.

3.3.1 A Heuristic Derivation of the Long-Term Debt Issuance Policy

Let us provide a heuristic derivation of the equilibrium issuance policy based on a local-perturbation approach. This derivation also helps illustrate the various economic forces at play.

Generally, the borrower would like to issue debt for two reasons: tax shields and the creditors' cheaper cost of capital. Think about a borrower in an upturn $\theta = H$, and the existing long-term debt is $f = f_0 \in [0, f_\dagger)$. Consider an adjustment in which the borrower issues an extra amount of long-term debt Δ at time t and proceeds to buy it back at $t + dt$. Hence, this deviation in the issuance policy lasts only for "one period." If the borrower still wants to borrow risk-free short-term debt, she must reduce its amount from $j_L(f_0)$ to $j_L(f_0 + \Delta)$. Figure 2(a) illustrates how this adjustment affects the tax-shield benefits between t and $t + dt$. The red line represents the tax shields from riskless short-term debt $j_L(f)$, whereas the blue the tax shields from long-term debt. The slope of the green line is $-p_L(f_0) = j_L'(f_0)$. Note that while this adjustment increases long-term debt's tax shields by $\pi r \Delta$, it reduces the short-term debt's tax shields by

$$\pi r\big(j_L(f_0) - j_L(f_0 + \Delta)\big) \approx -\pi r j_L'(f_0)\Delta = \pi r p_L(f_0)\Delta.$$

The net impact on tax shields is[8]

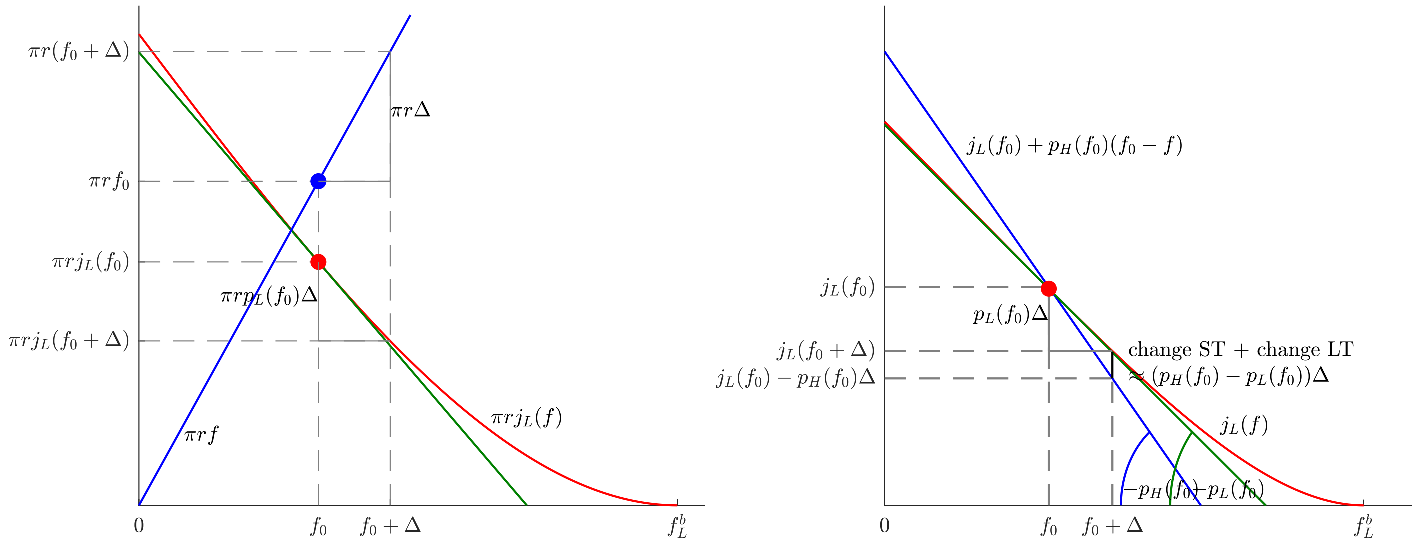
$$\text{Net change in TS} \approx \pi r\big(1 - p_L(f)\big)\Delta.$$

Given $p_L(f) < 1$, the borrower can increase the tax shields by replacing short-term debt with long-term debt.

Next, we consider how this adjustment affects the market value of total leverage. Before the adjustment, the market value is $j_L(f_0) + p_H(f_0)f_0$, whereas post the adjustment the market value becomes $j_L(f_0+\Delta)+p_H(f_0+\Delta)(f_0+\Delta)$. In Figure 2(b), we separate the change in the long-term debt's market value from that of the short-term debt. The red curve represents the amount of risk-free short-term debt; the blue line represents a first-order approximation to the value of long-term debt for an adjustment of Δ. The tangent green line illustrates the first-order approximation for the associated adjustment of short-term debt. Because $d_H = j_L(f)$, the slope of the green line is given by the marginal cost of long-term debt in the low state, which in equilibrium is equal to $-p_L(f_0)$. Thus, the market value of long-term debt increases by

$$p_H(f_0 + \Delta)(f_0 + \Delta) - p_H(f_0)f_0 \approx p_H(f_0)\Delta,$$

[8] If long-term debt has a coupon rate $c \neq r$, this expression becomes $\pi\left(c - y_L(f_0, d_H(f_0))p_L(f_0)\right)\Delta$.



(a) Marginal Effect on Tax Shield (b) Marginal Effect on Leverage

Figure 2: Benefit of long-term debt

whereas the value of short-term debt is reduced by

$$j_L(f_0) - j_L(f_0 + \Delta) \approx \underbrace{-j_L'(f_0)}_{p_L(f_0)} \Delta.$$

The net change in market leverage, illustrated by the wedge between the blue line and the green line in Figure 2(b), corresponds to

$$\text{Net change in market leverage} \approx \big(p_H(f_0) - p_L(f_0)\big)\Delta.$$

The crux of the matter is that while the marginal value of long-term debt determines the effect of the adjustment on long-term debt in the *high* state, the impact on short-term borrowing is determined by the marginal value in the *low* state.

Given each dollar of (market value) debt generates a benefit of $(\rho - r)$, the total benefit of the adjustment in long-term debt is

$$\text{Net benefit adjustment} \approx \underbrace{(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)}_{\text{difference in discounting}} \Delta + \underbrace{\pi r\big(1 - p_L(f_0)\big)}_{\text{tax shield}} \Delta.$$

The cost of this adjustment depends on the price impact of such trade, which is

$$\text{Cost of one period adjustment} \approx -p_H(f_0+\Delta)\Delta + p_H(f_0)\Delta = -p_H'(f_0)\,(\Delta)^2\,.$$

In equilibrium, the marginal benefit is equal to the marginal cost, so

$$\begin{aligned}
&(\rho - r)\big(p_H(f_0) - p_L(f_0)\big) + \pi r\Big(1 - p_L(f_0)\Big)\Delta \approx -p_H'(f_0)\,(\Delta)^2 \\
\implies\quad &\Delta = \frac{(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)}{-p_H'(f_0)} + \frac{\pi r\big(1 - p_L(f_0)\big)}{-p_H'(f_0)}.
\end{aligned}$$

The issuance function in other cases of Proposition 3 can be derived using similar heuristic arguments. Note that a minor change to the tax shield is that the interest rate becomes either $r+\eta$ or $r+\lambda$ instead of r – the effect coming from the benefit of additional leverage is different. Moreover, in these cases, an increment in long-term debt's market value is fully offset by the reduction in short-term debt. In this case, the borrower only issues long-term debt for tax shield benefits.

3.3.2 Repurchase of Long-Term Debt

Figure 3 illustrates the equilibrium issuance of long-term debt. Consistent with Proposition 3, the rate of long-term debt issuance in the high state g_H has a discontinuity at $f_\dagger$, which marks the switch between borrowing riskless and risky short-term debt. A notable feature is that the issuance could be negative in both states, implying that the borrower actually buys back long-term debt. This result differs from the literature on the leverage-ratchet effect, which predicts a borrower without commitment to debt issuance would never actively buy back the outstanding debt (DeMarzo and He, 2021; Admati et al., 2018). There are two reasons behind this result. First, for f close to $f_\dagger$, it might be that long-term debt is riskier in the upturn compared to the downturn, so $p_H(f) < p_L(f)$. Second, long- and short-term debt receives different tax shield benefits.

According to Proposition 3, for $f \in [0, f_\dagger]$, the borrower buys back long-term debt if $p_H(f) < p_L(f)$ in the absence of tax shields, which could happen for f sufficiently close to (but still below) $f_\dagger$. In this case, long-term debt's default risk is higher in the high state than in the low state. Intuitively, in the high state, a sequence of bad shocks could push f above $f_\dagger$, leading to default following a regime switch. In the low state, however, default happens if there is a disaster or if f reaches the default boundary f_L^b, which could be less likely. Note that this result may not hold when the cash flows in the low state are sufficiently risky, for example, when the arrival intensity of the disaster η is sufficiently high, as we show below.

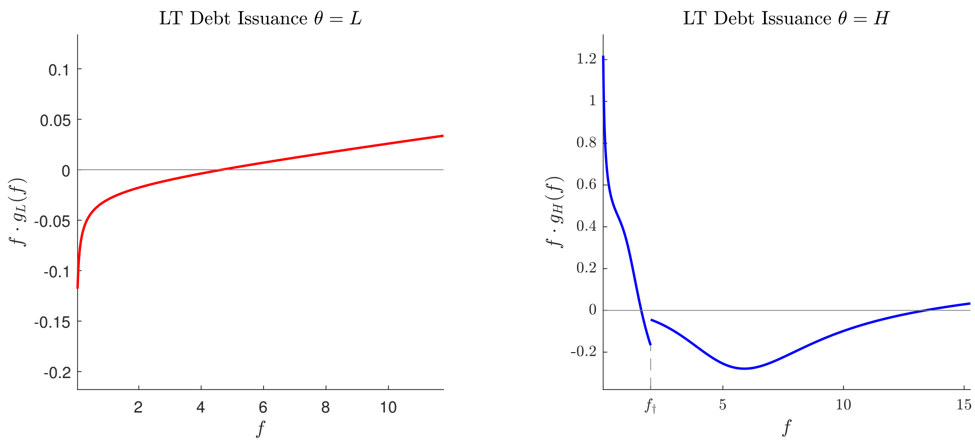


Figure 3: Long-term debt issuance

The baseline parameters in this figure are as follows: $\rho = 0.9$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\xi = 0.1$, $\lambda = 0.17$, $\eta = 0.02$, $\sigma = 0.3$, $\pi = 0.1$.

Corollary 1. *Without tax shields ($\pi = 0$), the borrower never repurchases its long-term debt if $\eta \geq \lambda$.*

Next, let us compare the tax shields from long- and short-term debt. For long-term debt, only the coupon payments are tax-deductible, so the tax-shield benefit is $\pi r F$. On the other hand, the interest payments are tax-deductible for the short-term debt, so the resulting tax-shield benefit is $\pi y D$. Therefore, whenever the short rate exceeds the coupon rate of the long-term bond, the borrower would prefer to repurchase long-term debt for tax-shield benefits. In the low state, this happens when $r + \eta > r/p_L(f)$. Similarly, this happens in the high state for $f > f_\dagger$ when $r + \lambda > r/p_H(f)$.

Remark 1 (Tax shields of long-term debt)**.** *We assume that only the coupon payments of the long-term debt are tax-deductible. This assumption follows the standard approach in the dynamic capital structure literature with debt rollovers (Leland and Toft, 1996; He and Xiong, 2012; DeMarzo and He, 2021). However, this is a simplification of the tax treatment of interest expenses in practice, where the tax shields apply to total interest expenses, including the amortization of any discount/premium at the time of issuance. The reason the existing literature does not consider the amortization of the discount/premium is that this would require keeping track of the discount/premium of all bonds issued in the past, making the analysis intractable. Our normalization of the coupon rate to the risk-free rate implies that the bond trades at a discount. Thus, the model's resulting tax-shield benefits of long-term debt are a lower bound of the tax benefit in practice. More* *generally, when the short-term rate is higher than the yield of long-term debt, the tax benefit of short-term borrowing is higher, so equity holders have incentives to substitute long-term debt for short-term debt.*

3.3.3 Mechanism: Commitment and Hedging

In our model, the borrower faces a time-inconsistency problem: she does not commit to future debt issuance policies and can dilute existing creditors. If the borrower only issues long-term debt, she always has incentives to issue more and dilute legacy long-term creditors. Moreover, as shown by Admati et al. (2018), the borrower financed by long-term debt never voluntarily reduces leverage even after negative shocks to the fundamentals. By contrast, short-term debt resolves the commitment problem because all outstanding debt must be rolled over continuously; that is, the existing short-term debt must be retired before issuing any new debt. In addition, following negative shocks, short-term debt forces the borrower to reduce the leverage, even though she does not commit to doing so. As DeMarzo (2019) shows, the borrower's problem when she only issues long-term debt is related to the Coase conjecture on the durable-goods monopoly (Coase, 1972). In our context, the borrower is the monopolist, and long-term debt is the durable goods. However, the issuance of short-term debt echoes the leasing solution to the Coase conjecture, initially proposed by Coase (1972) and later formalized by Bulow (1982). Short-term debt achieves commitment and prevents dilution because it needs to be continuously rolled over and repriced every time it is adjusted. In our model, if there is no regime switch, the borrower issues exclusively short-term debt.

To better explore the mechanism behind the issuance of long-term debt, we study a limiting model whereby the cash flow volatility converges to zero, i.e., $\sigma \to 0$. In this limiting model, the cash flows are (almost) deterministic conditional on the aggregate state. For simplicity, we also omit tax shields by letting $\pi = 0$ and assume $\mu_H + \xi > 0$ so that in the high state, f_t decreases in the absence of issuance. In the low state, there are two cases, depending on the comparison between $\mu_L + \xi$ and 0. If $\mu_L + \xi < 0$, the decline of cash flows exceeds the speed that the existing long-term debt matures, and long-term debt is risky. Specifically, f_t keeps increasing without any new issuance of long-term debt, eventually leading to the borrower defaulting. The next proposition summarizes the issuance policy, and detailed expressions for debt prices and the boundaries are provided in the appendix.

Proposition 4 (Limiting long-term debt issuance policy)**.** *Suppose* $\pi = 0$*,* $\lambda > \bar{\lambda}$*,* $\mu_L + \xi < 0$*, and*

$\mu_H + \xi > 0$. *Let*

$$\gamma = \frac{\hat{r} + \hat{\eta} - \mu_L}{-(\xi + \mu_L)} > 1 \quad \psi = \frac{\hat{r} + \hat{\lambda} - \mu_H}{\xi + \mu_H} > 0 \quad \delta = \frac{\rho + \lambda - \hat{r}}{\rho + \lambda - \hat{r} + (\mu_H - \mu_L)(\gamma - 1) - \hat{\eta}} \in (0, 1).$$

When $\sigma \to 0$*, the equilibrium issuance policy converges to:*

$$g_\theta(f) = \frac{\rho - r}{\rho + \lambda - r} \left[\frac{\eta(\xi + \mu_H) + (\mu_H - \mu_L)(r + \xi)}{r + \eta + \xi} + \eta \left(\frac{\xi + \mu_H}{\rho + \lambda + \xi} + \frac{-(\xi + \mu_L)}{r + \eta + \xi} \right) \left(\frac{f}{f_L^b} \right)^{-(\gamma - 1)} \right] \mathbb{1}_{\{f < f_\dagger, \theta = H\}}. \quad (23)$$

Results are different under $\mu_L + \xi > 0$. Now, if the borrower does not default right upon the downturn arrives, she will never default in the low state because the cash flow growth rate exceeds the speed that the existing long-term debt matures. The next proposition summarizes the issuance policy.

Proposition 5 (Limiting long-term debt issuance policy)**.** *Suppose* $\pi = 0$*,* $\lambda > \bar{\lambda}$*,* $\mu_L + \xi > 0$*, and* $\mu_H + \xi > 0$*. Consider the limit when* $\sigma \to 0$*. If* $\eta = 0$ *so there is no disaster risk, then the issuance policy converges to* $g_H(f) = g_L(f) = 0$*.*

Let us compare the results in Proposition 4 and 5. If $\mu_L + \xi < 0$, the borrower will eventually default in the low state and given so, the default option embedded in long-term debt is valuable because it delays default by allowing the borrower to pay later. This motivation leads to the issuance of long-term debt in the high state. By contrast, if $\mu_L + \xi > 0$, the borrower can only default in the low state due to the disaster risk. Without the disaster, the borrower defaults immediately after the regime shift or never defaults, so the longer maturity in the embedded default has zero value. As a result, the borrower does not issue long-term debt in the high state.[9]

These results also explain the long-term debt issuance policy in Proposition 3. While short-term debt resolves the dilution problem, it needs to be fully repaid at any instant; otherwise, the borrower is forced to default immediately. In other words, the default option embedded in short-term debt expires immediately. By contrast, long-term debt matures in the future, so the default option embedded also expires later. In downturn, the borrower never defaults if it only uses short-term debt. Therefore, there is no value associated with the default option and, thus, no need to issue long-term debt. In the high state when $f > f_\dagger$, the default option is also not valued because the borrower finds it optimal to default immediately after the transition.

[9] If $\eta > 0$, for any $f_t \in [0, f_\dagger)$, long-term debt immediately jumps such that $f_{t+} = f_\dagger$.

Matters are different in the high state when short-term debt is riskless. In this case, default does not occur after the regime switch, but the firm value experiences a discontinuous jump. Long-term debt offers the borrower an option to postpone default in this situation. After the regime shift, the default option becomes more valuable, so the price of the outstanding long-term debt goes down because the price of the debt is equivalent to the discounted value of risk-free payments subtracting the value of the default option. Figure 4 provides a graphical illustration of the impact of the state transition. Note that a transition from the high to the low state reduces the equity value by $j_H(f) - j_L(f)$ and the long-term debt price by $p_H(f) - p_L(f)$. However, it leaves the value of short-term debt intact.[10]

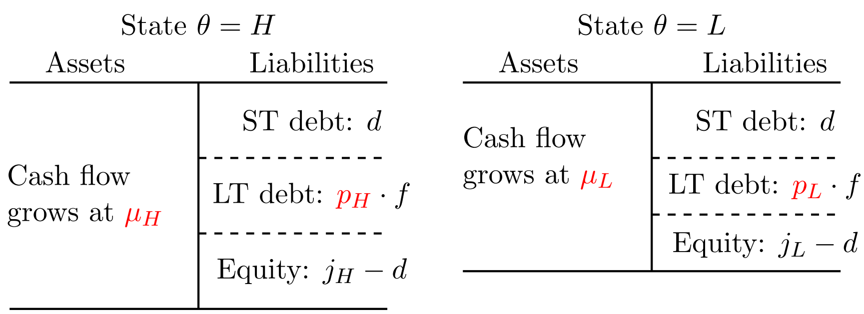


Figure 4: Balance sheet upon the state transition without immediate default

So far we have focused on the limiting model with $\sigma \to 0$. For the case of $\sigma > 0$, there is a second related benefit associated with long-term debt. Specifically, by postponing default after the downturn has arrived, long-term debt also increases the chances that the borrower can avoid default. In other words, long-term debt allows the borrower time to recover after the downturn.

To summarize, the maturity choice is determined by the trade-off between commitment and hedging. Short-term debt resolves dilution and forces the borrower to reduce leverage after negative shocks; long-term debt offers a better hedge against the downside risk by providing an option to default later and allowing time to recover. This insight relates to the previous work in the fiscal-policy literature that emphasizes how long-term debt allows for more state contingency (Angeletos, 2002).

3.3.4 The Maturity of Long-term Debt

We start by considering the impact of ξ on $f_\dagger$, which captures the incentives to issue long-term debt for purposes other than a tax shield. As shown in Figure 1, this threshold is determined by

[10]Given the presence of the disaster shock, all results in our model continue to hold under $\mu_H = \mu_L$, because the firm value experiences a discontinuous then the disaster risk becomes more likely following the state transition.

the trade-off between maximizing the benefits of leverage versus the bankruptcy cost. Thus, any parameter that increases the continuation value after the regime switch (thus, increasing bankruptcy cost) should increase $f_\dagger$, whereas parameters that increase debt capacity in the high state should decrease $f_\dagger$. Figure 1 shows that, keeping $j_L(f)$ constant, an increase in $j_H(f)$ reduces $f_\dagger$. Hence, it is immediate that $f_\dagger$ is increasing λ and decreasing in μ_H. Figure 5 presents some further comparative statics. The left panel shows that $f_\dagger$ increases in μ_L. Intuitively, a higher growth rate of cash flows in the low state increases the expected bankruptcy cost upon regime switching and makes risky short-term debt more costly. The right panel confirms the earlier result that $f_\dagger$ increases in λ. A comparison across different curves in both panels shows that $f_\dagger$ decreases in ξ or equivalently increases in the maturity of long-term debt. Intuitively, debt with longer maturity is more sensitive to changes in firm value. As a result, the embedded default option is more valuable and becomes more attractive to the borrower.

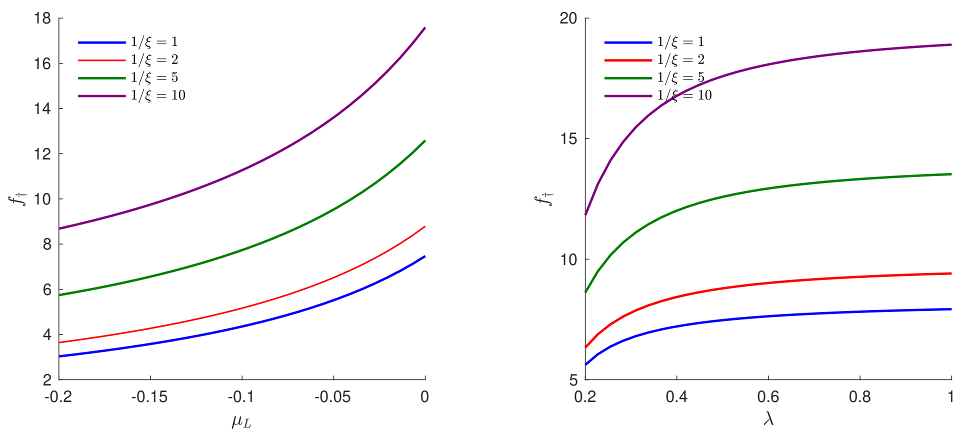


Figure 5: Comparative statics $f_\dagger$.

The baseline parameters in this figure are as follows: $\rho = 0.1$, $r = 0.05$, $\mu_H = 0.1$, $\mu_L = 0$, $\lambda = 0.5$, $\eta = 0.1$, $\sigma = 0.5$, $\pi = 0$.

The left panel Figure 6 describes how the maturity of long-term debt affects firm value, and the results show some interesting non-monotonic patterns. Intuitively, when the maturity of long-term debt gets longer, the dilution problem gets more severe. Meanwhile, the embedded default option becomes more valuable, so the overall effect can be non-monotonic. The right panel shows that in the economy with only long-term debt (such as in DeMarzo and He (2021) and Proposition 10 in Section 4), the firm value decreases with maturity because longer-maturity debt is subject to more future dilution.

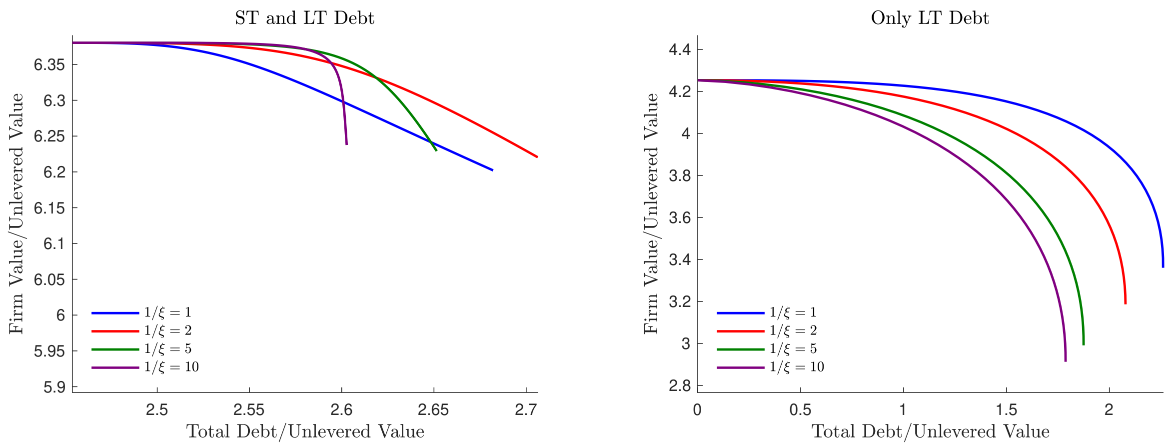


Figure 6: Comparative statics firm value and debt capacity.

The baseline parameters in this figure are as follows: $\rho = 0.075$, $r = 0.05$, $\mu_H = 0.2$, $\mu_L = 0.1$, $\lambda = 0.25$, $\eta = 0.1$, $\sigma = 1$, $\pi = 0$. The figures consider the relation between total debt and firm value. They are constructed by considering the upper branch of the graph $\{(p_H(f)f + d_H(f), p_H(f)f + j_H(f))\}$. In the case of both short- and long-term debt, we restrict attention to the interval $(0, f_\dagger)$. On $(f_\dagger, f_H^b)$, both total debt and firm value are decreasing in f.

3.3.5 Initial Debt Issuance

So far, our analysis has shown that for a given level $f > 0$, the borrower could have incentives to issue/repurchase some long-term debt. However, it remains a question whether an initially unlevered borrower would issue any long-term debt. The next proposition provides sufficient and necessary conditions for the issuance of long-term debt when $f = 0$.

Proposition 6. *In the high state, an initially unlevered borrower will issue long-term debt, that is* $\lim_{f \to 0} g_H(f)f > 0$, *if only if* $\eta > 0$ *and* $p'_H(0) > -\infty$. *The latter condition is satisfied if and only if the coefficients* γ *and* ϕ *in Proposition 2 are greater or equal than 2.*

Without the disaster shock, an unlevered borrower would never issue long-term debt. Intuitively, if there is no disaster, a marginal unit of long-term debt is riskless for the unlevered borrower in both the upturn and the downturn. Therefore the option to default later is not needed. The second condition on $p'_H(0) > -\infty$ is needed so that the price impact from issuing a marginal unit of long-term debt is not too high for the unlevered borrower.

3.4 Comparative Statics and Debt Dynamics

The analytical solutions allow us to conduct comparative statics and study debt structure dynamics. For simplicity, we restrict attention to the model presented in section 3.3.3, under the limit $\sigma \to 0$ and no tax shields ($\pi = 0$). As shown by (22), the borrower only issues long-term debt in the high state when $f \in [0, f_\dagger]$. In this region, several comparative static results immediately follow.

Proposition 7. *Consider the limiting model ($\sigma \to 0$) without tax shields ($\pi = 0$). Under the parametric conditions in Proposition 4, for any $f \in (0, f_\dagger)$, the issuance function $g_H(f)$ is:*

- *Increasing in ρ, η, and μ_H, and decreasing in λ.*
- *If $\eta > 0$, there is $\tilde{f} \in (0, f_\dagger]$ such that $g_H(f)$ is increasing in μ_L for $f < \tilde{f}$ and decreasing in μ_L for $f > \tilde{f}$. If $\eta = 0$, $g_H(f)$ is decreasing in μ_L for all $f \in [0, f_\dagger]$.*

The results in Proposition 7 are straightforward. Higher ρ increases the benefits of leverage; higher η and μ_H both increase the difference between $p_H(f) - p_L(f)$ for any given f. Therefore, the borrower issues more long-term debt. Meanwhile, when λ gets higher, $p_L(f)$ stays unchanged, whereas $p_H(f)$ gets lower. As a result, the borrower should issue less long-term debt.

The closed-form solution for the issuance policy allows us to fully characterize the dynamics of f_t. We focus on the case $\mu_L + \xi < 0$.

Proposition 8. *In the limiting model ($\sigma \to 0$) without tax shields ($\pi = 0$). Under the parametric conditions in Proposition 4, the ratio of long-term debt to cash flows f_t follows the piecewise-deterministic process*

$$\frac{df_t}{dt} = \begin{cases} -(\xi + \mu_L) f_t & \text{if } \theta_t = L \\ -\frac{\nu}{\gamma - 1}\left[1 - \kappa \left(\frac{f_L^b}{f_t}\right)^{\gamma-1}\right] f_t & \text{if } \theta_t = H \text{ and } f_t \in (0, f_\dagger) \\ -(\xi + \mu_H) f_t & \text{if } \theta_t = H \text{ and } f_t \in (f_\dagger, f_H^b), \end{cases}$$

where $\nu \cdot \kappa \geq 0$ and $\nu > 0$ only if

$$\frac{\xi + \mu_H}{-(\xi + \mu_L)} \frac{\lambda}{\rho - r} > \frac{r + \xi}{r + \eta + \xi}$$

If $\nu > 0$, let $f^\intercal \equiv \kappa^{\frac{1}{\gamma-1}} f_L^b$ be the unique solution to $g_H(f^\intercal) = \mu_H + \xi$. In the high state:

- *If $\nu < 0$ or $f^\intercal > f_\dagger$, f_t converges to $f_\dagger$.*

- *If $\nu \geq 0$ and $f^{\intercal} < f_{\dagger}$, f_t converges to $f^{\intercal}$. In this case,*
  - *The speed of adjustment ν is increasing in η, λ, μ_H, and μ_L, and it is decreasing in ρ. It is increasing in ξ if and only if*

$$\xi > -\mu_L - \sqrt{\frac{\lambda\,(\mu_H - \mu_L)\,(r + \eta - \mu_L)}{\rho + \lambda - r}}.$$

  - *The target $f^{\intercal}$ is increasing in ρ, η, and decreasing in λ and μ_H.*

One can interpret $f^{\intercal}$ as the target ratio of long-term debt to cash flow. Figure 7 illustrates the dynamics when $f^{\intercal} \in (0, f_{\dagger})$, with the left and right panels, respectively describe the evolution of long- and short-term debt. Starting in the high state, the path of f_t converges towards the target $f^{\intercal}$ until the regime switches. The convergence path for $f_0 < f^{\intercal}$ (red path in the figure) is straightforward. For $f_0 > f_{\dagger}$ (blue path in the figure), the equity holder initially borrows risky short-term debt and retires maturing long-term debt until $f_{\dagger}$ (this corresponds to time $\tau_{\dagger}$ in the figure). Once this threshold is reached, the borrower reduces the amount of short-term debt and starts to issue long-term debt. After the regime shift, which occurs at τ_L, the borrower stops issuing long-term debt and only borrows short-term, and f_t increases until the firm eventually defaults.

If either $\nu < 0$ or $f^{\intercal} > f_{\dagger}$, then f_t converges towards $f_{\dagger}$, after which it stays there until the state transition.[11]

3.5 Empirical Implications

Stock versus flow Static models of debt maturity tend to make the same predictions regarding the stock (outstanding) and the flow (issuance) of debt. One merit of constructing a dynamic model of debt maturity is to differentiate between the two. Our paper implies that the relationship between credit risk and maturity is dramatically different depending on whether we consider outstanding debt or new issuance as the dependent variable. For example, in the upturn, credit risk is high when

[11] This last case presents some technical complications because there is a difference between the limit equilibrium when $\sigma \to 0$ and the equilibrium in a model with $\sigma = 0$. At $f_{\dagger}$, we have that $g_H(f_{\dagger}-) > \mu_H + \xi$ and $g_H(f_{\dagger}+) < \mu_H + \xi$. A classical solution for the path of f_t only exists if we set $g_H(f_{\dagger}) = \xi + \mu_H$ – so the threshold $f_{\dagger}$ is absorbing. If $\sigma = 0$, this policy is consistent with the equilibrium price $p_H(f_{\dagger}) = j'_H(f_{\dagger})$ only if the probability of defaulting upon a transition is positive but less than one. We can construct an equilibrium with this property by setting $d_H(f_{\dagger}) = j_L(f_{\dagger})$ and specifying a mixed strategy of default (upon a transition to the low state) so the price of long-term debt satisfies no-arbitrage at $f_{\dagger}$. Such construction is possible because at $f_{\dagger}$ the equity holders are indifferent. When $\sigma > 0$, the particular issuance policy at $f_{\dagger}$ is not a problem because f_t fluctuates around the threshold $f_{\dagger}$. If we set $g_H(f_{\dagger}) = g_H(f_{\dagger}-)$, the existence result in Nakao (1972) implies the existence of a unique strong solution to the SDE for f_t for any $\sigma > 0$. We can interpret the path of f_t in the limit as an approximation for small $\sigma > 0$ where in the high state f_t mean reverts to $f_{\dagger}$.

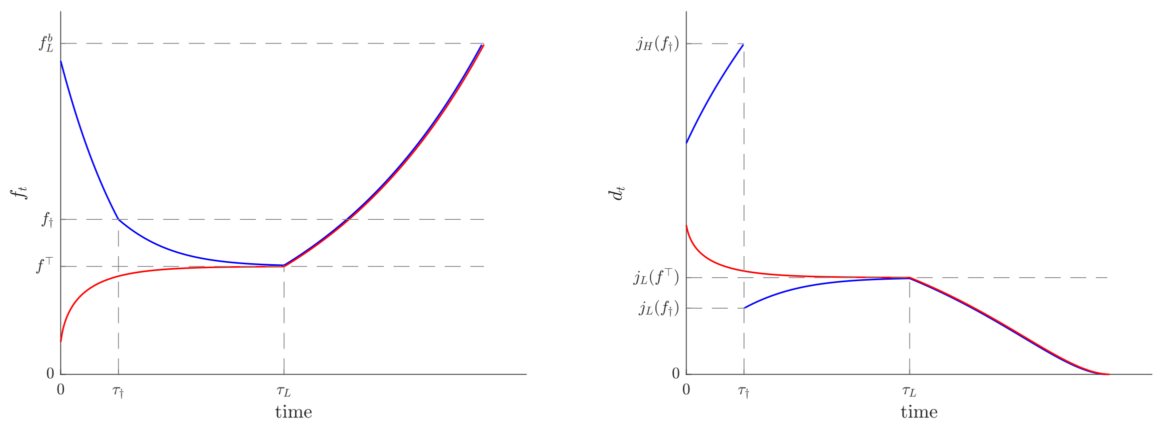


Figure 7: Sample path f_t and d_t for different values of f_0.

The parameters in this figure are as follows: $\rho = 0.2$, $r = 0.1$, $\mu_H = 0.045$, $\mu_L = -0.75$, $\xi = 0.5$, $\lambda = 0.4$, $\eta = 0.1$. The shock arrives at $\tau_L = 5$, the issuance threshold is $f_\dagger = 0.78$, the default threshold in the low state is $f_L^b = 1.66$, and the target is $f^\intercal = 0.54$. The blue corresponds to an initial value $f_0 = 1.57$, while the red line corresponds to an initial value $f_0 = 0.16$.

the borrower has a significant amount of long-term debt outstanding (i.e., f is very close to f_H^b). Thus, there is a positive relationship between credit risk and the maturity of outstanding long-term debt. However, close to the default boundary, newly issued debt is exclusively short-term, so there is a negative relation between credit risk and the maturity of the newly issued debt.[12]

Gradual and sudden defaults. Our model generates some novel empirical implications on debt maturity structure and defaults. In particular, the borrower defaults in two circumstances. First, the ratio of long-term debt to cash flow f_t gets sufficiently high such that the borrower approaches the default boundary gradually from below ($f_t \uparrow f_L^b$ or $f_t \uparrow f_H^b$). In this case, default occurs *gradually* after the deterioration of the fundamental cash flows relative to the outstanding long-term debt. In the second circumstance, default occurs after a transition from the upturn to the downturn, and the borrower has taken too much risky short-term debt before the transition. In this case, default follows *suddenly* upon the state transition. Our model implies that defaults in the real world can be classified into more gradual and more sudden ones. Moreover, gradual defaults are driven by the accumulation of long-term debt relative to the fundamental cash flows. In contrast, sudden defaults are triggered by a combination of a large shock to the borrower's fundamentals and the excessive short-term debt taken prior to the shock. These can be implications for future

[12]Note that the binary-state setup and the fact that the low state is absorbing imply that results in the upturn should be interpreted more broadly.

empirical tests.

Cross-sectional implications. We have shown that with only small shocks (diffusion risks), the borrower borrows exclusively short-term debt. By contrast, the borrower issues a combination of long- and short-term debt when there are large downside risks (such as the regime switch and the jump risks introduced in subsection 5.3 below) to hedge. Cross-sectionally, one should observe that firms more exposed to large downside shocks use more long-term debt. To some extent, one can interpret the small shocks as diversifiable risks, whereas large negative shocks as non-diversifiable. Under such an interpretation, our paper implies that firms with more non-diversifiable risk use more long-term debt.

Time series implications. In the main model, we have assumed that the low state $\theta_t = L$ is absorbing. If we interpret the changes in regime as business-cycles, it is natural to assume that states are transitory. We can extend the model to consider this situation. Such a model is solved in closed forms in Internet Appendix B.3, and we explore a few patterns as follows.

If we interpret the state transition as business-cycle shocks, our result implies that market leverage is countercyclical, which is consistent with the evidence provided in Adrian and Shin (2014). Our result that long-term debt is only issued in the high state $\theta_t = H$ immediately implies the borrower's debt maturity is pro-cyclical, if one interprets these two states as business-cycle frequency boom and bust. This prediction is consistent with the findings in Chen et al. (2021).

In Figure 8, we simulate a sample path and plot the time series of the cash-flow rate and debt maturity. Here, debt maturity is defined as the average maturity of total debt outstanding weighted by their book value:

$$\text{Maturity}_t \coloneqq \frac{F_t}{F_t + D_t}\frac{1}{\xi}.$$

In the absence of a regime shift, the maturity of debt seems to move in the opposite direction to cash flows. In other words, the borrower expands the average debt maturity following a negative Brownian shock to X_t. Intuitively, this pattern holds because, after a negative Brownian shock to X_t, the borrower immediately rolls over less short-term debt, whereas she only reduces long-term debt outstanding gradually over time. Meanwhile, when the regime shifts and the downturn arrives, the borrower exclusively borrows short-term debt and the average maturity goes down.[13] Therefore, our model implies that within a regime, cash flows and debt maturity negatively comove with each other. However, if we compare across regimes, the regime with higher cash-flow growth

[13]This result depends on the binary-state setup, where no additional downside risk exists in the low state. With more than two states, the borrower may still issue long-term debt in the low state. The broader message is the transition to a worse state, the borrower may only issue short-term debt for a while.

rates has on average longer debt maturity.

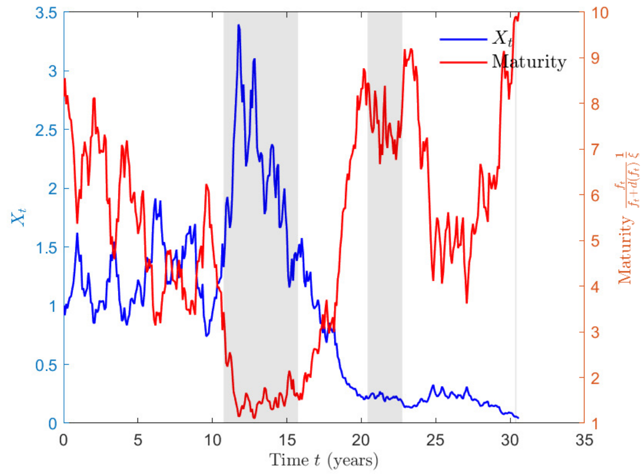


Figure 8: Sample path of leverage and maturity

This figure simulates the sample path of one firm and plots the time series of X_t, maturity, and market leverage, with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\eta = 0.05$.

Anecdotal examples We can illustrate the predictions of our model by considering real-world examples. Figure 9 plots debt maturity structure from 1990 onward for the Pacific Gas and Electric Company (PG&E) and General Motors (GM). PG&E entered bankruptcy twice in the last two decades. It initially entered Chapter 11 bankruptcy on April 6, 2001, and emerged from bankruptcy in April 2004. In 2019, it filed for bankruptcy on January 29 again and successfully exited on June 20. The left panel plots the maturity of newly issued long-term debt, weighted by the offering amount. The red-shaded areas marked the two bankruptcies, and the gray areas are NBER recessions. Consistent with our model, the newly-issued bonds have shorter maturities in the NBER recessions and shortly before the bankruptcies.[14] The right panel displays similar patterns for GM, which filed for bankruptcy on June 8, 2009.

[14] The maturity of newly-issued debt was also short in 2011, which might be due to the 2010 San Bruno explosion: PG&E was on probation after being found criminally liable in the fire. In the context of our model, the San Bruno explosion can be thought of as a transition from a high to a low state after a Poisson shock.

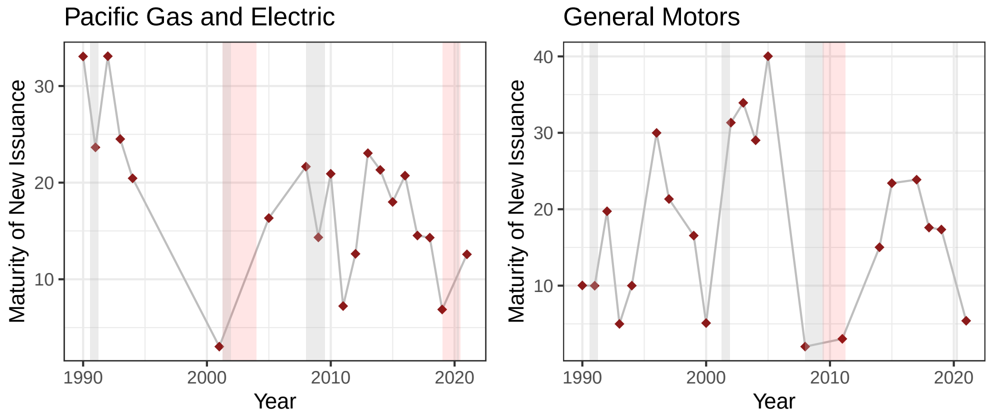


Figure 9: Time series maturity of new issuance for PG&E and GM

This figure shows the average maturity of bonds in a year (weighted by market value at issuance) and the share of long-term debt maturing within one year for Pacific Gas & Electric (PG&E) and General Motors (GM). The gray-shaded area indicates NBER recession, whereas the red-shaded area indicates periods over which these companies were in bankruptcy procedures. Source: Mergent FISD.

4 Equilibrium with Only Long- or Short-Term Debt

As a comparison, we describe in this section the equilibrium when only long- or short-term debt is allowed. The case with only short-term debt relates to Abel (2018), while the case with only long-term debt corresponds to DeMarzo and He (2021) (adapted to our setting with regime shift and the disaster shock).

Proposition 9 (Equilibrium with only short-term debt)**.** *If only short-term debt is allowed, the unique equilibrium is the following. The value function is given by $J^s_\theta(X) = Xj^s_\theta$ and the amount of short-term debt is $D^s_\theta(X) = Xd^s_\theta$.*

1. *In the low state L,*
$$j^s_L = \frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}.$$
Short-term debt is $d^s_L = j^s_L$. The borrower only defaults upon the disaster shock, so the short rate is $y_L = r+\eta$.

2. *In the high state H, the value function is*
$$j^s_H = \max\left\{\frac{1-\pi}{\hat{r}+\hat{\lambda}-\mu_H}, \frac{1-\pi}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}-\mu_L}\right)\right\}.$$

Let $\bar{\lambda}$ be the threshold in Proposition 2.

- *If $\lambda \leq \bar{\lambda}$, short-term debt is $d_H^s = j_H^s$. The borrower defaults as soon as θ switches from H to L, and the short rate is $y_H = r + \lambda$.*
- *If $\lambda > \bar{\lambda}$, $d_H^s = j_L^s$, the borrower never defaults, and $y_H = r$.*

3. *The total firm value is higher than in the case where the borrower can issue both types of debt, that is, $j_\theta^s \geq j_\theta(f) + p_\theta(f)f$, $\forall \theta \in \{L, H\}$*

If only short-term debt is allowed, the commitment problem in debt issuance no longer exists. Instead, capital structure choice is a static problem and follows the standard trade-off theory whereby equity holders balance cheap debt against costly bankruptcy. Interestingly, the total firm value is higher if the borrower is prohibited from issuing long-term debt. This result may appear paradoxical, given the earlier discussion on the hedging benefits of long-term debt. Due to the borrower's lack of commitment to future issuance policies, long-term debt prices drop to the level at which the hedging benefits are completely depleted. In other words, given the pricing function of long-term debt $\{p_H(f), p_L(f)\}$, the borrower always has incentives to borrow long to explore hedging benefits. However, due to a lack of commitment, the benefits from hedging are completely dissipated. Additionally, long-term debt leads to more defaults on the equilibrium path.

Proposition 10 (Equilibrium with only long-term debt)**.** *If only long-term debt is allowed, the unique equilibrium is the following.*

1. *In state L, the value function is*

$$v_L^\ell(f) = \frac{1-\pi}{\rho+\eta-\mu_L} - \frac{\hat{r}+\xi}{\rho+\eta+\xi} f + \frac{1-\pi}{\rho+\eta-\mu_L}\frac{1}{\gamma_\ell - 1}\left(\frac{f}{f_L^{b\ell}}\right)^{\gamma_\ell},$$

where γ_ℓ is given in (A.23) *and the default boundary is given in* (A.22)*.*

2. *In state H, the value function is*

$$v_H^\ell(f) = u_0^\ell(f) - u_0^\ell(f_H^{b\ell})\left(\frac{f}{f_H^{b\ell}}\right)^{\phi},$$

and

$$u_0^\ell(f) = \frac{1-\pi}{\rho+\lambda-\mu_H}\left(1+\frac{\lambda}{\rho+\eta-\mu_L}\right) - \frac{\hat{r}+\xi}{\rho+\lambda+\xi}\left(1+\frac{\lambda}{\rho+\eta+\xi}\right) f$$
$$+ \delta^\ell \frac{1}{\gamma_\ell - 1}\frac{1-\pi}{\rho+\eta-\mu_L}\left(\frac{f}{f_L^{b\ell}}\right)^{\gamma_\ell},$$

where

$$\delta^{\ell} = \frac{\lambda}{\lambda - \eta + (\mu_H - \mu_L)(\gamma_\ell - 1)}.$$

The borrower defaults upon the state transition if and only if $f > f_L^{b\ell}$.

3. *In both states* $\theta \in \{L, H\}$*, the debt price is* $p_\theta^\ell = -v_\theta^{\ell\prime}$*, and the issuance function follows*

$$g_\theta^\ell = \frac{\pi r + (\rho - r)\, p_\theta^\ell}{-f p_\theta^{\ell\prime}}.$$

As in DeMarzo and He (2021), equity holders do not reap the benefits of issuing cheaper debt without commitment to the issuance policy. In equilibrium, long-term debt is issued smoothly. The next proposition compares the equilibrium with only long-term debt with the one in which the borrower can issue both types of debt.

Proposition 11 (Comparison of equilibrium)**.**

1. *The total firm value is lower than the case in which the borrower can issue both types of debt, that is,* $v_\theta^\ell(f) + p_\theta^\ell(f)\, f \le j_\theta(f) + p_\theta(f)\, f$*,* $\forall f$*.*

2. *The default boundary is higher in the presence of short-term debt. That is,* $f_\theta^b > f_\theta^{b\ell}$*.*

3. *In the low state, the price of debt is higher in the presence of short-term debt. That is,* $p_L^\ell(f) < p_L(f)$*,* $\forall f \in [0, f_L^{b\ell}]$*. In the high state, if* $\rho > r + \lambda$*, there are thresholds* $\underline{f} \in [0, f_\dagger]$ *and* $\overline{f} \in [f_\dagger, f_H^{b\ell}]$ *such that* $p_H^\ell(f) \le p_H(f)$ *on* $[0, \underline{f}] \cup [\overline{f}, f_H^{b\ell}]$*.*

Figure 10 illustrates the comparison. The top panels show that the firm value is higher in both states when the borrower can issue both types of debt. Intuitively, borrowing short mitigates dilution and forces the borrower to reduce leverage following negative shocks. Consequently, the default boundaries are higher. The bottom-left panel compares long-term debt's price, which can be either higher or lower when the borrower can issue both types of debt. On the one hand, the availability of short-term debt increases the firm value and pushes up the default boundary. On the other hand, in the high state, when f gets above $f_\dagger$, short-term debt introduces the rollover risk, i.e., the borrower may default following the state transition from high to low. Without short-term debt, the borrower will not default following the same transition. Therefore, for f exceeds $f_\dagger$ but is still far from $f_L^{b\ell}$, the price of long-term debt is lower when the borrower can issue both types of debt. Finally, the bottom-right panel shows that in the low state, long-term debt prices are always higher with two types of debt.

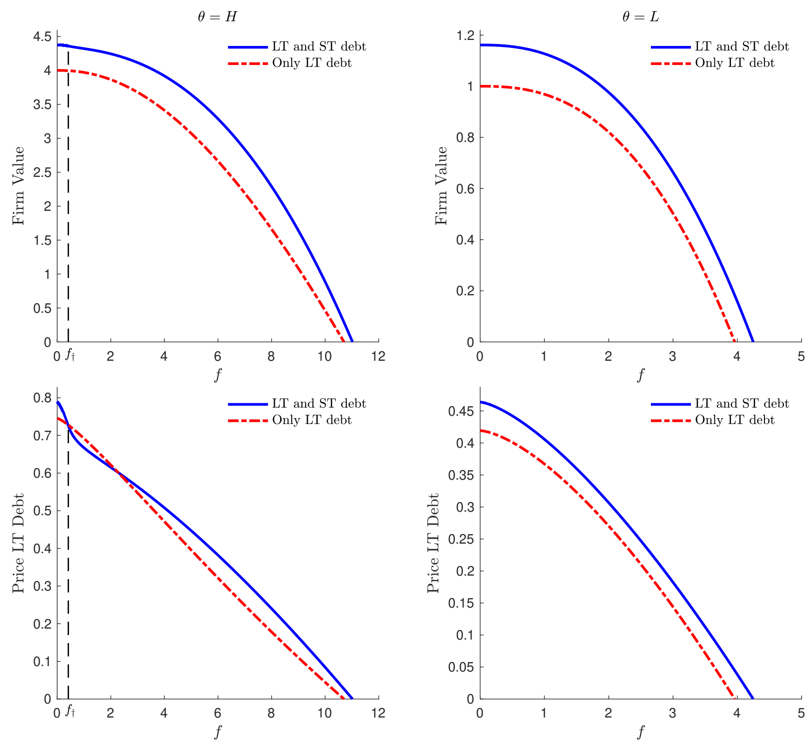


Figure 10: Comparison of equilibrium.

The baseline parameters in this figure are as follows: $\rho = 0.1$, $r = 0.05$, $\mu_H = 0.1$, $\mu_L = -0.1$, $\lambda = 0.3$, $\eta = 0.7$, $\sigma = 1$, $\pi = 0.1$, $\xi = 0.5$. This figure compares the equilibrium firm value (top panels) and the price of long-term debt (bottom panels) when both types of debt are available and when only long-term debt is available.

5 Extensions and Robustness

We consider the several extensions to further illustrate the main economic mechanisms. Through this section, we restrict attention to the case without tax shields, i.e., $\pi = 0$. In subsection 5.1, we explore the role of market incompleteness by allowing the borrower to issue financial instruments to partially hedge the regime shift. In subsection 5.2, we study short-term debt renegotiation and show that short-term debt dominates long-term debt when renegotiation is frictionless. In subsection 5.3 we show that results are similar if large shocks are instead modeled as jumps to the cash flows.

5.1 The Role of Market Incompleteness

In this subsection, we introduce derivative contracts that allow the borrower to insure against the regime shift. Results show that these derivative contracts can partially substitute for long-term debt.

Suppose a short-term derivative contract is written on a variable $\hat{\theta}_t$ that is correlated with θ_t. In the absence of a regime switch, $\hat{\theta}_t$ remains a constant. However, if there is a regime switch, then $\hat{\theta}_t$ switches with probability $q \in [0, 1]$ and remains a constant otherwise. The case of $q = 1$ corresponds to perfect insurance. If so, markets are dynamically complete, and shocks to θ_t can be perfectly insured.

The buyers of this derivative pay a premium $\varsigma \cdot dt$ over the period $[t, t + dt)$, in exchange of a payment of \$1 at time $t + dt$ if there is a change in $\hat{\theta}_t$. The expected payoff of this contract over the period $[t, t + dt)$ is $e^{-rdt}q(1 - e^{-\lambda dt})$, so no arbitrage implies

$$\varsigma = \lim_{dt \to 0} \frac{e^{-rdt}(1 - e^{-\lambda dt})q}{dt} = \lambda q.$$

Let z_t denote the number of contracts bought by the equity holder at time t. The analysis in the low state is unchanged. In the high state, upon the regime shifting, the borrower receives z_t with probability q and nothing with probability $1 - q$. In the first case, default occurs if and only if $j_L(f_{t-}) + z_{t-} \leq d_{t-}$, whereas in the second case, default happens if and only if $j_L(f_{t-}) \leq d_{t-}$. Consistent with the assumption of zero recovery, we assume that in the event of default, the payment from the derivative contract cannot be used to pay long-term creditors. Given the position z_t, the short rate is given by

$$y_H(f,d,z) = \begin{cases} r & \text{if } d \le j_L(f) \\ r + \lambda(1-q) & \text{if } j_L(f) < d \le j_L(f) + z \\ r + \lambda & \text{if } d > j_L(f) + z. \end{cases}$$

In the high state, the HJB equation is

$$(\rho + \lambda - \mu_H)\, j_H(f) = \max_{d \in [0, j_H(f)], z \ge 0} 1 - (r + \xi)\, f - q\lambda z + (\rho + \lambda - y_H)\, d - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) + \lambda q \max\{j_L(f) + z - d, 0\} + \lambda(1-q)\max\{j_L(f) - d, 0\}. \quad (24)$$

The following Lemma characterizes the solution to the maximization problem in equation (24)

Lemma 1. *The optimal short-term debt and hedging policy $d_H(f), z(f)$ is*

$$d_H(f) = \begin{cases} j_L(f) & \textit{if } j_L(f) \ge \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) & \textit{Otherwise.} \end{cases}$$

$$z(f) = \begin{cases} 0 & \textit{if } j_L(f) \ge \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) - j_L(f) & \textit{Otherwise.} \end{cases}$$

When $d_H(f) = j_L(f)$, the firm will survive the regime switch anyway, so insurance is unnecessary. By contrast, when $d_H(f) = j_H(f)$ so that short-term debt is risky, the equity holder buys enough derivative contracts to insure against the regime shift. The equilibrium takes a similar form as the one in section 3. The amount of short term debt is $d_H(f) = j_L(f)$ when $f < f_\dagger$, and $d_H(f) = j_H(f)$ if $f > f_\dagger$, with the threshold $f_\dagger$ given by the indifference condition

$$f_\dagger = \min\left\{f \ge 0 : (\rho + \lambda(1-q) - r) j_L(f) \le (\rho - r) j_H(f)\right\}.$$

It immediately follows from this indifference condition that when $q = 1$, the threshold $f_\dagger$ is equal to zero. In other words, the borrower does not issue any long-term debt if she can perfectly insure against the regime shift. We leave the rest of the analysis in Appendix B.1.

Proposition 12. *The firm never issues long-term debt if markets are complete. That is, if $q = 1$, then $g_\theta(f) = 0$ for all f. If $f_\dagger > 0$, then $f_\dagger$ is decreasing in q.*

The first part of this proposition shows that long-term debt is not issued if the borrower can fully insure against the regime switch using the derivative contracts. Long-term debt is more costly due to future dilution and potential bankruptcy. Therefore, derivative contracts serve as a substitute for long-term debt. The second part of the result shows this substitution between the derivative contracts and long-term debt is monotonic: there is more substitution as the derivative contract offers better insurance.

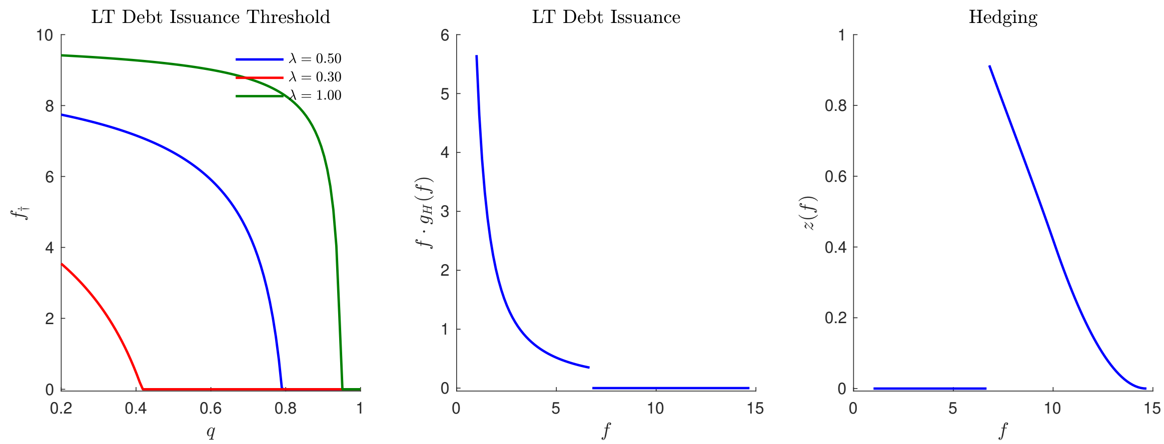


Figure 11: Debt Issuance under Derivative Contracts.

The baseline parameter values are: $\rho = 0.1$, $r = 0.05$, $\mu_H = 0.1$, $\mu_L = 0.05$, $\sigma = 0.5$, $\xi = 0.1$, $\lambda = 0.5$, $\eta = 1$, $q = 0.5$.

Figure 11 describes the results in this subsection. The left panel confirms that as q goes up, $f_\dagger$ decreases and eventually gets to zero. Comparing across lines, it is clear that when λ gets higher so that the regime shift is more likely, $f_\dagger$ is also higher. The middle and right panels further illustrate the decision between issuing long-term debt and buying the derivative contracts, with the two being substitutes.

5.2 Restructuring of Short-Term Debt

Our benchmark model has established the hedging benefits of long-term debt. Meanwhile, bankruptcy could also be avoided if the short-term debt can be restructured when the borrower gets distressed. In this subsection, we show that the restructuring of short-term debt reduces the issuance of LT debt.

To model renegotiation, we distinguish between default and bankruptcy. Whenever the borrower announces a default, and there is outstanding short-term debt, short-term debt can be restructured with some probability. Notice that in both states $\theta \in \{H, L\}$, when the borrower defaults at f^b_θ, the amount of short-term debt is zero. Therefore, renegotiating short-term debt is only relevant upon a regime shift from H to L. The renegotiation game goes as follows. With probability $1 - \alpha$, it is impossible to renegotiate, and the firm goes bankrupt. With probably α, the firm enters into a renegotiation process. In this case, the equity holder makes the offer with probability β and short-term creditors with probability $1-\beta$. If the short-term creditors make the offer, and this offer is rejected, the firm goes bankrupt. If the equity holder makes the offer and the offer is rejected, she can still choose between repaying the original short-term debt and bankruptcy. Figure 12 presents the timing of events.

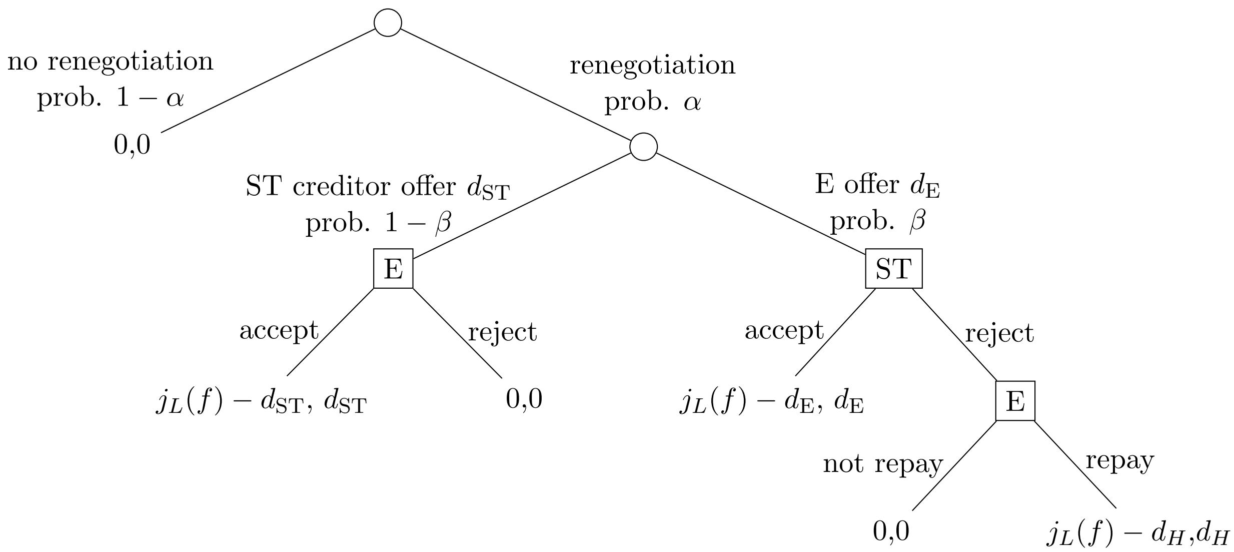


Figure 12: Renegotiation process.

The game tree illustrates the renegotiation process. If the firm defaults, renegotiation is triggered with probability α; otherwise, there is bankruptcy. In the event of renegotiation, the equity holder gets to make an offer with probability β, in which case she offers d_{E}. Otherwise, the offer is made by short-term creditors, in which case they offer d_{ST}. In the tree, E indicates nodes where the equity holder moves and ST nodes where short-term creditors move. At the end of the tree, the first coordinate indicates the payoff to the equity holder, and the second coordinate indicates the payoff to short-term creditors.

Following the state transition, the borrower receives $j_L(f)-d_H$ if she does not default. If there is a default, with probability $1-\alpha$, there is no renegotiation, and she receives zero. With probability α, there is renegotiation. In this case, if short-term creditors make an offer, they receive $j_L(f)$ while the borrower receives 0. If the equity holder makes an offer, they offer 0 and obtain $j_L(f)$, while the short-term creditors get 0; however, such an offer is credible only if $j_L(f)<d_H$. If $j_L(f)\geq d_H$, then the only credible offer is d_H.[15] It is easy to verify that renegotiation is triggered

[15]When there is indifference, we break ties in favor of the efficient outcome of continuation.

only after the regime switch and $j_L(f_t) < d_H(f_t)$. The firm goes bankrupt only if the restructuring process fails, which happens with probability $1 - \alpha$. To determine the interest rate, we need to analyze the expected recovery. If $d_H(f) \leq j_L(f)$, there is no default, and short-term creditors are paid in full. The equity holder does not have incentives to default because no credible offer would allow paying less than d_H. If $d_H > j_L(f)$, the equity holder default so the expected payoff is $(1 - \alpha\beta) \times 0 + \alpha\beta \times j_L(f)$. In the event of default, each creditor gets zero with probability $1 - \alpha(1 - \beta)$; that is, if either renegotiation is not possible or if it is possible, but equity holder is the one to make the offer. With probability $\alpha(1 - \beta)$, the short-term debt recovery per dollar is $j_L(f)/d_H$. Hence, the short rate is given by

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \leq j_L(f) \\ r + \lambda\left(1 - \alpha(1 - \beta)\frac{j_L(f)}{d_H}\right) & \text{if } d_H > j_L(f). \end{cases}$$

The analysis in state L is unchanged. In state H, we try to construct a similar equilibrium. The HJB in the high state follows

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = &\max_{d_H \in [0, j_H(f)]} (1 - \pi) - (r + \xi)\, f + (\rho + \lambda - y_H)\, d_H \\ &+ \lambda\Big(\big(j_L(f) - d_H\big)\mathbb{1}_{d_H \leq j_L(f)} + \alpha\beta j_L(f)\mathbb{1}_{d_H > j_L(f)}\Big) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \end{aligned} \tag{25}$$

The optimal solution for short-term debt is

$$d_H(f) = \begin{cases} j_L(f) & \text{if } j_L(f) \geq \frac{\rho - r}{\rho + \lambda(1-\alpha) - r} j_H(f) \\ j_H(f) & \text{Otherwise.} \end{cases}$$

Note that when $\alpha = 0$, we are back to the benchmark model. Interestingly, α and β serve different purposes: the former leads to efficiency loss, and the latter is only about how to redistribute the surplus across the coalition. The threshold now is determined by the indifference condition

$$f_\dagger = \min\left\{f \geq 0 : (\rho + \lambda(1 - \alpha) - r) j_L(f) \leq (\rho - r) j_H(f)\right\}. \tag{26}$$

We supplement the remaining details in Appendix B.2.

If renegotiation is perfect (that is, $\alpha = 1$), there is no threshold $f_\dagger \in (0, f_H^b)$ satisfying the indifference condition (26), so $f_\dagger = 0$ and the optimal amount of short-term debt is always $d_H(f) = j_H(f)$.

Proposition 13. *The firm never issues long-term debt if the short-term debt can be renegotiated* *without friction. That is, if $\alpha = 1$, then $g_\theta(f) = 0$ for all f. If $f_\dagger > 0$, then $f_\dagger$ is decreasing in α.*

The intuition behind this result is similar to the one with derivative contracts. The main purpose of long-term debt is its state contingency that reduces the probability of default. However, if the short-term debt is also contingent, thanks to renegotiation, then there is no benefit of using long-term debt.

5.3 Jump Risk

In the benchmark model, the borrower is subject to two types of risks. The Brownian motion captures small frequent shocks to the cash flow, which has a continuous effect on the firm value. By contrast, a transition from the high to the low state, that is, the regime shift, captures large infrequent shocks that reduce the firm value discontinuously. In this subsection, we show the modeling choice of a regime shift is unimportant. In particular, our mechanism continues to hold if large infrequent shocks are modeled as downward jump risks to the cash flow. Specifically, we assume the cash flow follows a jump-diffusion process:

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \omega^{-1}\right) X_{t-} dN_t, \tag{27}$$

where N_t is a Poisson process with intensity λ and $\omega \in (1, \infty)$ is a constant. We can construct an equilibrium characterized by thresholds $f_\dagger$ and f^b. The value function $j(f)$ satisfies the HJB equation. Thus, the scaled value function satisfies the delay differential equation

$$\begin{aligned}(\rho + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) \\ + \max\left\{(\rho + \lambda - r)\frac{j(\omega f)}{\omega}, (\rho - r) j(f)\right\},\end{aligned}$$

with value matching and smooth pasting conditions $j(f^b) = j'(f^b) = 0$. The optimal short-term debt policy is given by

$$d(f) = \begin{cases} \frac{j(\omega f)}{\omega} & \text{if } f \in [0, f_\dagger) \\ j(f) & \text{if } f \in [f_\dagger, f^b], \end{cases}$$

where the threshold $f_\dagger$ satisfies the condition $(\rho + \lambda - r)\frac{j(\omega f_\dagger)}{\omega} = (\rho - r) j(f_\dagger)$. The issuance of long-term debt satisfies $g(f) = 0$, for $g(f) = 0\ \forall f \in (f_\dagger, f^b]$, where f^b is the endogenous default boundary. The issuance of long-term debt follows

$$g\left(f\right) = \frac{(\rho - r)\left(p\left(f\right) - p\left(\omega f\right)\right)}{-f p'\left(f\right)} \mathbb{1}_{\{f < f_\dagger\}}. \tag{28}$$

In other words, long-term debt is issued if and only if the amount of outstanding long-term debt is low relative to the operating cash flow. Equation (28) resembles (22): the difference in prices $p(f) - p(\omega f)$ reflects the drop in the long-term debt's price following the downward jump, and the denominator captures the sensitivity of long-term debt price to issuance.

The issuance of short-term debt is also similar to that in section 3. Short-term debt is riskless when $f \leq f_\dagger$ and the amount of issuance is $d(f) = \frac{j(\omega f)}{\omega}$. On the other hand, when $f > f_\dagger$, short-term debt becomes risky and the amount of issuance becomes $d(f) = j(f)$. The scaled-value function $j(f)$ satisfies a second-order delay differential equation, which cannot be solved in closed form. The detailed analysis of the problem is available in the Internet Appendix B.4. Figure 13 illustrates the dynamics of leverage and short-term debt. Note that the ratio of long-term debt to cash flow f_t experience upward jumps following downward jumps to cash flows. The ratio of short-term debt to cash flow d_t only significantly jumps downwards if $f_t \leq f_\dagger$ before and after the cash-flow jump.

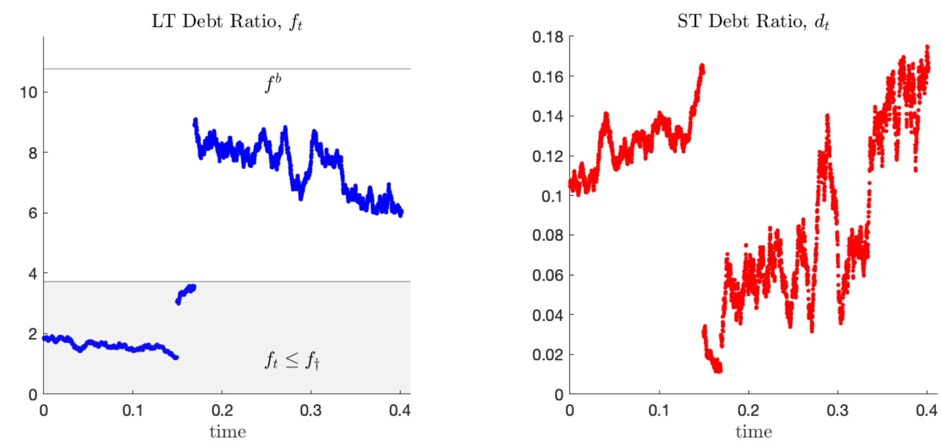


Figure 13: Example of a sample path with jumps

The parameter values are: $\rho = 0.1$, $r = 0.05$, $\mu = 0.2$, $\sigma = 0.75$, $\xi = 0.1$, $\lambda = 2$, $\omega^{-1} = 0.4$. With these parameters, $f_\dagger = 3.72$ and $f^b = 10.76$. For the simulation, we set the initial value $f_0 = 0.5 \times f_\dagger$.

6 Final Remarks

Our paper offers a theory of debt maturity based on a tradeoff between commitment and hedging. Short-term debt mitigates the lack-of-commitment problem and forces the borrower to reduce leverage after negative shocks. Long-term debt offers the borrower an option to postpone default and allows some time to recover after downside risks.

We end this paper with discussions and interpretations on some key assumptions of the paper.

Risk. The Brownian motion captures continuous fluctuations in day-to-day operating cash flows, which are meant to be small and frequent. Meanwhile, a transition across the two states affects the expected growth in cash flow and captures large and infrequent shocks. In a model with only Brownian Motion, a borrower would only issue short-term debt (except for tax-shield benefits). The disaster shock offers a reason for an unlevered borrower to issue long-term debt.

Debt maturity. Our modeling choice of short- and long-term debt is motivated by the discrete-time microfoundation. There, short-term debt would last for one period and therefore mature simultaneously. In the continuous-time setup, this feature is captured by zero-maturity debt that needs to be continuously rolled over. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing it each period would lead to the staggered structure. This feature is well captured by exponentially-maturing debt in the continuous-time setup.

Zero recovery in default. The assumption that creditors do not recover any value once the borrower defaults is made for simplicity and does not affect our mechanism. It implies debt seniority becomes irrelevant, ruling out the theoretical channel highlighted in Brunnermeier and Oehmke (2013) whereby the equity holder dilutes existing creditors' recovery value in bankruptcy through issuing new debt.

Covenants In our paper, we do not consider covenants designed to mitigate dilutions. Introducing covenants would allow the borrower to reap more benefits from long-term debt issuance. For example, a covenant that restricts the issuance of long-term debt to be lower than some threshold can limit the extent of dilution. However, covenants do not eliminate the benefits from short-term debt for two reasons. First, covenants are written on imperfect proxies of the firm's fundamentals, and therefore they don't completely rule out dilution. Second, following small and frequent shocks to cash flows, it is more costly for the borrower to adjust long-term debt. By contrast, short-term debt is more flexible. Therefore, our main mechanism between commitment and hedging continues to work under covenants.

References

Abel, A. B. (2018). Optimal debt and profitability in the trade-off theory. Journal of Finance 73(1), 95–143.

Admati, A. R., P. M. DeMarzo, M. F. Hellwig, and P. Pfleiderer (2018). The leverage ratchet effect. Journal of Finance 73(1), 145–198.

Adrian, T. and H. S. Shin (2014). Procyclical leverage and value-at-risk. The Review of Financial Studies 27(2), 373–403.

Aguiar, M., M. Amador, H. Hopenhayn, and I. Werning (2019). Take the short route: Equilibrium default and debt maturity. Econometrica 87(2), 423–462.

Almeida, H., M. Campello, B. Laranjeira, S. Weisbenner, et al. (2011). Corporate debt maturity and the real effects of the 2007 credit crisis. Critical Finance Review 1(1), 3–58.

Angeletos, G.-M. (2002). Fiscal policy with noncontingent debt and the optimal maturity structure. Quarterly Journal of Economics 117(3), 1105–1131.

Arellano, C. and A. Ramanarayanan (2012). Default and the maturity structure in sovereign bonds. Journal of Political Economy 120(2), 187–232.

Benzoni, L., L. Garlappi, R. S. Goldstein, J. Hugonnier, and C. Ying (2019). Optimal debt dynamics, issuance costs, and commitment. Issuance Costs, and Commitment (September 16, 2019).

Bigio, S., G. Nuño, and J. Passadore (2021). Debt-maturity management with liquidity costs.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. Journal of Political Economy 81(3), 637–654.

Bolton, P., H. Chen, and N. Wang (2011). A unified theory of tobin's q, corporate investment, financing, and risk management. The journal of Finance 66(5), 1545–1578.

Bolton, P., N. Wang, and J. Yang (2021). Leverage dynamics under costly equity issuance. NBER Working Paper 26802.

Brunnermeier, M. K. (2009). Deciphering the liquidity and credit crunch 2007-2008. Journal of Economic perspectives 23(1), 77–100.

Brunnermeier, M. K. and M. Oehmke (2013). The maturity rat race. Journal of Finance 68(2), 483–521.

Brunnermeier, M. K. and M. Yogo (2009). A note on liquidity risk management. American Economic Review 99(2), 578–83.

Bulow, J. I. (1982). Durable-goods monopolists. Journal of Political Economy 90(2), 314–332.

Calomiris, C. W. and C. M. Kahn (1991). The role of demandable debt in structuring optimal banking arrangements. American Economic Review, 497–513.

Chen, H., Y. Xu, and J. Yang (2021). Systematic risk, debt maturity, and the term structure of credit spreads. Journal of Financial Economics 139(3), 770–799.

Coase, R. H. (1972). Durability and monopoly. Journal of Law and Economics 15(1), 143–149.

Dangl, T. and J. Zechner (2020). Debt maturity and the dynamics of leverage. Available at SSRN 890228.

DeMarzo, P. and Z. He (2021). Leverage dynamics without commitment. Journal of Finance 73(3), 1195–1250.

DeMarzo, P. M. (2019). Presidential address: Collateral and commitment. Journal of Finance 74(4), 1587–1619.

Diamond, D. W. (1991). Debt maturity structure and liquidity risk. Quarterly Journal of Economics 106(3), 709–737.

Diamond, D. W. and Z. He (2014). A theory of debt maturity: the long and short of debt overhang. The Journal of Finance 69(2), 719–762.

Diamond, D. W. and R. G. Rajan (2001). Liquidity risk, liquidity creation, and financial fragility: A theory of banking. Journal of Political Economy 109(2), 287–327.

Duffie, D., N. Gârleanu, and L. H. Pedersen (2005). Over-the-counter markets. Econometrica 73(6), 1815–1847.

Fama, E. F. and M. Miller (1972). The theory of finance. Holt Rinehart & Winston.

Flannery, M. J. (1986). Asymmetric information and risky debt maturity choice. The Journal of Finance 41(1), 19–37.

Froot, K. A., D. S. Scharfstein, and J. C. Stein (1993). Risk management: Coordinating corporate investment and financing policies. Journal of Finance 48(5), 1629–1658.

Gertner, R. and D. Scharfstein (1991). A theory of workouts and the effects of reorganization law. Journal of Finance 46(4), 1189–1222.

Goldstein, R., N. Ju, and H. Leland (2001). An ebit-based model of dynamic capital structure. The Journal of Business 74(4), 483–512.

He, Z. and K. Milbradt (2016). Dynamic debt maturity. Review of Financial Studies 29(10), 2677–2736.

He, Z. and W. Xiong (2012). Rollover risk and credit risk. Journal of Finance 67(2), 391–430.

Hu, Y. and F. Varas (2021). Intermediary financing without commitment. Technical report, Working paper, Duke University.

Krishnamurthy, A. (2010). How debt markets have malfunctioned in the crisis. Journal of Economic Perspectives 24(1), 3–28.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. The journal of finance 49(4), 1213–1252.

Leland, H. E. (1998). Agency costs, risk management, and capital structure. Journal of Finance 53(4), 1213–1243.

Leland, H. E. and K. B. Toft (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. The Journal of Finance 51(3), 987–1019.

Malenko, A. and A. Tsoy (2020). Optimal time-consistent debt policies. Available at SSRN 3588163.

Nakao, S. (1972). On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. Osaka Journal of Mathematics 9(3), 513–518.

Niepelt, D. (2014). Debt maturity without commitment. Journal of Monetary Economics 68, S37–S54.

Panageas, S. (2010). Bailouts, the incentive to manage risk, and financial crises. Journal of Financial Economics 95(3), 296–311.

Protter, M. H. and H. F. Weinberger (1967). Maximum Principles in Differential equations. Prentice-Hall.

Protter, P. E. (2005). Stochastic integration and differential equations. In Stochastic integration and differential equations, pp. 249–361. Springer.

Rampini, A. A. and S. Viswanathan (2010). Collateral, risk management, and the distribution of debt capacity. Journal of Finance 65(6), 2293–2322.

Villas-Boas, J. M. (1997). Comparative statics of fixed points. Journal of Economic Theory 73(1), 183–198.

Xu, Q. (2018). Kicking maturity down the road: early refinancing and maturity management in the corporate bond market. Review of Financial Studies 31(8), 3061–3097.

Zaitsev, V. F. and A. D. Polyanin (2002). Handbook of Exact Solutions for Ordinary Differential Equations. Chapman & Hall/CRC.

Appendix

A Proofs of Section 3

Proof of Proposition 1

Proof. We prove the result for J_H, and the one for J_L follows similar steps. Let $\theta_t = H$ and $\tau_\lambda \geq t$ be the time that the state switches from H to L. By the principle of dynamic programming,

$$\begin{aligned}
V_t = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t\bigg[& \int_t^{\tau_b \wedge \tau_\lambda} e^{-\rho(s-t)} \Big(((1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s - \hat{y}_{s-} D_{s-})\, ds + dD_s \Big) \\
& + e^{-\rho \tau_b \wedge \tau_\lambda} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \bigg] \\
= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t\bigg[& \int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(((1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s - \hat{y}_{s-} D_{s-})\, ds + dD_s \Big) \\
& + e^{-\rho \tau_\lambda} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \bigg] \\
= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t\bigg[& \int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(((1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s - \hat{y}_{s-} D_{s-})\, ds + dD_s \Big) \\
& + e^{-\rho \tau_\lambda} \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \bigg] \\
= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t\bigg[& \int_t^{\tau_b \wedge \tau_\lambda} e^{-(\rho+\lambda)(s-t)} \Big(((1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s - \hat{y}_{s-} D_{s-})\, ds + dD_s \\
& + \lambda \max\{J_s - D_{s-}, 0\} \Big) \bigg],
\end{aligned}$$

where we have used the definition $V_{\tau_\lambda} = \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\}$. Using the integration by parts formula for semi-martingales (Corollary 2 in Section 2.6 of Protter (2005)), we get

$$\mathbb{E}_t\bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} dD_s\bigg] = \mathbb{E}_t\Big[e^{-(\rho+\lambda)(\tau_b - t)} D_{\tau_b}\Big] - D_{t-} + \mathbb{E}_t\bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} (\rho+\lambda) D_{s-} ds\bigg].$$

At the time of default, $D_{\tau_b} = 0$. Hence

$$\begin{aligned}
V_t = \sup_{\tau_b, g_s} \mathbb{E}_t\bigg[& \int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ (1-\pi)X_s - (\hat{r}+\xi)F_s + p_s g_s F_s \\
& + (\rho + \lambda - \hat{y}_{s-}) D_{s-} + \lambda \max\{J_L(X_s, F_s) - D_{s-}, 0\} \Big\} ds \bigg] - D_{t-}.
\end{aligned}$$

□

A.1 Detailed Solutions

Solution to the HJB equation in the low state. Equation (10) is a second-order ODE, and a standard solution takes the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$\begin{aligned} A_0 &= \frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L} \\ A_1 &= \frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi} \\ \gamma_1 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(\hat{r}+\hat{\eta}-\mu_L\right)}}{\sigma^2} > 1 \\ \gamma_2 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(\hat{r}+\hat{\eta}-\mu_L\right)}}{\sigma^2} < 0. \end{aligned} \tag{A.1}$$

The condition $\lim_{f\to 0} j_L(f) < \infty$ implies $A_3 = 0$. We define $\gamma \equiv \gamma_1$. Combining with value-matkappang and smooth-pasting condition, we get the default boundary is

$$f_L^b = \frac{\gamma}{\gamma-1}\frac{(1-\pi)\big(\hat{r}+\hat{\eta}+\xi\big)}{\big(\hat{r}+\hat{\eta}-\mu_L\big)(\hat{r}+\xi)} \tag{A.2}$$

From here, we get that the price $p_L(f) = -j_L'(f)$ is given by

$$p_L\left(f\right) = \frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left[1-\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right].$$

Solution to the HJB equation in the high state for $f_\dagger > 0$. The value function satisfies equation (13) together with the boundary conditions

$$j_H(f\dagger-) = j_H(f\dagger+) \tag{A.3}$$

$$j'_H(f_\dagger-) = j'_H(f_\dagger+) \tag{A.4}$$

$$j_H(f^b_H) = 0 \tag{A.5}$$

$$j'_H(f^b_H) = 0 \tag{A.6}$$

$$\lim_{f\to 0} j_H(f) < \infty \tag{A.7}$$

$$j_H\left(f_\dagger\right) = \frac{\rho+\lambda-\hat{r}}{\rho-\hat{r}+\pi\lambda} j_L\left(f_\dagger\right). \tag{A.8}$$

First, we consider the solution for $f \in [0, f_\dagger]$, in which region the value function satisfies the equation

$$(\rho+\lambda-\mu_H)\, j_H(f) = (1-\pi) - (\hat{r}+\xi)\, f + (\rho+\lambda-\hat{r})\, j_L(f) - (\mu_H+\xi)\, f j'_H(f) + \frac{1}{2}\sigma^2 f^2 j''_H\left(f\right).$$

The unique solution to this ODE satisfying condition (A.7) takes the form

$$j_H\left(f\right) = u_0(f) + Bf^\phi,$$

where the coefficient ϕ is given by

$$\phi = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho+\lambda-\mu_H\right)}}{\sigma^2} > 1, \tag{A.9}$$

and a particular solution u_0 is given by

$$u_0(f) = \underbrace{\frac{1-\pi}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}-\mu_L}\right) - \frac{\hat{r}+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}+\xi}\right)f}_{\text{no default value}} + \underbrace{\delta\frac{1}{\gamma-1}\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}\left(\frac{f}{f^b_L}\right)^\gamma}_{\text{default option in low state}} \tag{A.10}$$

where the discount factor δ is

$$\delta \equiv \frac{\rho+\lambda-\hat{r}}{\rho+\lambda-\hat{r}-\hat{\eta}+(\mu_H-\mu_L)(\gamma-1)} \in (0,1). \tag{A.11}$$

From the Feynman-Kac formula, we know that the solution to the particular solution admits the following stochastic representation:

$$u_0(f) = \mathbb{E}_0\left[\int_0^\infty e^{-(\rho+\lambda-\mu_H)t}\left((1-\pi) - (\hat{r}+\xi)\,\tilde{f}_t + (\rho+\lambda-\hat{r})\,j_L(\tilde{f}_t)\right)dt\right]$$

where $\tilde{f}_t$ corresponds to the process

$$d\tilde{f}_t = -(\mu_H+\xi)f_t dt - \sigma f_t d\tilde{B}_t, \quad \tilde{f}_0 = f$$

for some Brownian motion $\tilde{B}_t$. Equation (17) follows by Girsanov's theorem after a change of measure using the Radon-Nikodym derivative $e^{-\mu_H t}(X_t/X_0)$.

The coefficient B is pinned down from the value at $j_H\left(f_\dagger\right)$

$$B = f_\dagger^{-\phi}\left(j_H(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H(f) = u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi}, \qquad \forall f \in [0, f_\dagger],$$

where $j_H\left(f_\dagger\right) = \frac{\rho+\lambda-\hat{r}}{\rho-\hat{r}+\pi\lambda} j_L\left(f_\dagger\right)$. The solution on the interval $\left[f_\dagger, f_H^b\right]$ can be obtained in a similar way. In this interval, the value function satisfies the equation

$$\left(\hat{r}+\hat{\lambda}-\mu_H\right) j_H(f) = (1-\pi) - (\hat{r}+\xi)\,f + \mathcal{D}^H j_H(f).$$

The homogeneous equation

$$\left(\hat{r}+\hat{\lambda}-\mu_H\right)\varphi = \mathcal{D}^H\varphi$$

has two solution f^{β_1} and f^{β_2}, where

$$\begin{aligned}
\beta_1 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\hat{r}+\hat{\lambda}-\mu_H\right)}}{\sigma^2} > 1 \\
\beta_2 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\hat{r}+\hat{\lambda}-\mu_H\right)}}{\sigma^2} < 0.
\end{aligned} \tag{A.12}$$

Hence, the value function takes the form

$$j_H\left(f\right) = u_1(f) + D_1 f^{\beta_1} + D_2 f^{\beta_2}.$$

As before, the particular solution

$$u_1(f) = \frac{1-\pi}{\hat{r}+\hat{\lambda}-\mu_H} - \frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi} f$$

admits the representation

$$u_1(f) = \mathbb{E}_0\left[\int_0^\infty e^{-(\hat{r}+\hat{\lambda}-\mu_H)t}\left(1-\pi-(\hat{r}+\xi)\tilde{f}_t\right)dt\right],$$

which, after an appropriate change of measure, can be written as equation (18). Finally, by combining equations (A.3) and (A.5), we get

$$D_1 = \frac{j_H(f_\dagger) + u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}$$
$$D_2 = (f_H^b)^{-\beta_2}\left(-u_1(f_H^b) - D_1(f_H^b)^{\beta_1}\right).$$

It follows that the solution to the value function on this interval is given by

$$j_H(f) = u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)h_0\left(f, f_\dagger, f_H^b\right) - u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right),$$

where

$$h_0\left(f\middle|f_\dagger, f_H^b\right) = \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}$$
$$h_1\left(f\middle|f_\dagger, f_H^b\right) = \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}. \tag{A.13}$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions

(A.4) and (A.6). These two conditions lead to the two-variable, non-linear equation system below

$$u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}-\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right]=u_1'(f_H^b)f_H^b+\big(j_H(f_\dagger)-u_1(f_\dagger)\big)\frac{\beta_1-\beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \tag{A.14}$$

$$\big(u_0'(f_\dagger)-u_1'(f_\dagger)\big)f_\dagger+\phi\big(j_H(f_\dagger)-u_0(f_\dagger)\big)=$$
$$u_1(f_H^b)\frac{\beta_1-\beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2}+\big(j_H(f_\dagger)-u_1(f_\dagger)\big)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}. \tag{A.15}$$

From here, we get the following expressions for the price. For $f \in [0, f_\dagger]$, the price is

$$p_H(f)=\frac{\hat{r}+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}+\xi}\right)-\delta\frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma-1}-\frac{\phi\big(j_H(f_\dagger)-u_0(f_\dagger)\big)}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{\phi-1},$$

while for $f \in (f_\dagger, f_H^b]$, the price is

$$p_H(f)=\frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi}\left[1-\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1-1}-\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2-1}}{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}-\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}\right]$$
$$+\frac{\big(j_H(f_\dagger)-u_1(f_\dagger)\big)\beta_1\beta_2}{f_H^b}\left[1-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1-\beta_2}\right]\frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left[\frac{\left(\frac{f}{f_H^b}\right)^{\beta_2-1}-\left(\frac{f}{f_H^b}\right)^{\beta_1-1}}{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}-\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}\right]$$

Proof of Proposition 2

The proof includes three parts. In the first part, we show the existence and uniqueness of a solution. In the second part, we prove a single-crossing property and therefore show that it is optimal for the borrower to issue riskless short-term debt $d_H = j_L(f)$ if $f \leq f_\dagger$. Finally, we verify that $j_H(f)$ is a convex function on $\left[0, f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly. We start by establishing the uniqueness of the equilibrium.

Existence and Uniqueness: For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^{\tau} e^{-\hat{\rho}t}\left(1 - \pi - (\hat{r}+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f\right]$$
$$\text{subject to} \quad dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t,$$

where

$$\nu(z, \tilde{j}) \equiv \max_{d \in [0, \tilde{j}]} \left(\rho + \lambda - \hat{y}_H(z, d_H)\right) d_H + \lambda \max\{j_L(z) - d_H, 0\} = \max\{(\rho + \lambda - \hat{r})\, j_L(z), (\rho - \hat{r} + \pi\lambda)\, \tilde{j}\}.$$

It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f, \tilde{j}_1) \geq \nu(f, \tilde{j}_0)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\nu(z, \tilde{j} + a) = \max\{(\rho + \lambda - \hat{r})\, j_L(z), (\rho - \hat{r} + \pi\lambda)\, (\tilde{j} + a)\} \leq (\rho - \hat{r} + \pi\lambda)a + \nu(z, \tilde{j}),$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j} + a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1 - \pi - (\hat{r}+\xi)z_t + \nu(z_t, \tilde{j}(z_t) + a)\right) dt \Big| z_0 = f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1 - \pi - (\hat{r}+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f\right] \\
&\quad + \frac{\rho - \hat{r} + \pi\lambda}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f\right] a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1 - \pi - (\hat{r}+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f\right] \\
&\quad + \frac{\rho - \hat{r} + \pi\lambda}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f\right] a \\
&= \Phi(\tilde{j})(f) + \frac{\rho - \hat{r} + \pi\lambda}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f\right] a \leq \Phi(\tilde{j})(f) + \frac{\rho - \hat{r} + \pi\lambda}{\rho + \lambda - \mu_H} a.
\end{aligned}$$

Thus, the operator Φ is monotone and satisfies discounting, it follows then by Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f) = \Phi(j_H)(f)$.

Optimality Short-term Debt Policy We start with the following result, which will be used later on. First, let

$$\bar{\lambda} \equiv -\frac{1}{2}\left(\rho - \pi\eta + \frac{\pi\mu_L - \mu_H}{1-\pi}\right) + \sqrt{\left(\frac{\rho - \pi\eta}{2} + \frac{\pi\mu_L - \mu_H}{2(1-\pi)}\right)^2 + \frac{(\rho - \hat{r})(\mu_H - \mu_L + \hat{\eta})}{1-\pi}}. \quad \text{(A.16)}$$

Lemma 2. *The condition*

$$(\rho + \lambda - \hat{r})\, j_L(0) > (\rho - \hat{r} + \pi\lambda)\, j_H(0).$$

is satisfied if and only if $\lambda > \bar{\lambda}$.

Proof. See online appendix. □

Next, the following result shows that it is optimal for the borrower to issue $d_H = j_L(f)$ when $f \leq f_\dagger$ and $d_H = j_H(f)$ otherwise.

Lemma 3 (Single-crossing)**.** *There exists a unique* $f_\dagger \in \left(0, f_L^b\right)$ *such that* $(\rho + \lambda - \hat{r})\, j_L(f) \geq (\rho - \hat{r} + \pi\lambda)\, j_H(f)$ *if and only if* $f \leq f_\dagger$.

Proof. See online appendix. □

Strict convexity of $j_H(f)$ on $\left[0, f_H^b\right]$. The proof relies on a few auxiliary lemmas.

Lemma 4.

$$j_H'(f) \geq -1, \qquad \forall f \in \left[0, f_H^b\right],$$

Proof. See online appendix. □

Lemma 5.

$$f_H^b > \frac{1-\pi}{\hat{r} + \xi} \qquad \textit{and} \qquad \min\left\{j_H''(0), j_H''(f_H^b)\right\} > 0,$$

Proof. See online appendix. □

Lemma 6.

$$j_H'''(f_\dagger^-) > j_H'''(f_\dagger^+).$$

Proof. See online appendix. □

Now we are ready to verify that the solution to the HJB equation is convex. We differentiate the HJB twice and let $\tilde{u} \equiv f j_H''$ to get

$$(\rho + \lambda + \xi)\,\tilde{u} = (\rho + \lambda - r)\, f j_L'' - \left(\mu_H + \xi - \sigma^2\right) f \tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in (0, f_\dagger) \tag{A.17}$$

$$\left(\hat{r} + \hat{\lambda} + \xi\right)\tilde{u} = -\left(\mu_H + \xi - \sigma^2\right) f \tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in (f_\dagger, f_H^b). \tag{A.18}$$

By the maximum principle in Theorem 1, $\tilde{u}$ cannot have an interior nonpositive local minimum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonpositive minimum is that $\tilde{u}(f_\dagger) < 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are positive, this requires that $j_H''(f_\dagger-) + f_\dagger j_H'''(f_\dagger-) = \tilde{u}'(f_\dagger-) < \tilde{u}'(f_\dagger+) = j_H''(f_\dagger+) + f_\dagger j_H'''(f_\dagger+)$. From the HJB equation it follows that j_H is twice continuously differentiable at $f_\dagger$, so such a kink would require $j_H'''(f_\dagger^-) < j_H'''(f_\dagger^+)$, which is ruled out by Lemma 6. We can conclude that $\tilde{u}$ does not have an interior nonpositive minimum, so it follows that $\tilde{u}(f) = f j_H''(f) > 0$ on $(0, f_H^b)$.

Solution HJB Equation when $f_\dagger = 0$: In the case that $\lambda \le \bar{\lambda}$, the firm never issues long-term debt, so the analysis reduces to the one in Case 1 for $f' > f_\dagger$ in the proof of Lemma 3.

A.2 Proof of Proposition 3

Proof. First, we consider the slow state. The debt price satisfies the asset pricing equation

$$(r + \xi + \eta)\, p_L(f) = (r + \xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f). \tag{A.19}$$

The indifference condition (9) implies that the price of debt is $p_L(f) = -j_L'(f)$. Substituting in (A.19) we get

$$(r + \xi + \eta)\, j_L'(f) = -(r + \xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f). \tag{A.20}$$

Combining (A.19) and (A.20) we get

$$g_L(f) = \frac{\pi r - \pi\,(r + \eta)\, p_L(f)}{-f p_L'(f)}.$$

In the high state, the debt price follows

$$(r + \xi + \lambda)\, p_H(f) = (r + \xi) + \lambda p_L(f) \mathbf{1}_{\{d_H(f) \le j_L(f)\}} + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f).$$

First, consider the case when $f \in (0, f_\dagger)$. Differentiating the HJB equation for $j_H(f)$ we obtain,

$$(r+\lambda+\xi)\, j_H'(f) + (r+\xi) - \lambda j_L'(f) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) \\ = (\rho - (1-\pi)\, r)\, j_L'(f) + \pi r - (\rho - r)\, j_H'(f).$$

Combining with the indifference condition (11) – which requires that $p_H(f) = -j_H'(f)$ – we find that

$$g_H(f) = \frac{(\rho - r)\,(p_H(f) - p_L(f)) + \pi r\,(1 - p_L(f))}{-f p_H'(f)}.$$

Finally, for $f \in (f_\dagger, f_H^b)$, we differentiate the HJB equation (13) to obtain,

$$(r+\lambda+\xi)\, j_H'(f) + (r+\xi) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = \pi\,(r + \lambda_{HL})\, j_H'(f) + \pi r,$$

which combined with the optimality condition $p_H(f) = -j_H'(f)$ yields

$$g_H(f) = \frac{\pi r - \pi\,(r+\lambda)\, p_H(f)}{-f p_H'(f)}.$$

□

Proof of Proposition 6

Proof. In the region $f < f_\dagger$, the issuance function is given by

$$g_H(f) f = \frac{(\rho - r)\,(j_L'(f) - j_H'(f)) + \pi r\,(j_L'(f) + 1)}{j_H''(f)}.$$

The different terms in the previous expression are given by

$$\begin{aligned}
j_L'(f) + 1 &= \frac{\hat\eta}{\hat r + \hat\eta + \xi} + \frac{\hat r + \xi}{\hat r + \hat\eta + \xi}\left(\frac{f}{f_L^b}\right)^{\gamma - 1} \\
j_H'(f) &= -\frac{\rho + \lambda + \hat\eta + \xi}{\hat r + \hat\eta + \xi}\frac{\hat r + \xi}{\rho + \lambda + \xi} + \delta\frac{\hat r + \xi}{\hat r + \hat\eta + \xi}\left(\frac{f}{f_L^b}\right)^{\gamma - 1} + \phi\big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{\phi - 1}\frac{1}{f_\dagger} \\
j_L'(f) - j_H'(f) &= \frac{\hat\eta}{\rho + \lambda + \xi}\frac{\hat r + \xi}{\hat r + \hat\eta + \xi} + (1 - \delta)\frac{\hat r + \xi}{\hat r + \hat\eta + \xi}\left(\frac{f}{f_L^b}\right)^{\gamma - 1} - \phi\big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{\phi - 1}\frac{1}{f_\dagger} \\
f j_H''(f) &= (\gamma - 1)\delta\frac{\hat r + \xi}{\hat r + \hat\eta + \xi}\left(\frac{f}{f_L^b}\right)^{\gamma - 1} + \phi(\phi - 1)\big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{\phi - 1}\frac{1}{f_\dagger}
\end{aligned}$$

where

$$\gamma = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\hat{r} + \hat{\eta} - \mu_L\right)}}{\sigma^2}$$
$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda - \mu_H\right)}}{\sigma^2}$$

As $\gamma > 1$ and $\phi > 1$, we have that

$$\lim_{f\to 0} j_L'(f) + 1 = \frac{\hat{\eta}}{\hat{r} + \hat{\eta} + \xi}$$
$$\lim_{f\to 0} \left(j_L'(f) - j_H'(f)\right) = \frac{\hat{\eta}}{\rho + \lambda + \xi}\frac{\hat{r} + \xi}{\hat{r} + \hat{\eta} + \xi}$$

This means that the limit of $g_H(f)f$ is positive as long as the limit of $j_H''(f)$ is finite, which requires that $\gamma \geq 2$ and $\phi \geq 2$.

□

A.3 Benchmark with Only Short- and Long-term Debt

Proof of Proposition 9

Proof. In the low state, the borrower chooses short-term debt $D_t = Xj_L$ and only defaults upon the disaster shock. So the short rate is $y_L = r + \eta$, which implies the value of the firm is

$$J_L^s(X) = \frac{(1-\pi)X}{\hat{r} + \hat{\eta} - \mu_L}.$$

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage, in which case

$$J_H^s(X) = \frac{(1-\pi)X}{\hat{r} + \hat{\lambda} - \mu_H}.$$

On the other hand, if she borrows riskless debt up to $X_t j_L$, the firm value is

$$J_H^s(X) = \frac{(1-\pi)X}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - \hat{r}}{\hat{r} + \hat{\eta} - \mu_L}\right).$$

From here we get that the value of the firm is

$$J_H^s(X) = X \max\left\{\frac{(1-\pi)}{\hat{r}+\hat{\lambda}-\mu_H},\ \frac{(1-\pi)}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-\hat{r}}{\hat{r}+\hat{\eta}-\mu_L}\right)\right\}$$

Finally, $J_L^s(X) \geq J_L(X,F) + p_L(X,F)\,F$ is straightforward given the former is the first-best firm value. In the high state, this is equivalent to proving $j_H^s \geq j_H(f) + p_H(f)\,f$. It is easily verified that $j_H^s \geq j_H(0)$ (and the equality holds for both cases no matter the value of λ). The result follows from

$$\frac{d\left[j_H(f) + p_H(f)\,f\right]}{df} = p_H'(f)f < 0.$$

□

Proof of Proposition 10

Proof. Again, let $\tilde{V}_L = X\tilde{v}_L$ so that $\frac{\partial \tilde{V}_L}{\partial F} = \tilde{v}_L'$, $\frac{\partial \tilde{V}_L}{\partial X} = \tilde{v}_L - f\tilde{v}_L'$, and $X\frac{\partial^2 \tilde{V}_L}{\partial X^2} = f^2\tilde{v}_L''$. For notation convenience, we use $\tilde{v}_L = v_L^\ell$. The scaled HJB becomes

$$(\rho+\eta-\mu_L)\,\tilde{v}_L = (1-\pi) - (r(1-\pi)+\xi)\,f - (\mu_L+\xi)\,f\tilde{v}_L' + \frac{1}{2}\sigma^2 f^2\tilde{v}_L''.$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_L(f) < \infty$, $\tilde{v}_L\left(\tilde{f}_L^b\right) = 0$, and $\tilde{v}_L'\left(\tilde{f}_L^b\right) = 0$, we obtain the solution

$$\tilde{v}_L(f) = \frac{1-\pi}{\rho+\eta-\mu_L} - \frac{r(1-\pi)+\xi}{\rho+\eta+\xi}f + \frac{r(1-\pi)+\xi}{\rho+\eta+\xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}} \tag{A.21}$$

$$\tilde{f}_L^b = \frac{1-\pi}{\rho+\eta-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\eta+\xi}{r(1-\pi)+\xi} \tag{A.22}$$

where

$$\tilde{\gamma} = \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\eta-\mu_L)}}{\sigma^2} > 1. \tag{A.23}$$

In a smooth equilibrium, $\tilde{p}_L = -\tilde{v}_L'$, and $\tilde{p}_L$ satisfies

$$(r+\xi+\eta)\,\tilde{p}_L = (r+\xi) + \left(g_L-\xi-\mu_L+\sigma^2\right)f\tilde{p}_L' + \frac{1}{2}\sigma^2 f^2\tilde{p}_L''.$$

Differentiating once the HJB for $\tilde{v}_L$, we get $\tilde{g}_L = \frac{\pi r+(\rho-r)\tilde{p}_L}{f\tilde{v}_L''}$.

In the high state , the scaled HJB becomes

$$(\rho - \mu_H)\,\tilde{v}_H = (1-\pi) - (r\,(1-\pi)+\xi)\,f - (\mu_H+\xi)\,f\tilde{v}_H' + \frac{1}{2}\sigma^2 f^2 \tilde{v}_H'' + \lambda\,(\tilde{v}_L - \tilde{v}_H)\,.$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_H(f) < \infty$, $\tilde{v}_H\left(\tilde{f}_H^b\right) = 0$, and $\tilde{v}_H'\left(\tilde{f}_H^b\right) = 0$, we obtain the solution

$$\tilde{v}_H(f) = \tilde{u}_0(f) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi},$$

where

$$\tilde{u}_0(f) = \frac{(1-\pi)\,(\rho+\eta+\lambda-\mu_L)}{(\rho+\eta-\mu_L)\,(\rho+\lambda-\mu_H)} - \frac{(r\,(1-\pi)+\xi)\,(\rho+\eta+\lambda+\xi)}{(\rho+\eta+\xi)\,(\rho+\lambda+\xi)}f + \frac{\lambda\frac{r(1-\pi)+\xi}{\rho+\eta+\xi}}{\lambda-\eta+(\mu_H-\mu_L)(\tilde{\gamma}-1)}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

and

$$\phi = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\,(\rho+\lambda-\mu_H)}}{\sigma^2} > 1$$

Finally, the boundary $\tilde{f}_H^b$ is pinned down by the smooth-pasting condition

$$\tilde{f}_H^b\tilde{u}_0'\left(\tilde{f}_H^b\right) - \phi\tilde{u}_0\left(\tilde{f}_H^b\right) = 0.$$

In a smooth equilibrium, $\tilde{p}_H = -\tilde{v}_H'$, and $\tilde{p}_H$ satisfies

$$(r+\xi)\,\tilde{p}_H = (r+\xi) + \left(\tilde{g}_H - \xi - \mu_H + \sigma^2\right)f\tilde{p}_H' + \frac{1}{2}\sigma^2 f^2\tilde{p}_H'' + \lambda\,(\tilde{p}_L - \tilde{p}_H)\,.$$

Differentiating once the HJB for $\tilde{v}_H$, we get $\tilde{g}_H = \frac{\pi r + (\rho - r)\tilde{p}_H}{f\tilde{v}_H''}$. □

Proof of Proposition 11

Proof. The proof follows from the following lemmas (proofs available in online appendix):

Lemma 7. *$j_L(f) \geq \tilde{v}_L(f)$ and $f_L^b \geq \tilde{f}_L^b$*

Lemma 8. *$\tilde{p}_L(f) \leq p_L(f)$, and the inequality is strict for $\forall f > 0$.*

Lemma 9. *$j_H(f) \geq \tilde{v}_H(f)$ and $f_H^b \geq \tilde{f}_H^b$.*

Lemma 10. *There is $0 \leq \underline{f} \leq f_\dagger \leq \overline{f} \leq \tilde{f}_H^b$ such that $\tilde{p}_H(f) \leq p_H(f)$ on $[0, \underline{f}] \cup [\overline{f}, \tilde{f}_H^b]$*

$\square$

Internet Appendix for "Debt Maturity Management"

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This Internet Appendix contains additional analysis to accompany the manuscript. Section A provides the remaining proofs for the analysis in Section 3, including all technical lemmas. Section B provides the details for Section 5. Section C offers additional examples of how real world corporations manage debt maturity.

A Proofs of Section 3

Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

Theorem 1 (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{A.24}$$

in an interval $(0, b)$ with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Theorem 2 (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that u is a nonconstant solution of the differential inequality* (A.24) *having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

Corollary 2. *If u satisfies* (A.24) *in an interval (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u\,(a) \leq 0$, $u\,(b) \leq 0$, then $u\,(x) < 0$ in (a, b) unless $u \equiv 0$.*

A.1 Proofs of Auxiliary Lemmas

Proof of Lemma 2

Proof. The proof of Proposition 9 makes it clear that the condition $\lambda > \bar{\lambda}$ guarantees that

$$(\rho + \lambda - \hat{r})\, j_L(0) > (\rho - \hat{r} + \pi\lambda)\, j_H(0)\,.$$

This inequality is satisfied only if

$$\frac{\rho + \lambda - \hat{r}}{\rho - \hat{r} + \pi\lambda} > \frac{\rho + \lambda + \hat{\eta} - \mu_L}{\rho + \lambda - \mu_H}.$$

Combining terms, we can write this as the following quadratic inequality

$$(1-\pi)\,\lambda^2 + \left[(1-\pi)\,\rho - \pi\hat{\eta} + \pi\mu_L - \mu_H\right]\lambda - (\rho - \hat{r})\,(\mu_H - \mu_L + \hat{\eta}) > 0.$$

The left hand side is positive if and only if λ is greater than the unique positive root of the quadratic equation for $\bar{\lambda}$

$$(1-\pi)\,\bar{\lambda}^2 + \left[(1-\pi)\,\rho - \pi\hat{\eta} + \pi\mu_L - \mu_H\right]\bar{\lambda} - (\rho - \hat{r})\,(\mu_H - \mu_L + \hat{\eta}) = 0,$$

which is given by (A.16). □

Proof of Lemma 3

Proof. Define $a \equiv 1 + \frac{\hat{\lambda}}{\rho - \hat{r} + \pi\lambda}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function u let,

$$L^{0\dagger}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\, f u' - (\rho + \lambda - \mu_H)\, u$$
$$L^{\dagger b}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\, f u' - \left(\hat{r} + \hat{\lambda} - \mu_H\right) u.$$

The HJB in state $\theta = H$ can be written as

$$L^{0\dagger} j_H + 1 - \pi - (\hat{r} + \xi)\, f + (\rho + \lambda - \hat{r})\, j_L = 0,\ \ f \in (0, f_\dagger)$$
$$L^{\dagger b} j_H + 1 - \pi - (\hat{r} + \xi)\, f = 0,\ \ f \in (f_\dagger, f_H^b).$$

Similarly, the HJB in state $\theta = L$ can be written as

$$L^{0\dagger} a j_L + a(\mu_H - \mu_L) f j_L' + a\left(\rho + \lambda - (\hat{r} + \hat{\eta}) + \mu_L - \mu_H\right) j_L + a\left(1 - \pi - (\hat{r} + \xi) f\right) = 0$$
$$L^{\dagger b} a j_L + a(\mu_H - \mu_L) f j_L' - a\left(\mu_H - \mu_L + \hat{\eta} - \hat{\lambda}\right) j_L + a\left(1 - \pi - (\hat{r} + \xi) f\right) = 0.$$

Therefore, we have

$$L^{0\dagger}\left(a j_L - j_H\right) + H(f) = 0$$
$$L^{\dagger b}\left(a j_L - j_H\right) + H(f) = 0,$$

where the function $H(f)$ defined as

$$H(f) \equiv a(\mu_H - \mu_L) f j_L' - a\left(\mu_H - \mu_L + \hat{\eta} - \hat{\lambda}\right) j_L + (a-1)\left(1 - \pi - (\hat{r} + \xi) f\right),$$

and

$$\begin{aligned} H''(f) &= \left[(\mu_H - \mu_L) a \frac{f j_L'''}{j_L''} + (\mu_H - \mu_L) a + a\left(\hat{\lambda} - \hat{\eta}\right)\right] j_L'' \\ &= \left[(\mu_H - \mu_L)(\gamma - 1) + \hat{\lambda} - \hat{\eta}\right] a j_L''. \end{aligned} \tag{A.25}$$

We need to distinguish two cases. If $\hat{\lambda} \geq \hat{\eta} - (\mu_H - \mu_L)(\gamma - 1)$, $H''(f) \geq 0$, which implies $H(f)$ is convex and the maximum of $H(f)$ on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. Evaluating $H(f)$ at the two boundaries and using the hypothesis $\lambda > \bar{\lambda}$, we have

$$(\rho + \lambda - \mu_H)\, a - (\rho + \lambda - \hat{\eta} - \mu_L) > 0,$$

from Lemma 2. Then, we get

$$\begin{aligned} H(0) &= -a\left(\mu_H - \mu_L + \hat{\eta} - \hat{\lambda}\right) \frac{1 - \pi}{\hat{r} + \hat{\eta} - \mu_L} + (a-1)(1-\pi) \\ &= \frac{1-\pi}{\hat{r} + \hat{\eta} - \mu_L}\left((\rho + \lambda - \mu_H)\, a - (\rho + \lambda - \hat{\eta} - \mu_L)\right) > 0, \\ H(f_L^b) &= (a-1)\left(1 - \pi - (\hat{r} + \xi) f_L^b\right) < 0. \end{aligned}$$

Therefore, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. On the other hand, if $\hat{\lambda} < \hat{\eta} - (\mu_H - \mu_L)(\gamma - 1)$, $H''(f) < 0$, which implies $H(f)$ is concave and the minimum of $H(f)$ on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. Since $H(0) > 0$ and $H(f_L^b) < 0$, Therefore, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$.

Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.
  - On $f \in [0, f_\dagger]$, we know $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f_\dagger]$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f_\dagger)$. Moreover, Theorem 2 and Corollary 2 imply $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$.
  - On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}(aj_L - j_H) \geq 0$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Given that $aj_L\left(f_L^b\right) - j_H\left(f_L^b\right) < 0$, $aj_L(f) - j_H(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
  - On $f \in [f_\dagger, f']$. Suppose there exists a $f'' \in (f_\dagger, f')$ such that $aj_L(f'') - j_H(f'') > 0$. Given that $aj_L(f_\dagger) - j_H(f_\dagger) = 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$, it must be that $aj_L(f) - j_H(f)$ has a nonpositive interior minimum on $[f_\dagger, f'']$. Meanwhile, from $L^{\dagger b}(aj_L(f) - j_H(f)) \leq 0$ for $f \in (f_\dagger, f'')$, we know from Theorem 1 that $aj_L(f) - j_H(f)$ cannot have a nonpositive interior minimum on $(f_\dagger, f'')$, which constitutes a contradiction.
- Case 2: $f' \leq f_\dagger$.
  - On $f \in [f_\dagger, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}(aj_L - j_H) \leq 0$. From Theorem 1 and 2, we know $aj_L(f) - j_H(f) \leq 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$.
  - On $f \in [f', f_\dagger]$, $L^{0\dagger}(aj_L - j_H) \geq 0$ so that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Together with $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$, this shows $aj_L(f) - j_H(f) \geq 0$.
  - On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f')$.

□

Proof of Lemma 4

Proof. Let $\hat{u} = j_H'(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (16) that $\hat{u}(0) = 0$ and (A.6) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\hat{u} = -(\rho + \lambda - \hat{r})(j_L' + 1) < 0, \qquad f \in [0, f_\dagger]$$

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\hat{u} = -\hat{\lambda} < 0 \qquad f \in [f_\dagger, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. $\square$

Proof of Lemma 5

Proof. For any $f \leq \frac{1-\pi}{\hat{r}+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1 - \pi - (\hat{r} + \xi) f + (\rho + \lambda - y) d > 0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1-\pi}{\hat{r}+\xi}$. Plugging (A.5) and (A.6) into the HJB equation for $j_H(f)$, we get $j_H''\left(f_H^b\right)$ whenever $f_H^b > \frac{1-\pi}{\hat{r}+\xi}$.

Next, let us turn to prove that $j_H''(0) \geq 0$. Let us define $u \equiv j_H'$ and differentiate the HJB equation once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H + \xi - \sigma^2\right) f u' - (\rho + \lambda + \xi) u = (\hat{r} + \xi) - (\rho + \lambda - \hat{r}) j_L'.$$

Moreover, let z be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H + \xi - \sigma^2\right) f z' - (\rho + \lambda + \xi) z = (\hat{r} + \xi) - (\rho + \lambda - \hat{r}) j_L'(0)$$

with boundary conditions

$$\lim_{f \downarrow 0} z(f) < \infty$$
$$z(f_\dagger) = u(f_\dagger) = j_H'(f_\dagger).$$

The solution is

$$z(f) = -\frac{\hat{r} + \xi}{\rho + \lambda + \xi} + \frac{(\rho + \lambda - \hat{r}) j_L'(0)}{\rho + \lambda + \xi} + \left(j_H'(f_\dagger) + \frac{\hat{r} + \xi}{\rho + \lambda + \xi} - \frac{(\rho + \lambda - \hat{r}) j_L'(0)}{\rho + \lambda + \xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1 = \frac{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (\rho + \lambda + \xi)}}{\sigma^2} > 0.$$

Let $\delta(f) = z - u$. It is easily verified that $\delta(0) = 0$ and $\delta(f_\dagger) = 0$. Moreover, δ satisfies

$$\frac{1}{2}\sigma^2 f^2 \delta'' - \left(\mu_H + \xi - \sigma^2\right) f \delta' - (\rho + \lambda + \xi) \delta = (\rho + \lambda - \hat{r}) \left(j_L'(f) - j_L'(0)\right) \geq 0.$$

By Theorem 1, δ cannot have an interior nonnegative maximum, and the maximum is attained at $f = 0$. Theorem 2 further implies $\delta'(0) < 0$ so $u'(0) > z'(0)$. Finally, we know that

$$z'(f) = \omega_1 \left(j_H'(f_\dagger) + \frac{r(1-\pi)+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-\hat{r})\, j_L'(0)}{\rho+\lambda+\xi} \right) f_\dagger^{-\omega_1} f^{\omega_1 - 1} = \omega_1 \left(j_H'(f_\dagger) + 1 \right) f_\dagger^{-\omega_1} f^{\omega_1-1},$$

which implies $z'(f) \geq 0$ given that $j_H'(f_\dagger) \geq -1$. Therefore, $u'(0) = j_H''(0) > 0$. □

Proof of Lemma 6

Proof. We differentiate the HJB (13) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2 (j_H'''(f_\dagger+) - j_H''''(f_\dagger-)) = \left(\rho + \lambda - \left(\hat{r} + \hat{\lambda}\right)\right) \left[a j_L'(f_\dagger) - j_H'(f_\dagger)\right],$$

where $a \equiv 1 + \frac{\hat{\lambda}}{\rho+\lambda-(\hat{r}+\hat{\lambda})}$ The proof of Proposition 3 shows $a j_L'(f_\dagger) - j_H'(f_\dagger) < 0$ so that $j_H''''(f_\dagger-) > j_H'''(f_\dagger+)$. □

Proof of Lemma 7

Proof.

$$\begin{aligned}
J_L(X,F) &= \sup_{\tau_b, \{G_s, D_s \leq J_L(X_s,F_s)\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \left((1-\pi) X_s - (\hat{r}+\xi) F_s + (\rho+\eta-\hat{y}_{s-}) D_{s-} \right) ds + p_s dG_s \right) \right] \\
&> \sup_{\tau_b, \{G_s\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \left((1-\pi) X_s - (\hat{r}+\xi) F_s \right) ds + p_s dG_s \right) \right] \\
&= \sup_{\tau_b} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \left((1-\pi) X_s - (\hat{r}+\xi) F_s \right) ds \right) \right] = \tilde{V}_L(X,F) \qquad \text{(A.26)}
\end{aligned}$$

where the inequality comes from that $(\rho + \eta - \hat{y}_{s-}) D_{s-}$ is positive. This implies that the value function in the low state in the benchmark is higher than that in the economy with long-term debt only for any (X, F).

Let T_L and $\tilde{\tau}_{bL}$ be the default time in both economies in the low state. We have that $\tilde{v}_L(f_{\tilde{\tau}_{bL}}) = 0 \Rightarrow j_L(f_{\tilde{\tau}_{bL}}) > 0$, which means that $T_L > \tilde{\tau}_{bL}$. It follows then that $f_L^b > \tilde{f}_L^b$. □

Proof of Lemma 8

Proof. Given $f_L^b > \tilde{f}_L^b$ and $\tilde{g}_L(f) > g_L(f) = 0$ and $T_L > \tilde{\tau}_{bL}$, the result follows from the definition

$$p_L = \mathbb{E}\left[\int_t^{\tau_\xi \wedge T_L} e^{-(r+\eta)(s-t)} r ds + e^{-(r+\eta)(T_L - t)} \mathbb{1}_{T_L > \tau_\xi}\right]$$
$$\tilde{p}_L = \mathbb{E}\left[\int_t^{\tau_\xi \wedge \tilde{\tau}_{bL}} e^{-(r+\eta)(s-t)} r ds + e^{-(r+\eta)(\tilde{\tau}_{bL} - t)} \mathbb{1}_{\tilde{\tau}_{bL} > \tau_\xi}\right].$$

□

Proof of Lemma 9

Proof. We define an auxiliary process z_t that satisfies $z_0 = f_0$ and

$$dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t.$$

The value functions are equivalently

$$j_H(z) \equiv \sup_{\tau_b, d_t \in [0, j_H(z_t)]} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t}\left((1-\pi) - (r(1-\pi) + \xi) z_t + (\rho + \lambda - (1-\pi) y) d_t + \lambda (j_L(z_t) - d_t)^+\right) dt\right]$$
$$\tilde{v}_H(z) \equiv \sup_{\tau_b} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t}\left((1-\pi) - (r(1-\pi) + \xi) z_t + \lambda \tilde{v}_L(z_t)\right) dt\right],$$

where $\hat{\rho} \equiv \rho + \lambda - \mu_H$. Note that

$$\begin{aligned}
j_H(f) &\geq \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t}\left((1-\pi) - (r(1-\pi) + \xi) f_t + (\rho + \lambda - (1-\pi) r) j_L(f_t)\right) dt\right] \\
&> \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t}\left((1-\pi) - (r(1-\pi) + \xi) f_t + \lambda j_L(f_t)\right) dt\right] \\
&\geq \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t}\left((1-\pi) - (r(1-\pi) + \xi) f_t + \lambda \tilde{v}_L(f_t)\right) dt\right] \\
&= \tilde{v}_H(f).
\end{aligned}$$

It follows that $f_H^b > \tilde{f}_H^b$. □

Proof of Lemma 10

Proof. Define $\Delta_H(f) = p_H(f) - \tilde{p}_H(f)$. We get

- On $f \in (0, f_\dagger)$, from the HJB equations, $p_H = -j_H^{'}$ and $\tilde{p}_H = -\tilde{v}_H^{'}$, we can find

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \lambda + \xi)\Delta_H(f) = -\left(\rho - (1-\pi) r\right) p_L(f) - \lambda \triangle_L(f)$$

From here we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \lambda + \xi)\Delta_H(f) \leq 0.$$

By the maximum principle, $\Delta_H(f)$ cannot have a nonpositive minimum. In addition,

$$\Delta_H(0) = \frac{(1-\pi) r + \xi}{\rho + \lambda + \xi} \frac{\lambda\left(\rho + \eta - (1-\pi)(r+\eta)\right) + \left(\rho - (1-\pi) r\right)(\rho + \eta + \xi)}{\left[(1-\pi)(r+\eta) + \xi\right](\rho + \eta + \xi)} > 0.$$

Hence, $\Delta_H(f)$ single crosses 0 from above when f starts from $f = 0$.

- On $f \in (f_\dagger, \tilde{f}_H^b)$, from the HJB equations, $p_H = -j_H^{'}$ and $\tilde{p}_H = -\tilde{v}_H^{'}$, we can find

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \xi)\Delta_H(f) = -\left[\rho - (1-\pi)(r+\lambda)\right] p_H(f) - \lambda\left(\tilde{p}_H(f) - \tilde{p}_L(f)\right)$$

Given $\rho > (1-\pi)(r+\lambda)$ and $\tilde{p}_H(f) > \tilde{p}_L(f)$, we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \xi)\Delta_H(f) \leq 0.$$

It follows that $\Delta_H(f)$ cannot have a nonpositive minimum. In addition, $\Delta_H(\tilde{f}_H^b) \geq 0$ since $f_H^b \geq \tilde{f}_H^b$. Hence, $\Delta_H(f)$ single crosses 0 from below when f goes to $f = \tilde{f}_H^b$.

□

A.2 Analysis of Section 3.4: Limit when $\sigma \to 0$

Proof of Proposition 4

The first step in the analysis is to derive the limit of the value function. This is given in the following proposition

Proposition 14 (Limit value function)**.** *Suppose that $\mu_L + \xi < 0$ and $\mu_H + \xi > 0$. Consider the* *case when* $\lambda > \bar{\lambda}$, *where* $\bar{\lambda}$ *is given in equation* (A.16). *Let*

$$\gamma \equiv \frac{\hat{r} + \hat{\eta} - \mu_L}{-(\xi + \mu_L)} > 1$$
$$\psi \equiv \frac{\hat{r} + \hat{\lambda} - \mu_H}{\xi + \mu_H} > 0.$$

In the limit when $\sigma^2 \to 0$, *the value function converges to*

$$j_L(f) = \frac{1-\pi}{\hat{r} + \hat{\eta} - \mu_L} - \frac{\hat{r} + \xi}{\hat{r} + \hat{\eta} + \xi} f + \frac{1}{\gamma - 1} \frac{1-\pi}{\hat{r} + \hat{\eta} - \mu_L} \left(\frac{f}{f_L^b} \right)^{\gamma}$$
$$j_H(f) = \begin{cases} u_0(f) & f \in [0, f_\dagger] \\ u_1(f) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big) \left(\frac{f}{f_\dagger} \right)^{-\psi} & f \in \left(f_\dagger, f_H^b\right], \end{cases}$$

where $u_0(f)$ *and* $u_1(f)$ *are given in equations* (17) *and* (18). *The default boundary in the low state is* $f_L^b = \frac{1-\pi}{\hat{r}+\xi}$. *In the high state, the threshold* $f_\dagger$ *solves*

$$j_L(f_\dagger) = \frac{\rho - \hat{r} + \pi\lambda}{\rho + \lambda - \hat{r}} u_0(f_\dagger)$$

where the functions $u_0(f)$ *and* $u_1(f)$ *are given in equations* (17) *and* (18). *The default boundary solves*

$$u_1(f_H^b) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big) \left(\frac{f_\dagger}{f_H^b} \right)^{\psi} = 0.$$

Proof. Under the assumption that $\mu_L + \xi < 0$, the default boundary becomes

$$f_L^b = \frac{1-\pi}{\hat{r} + \xi}.$$

Using L'Hôpital rule, and noticing that $\sqrt{x^2} = \pm|x|$, we get

$$\begin{aligned} \lim_{\sigma^2 \to 0} \gamma &= \frac{1}{2} + \frac{1}{2} \left[(\mu_L + \xi)^2 \right]^{-1/2} \left[(\mu_L + \xi) + 2\left(\hat{r} + \hat{\eta} - \mu_L\right) \right] \\ &= \frac{1}{2} - \frac{1}{2} (\mu_L + \xi)^{-1} \left[(\mu_L + \xi) + 2\left(\hat{r} + \hat{\eta} - \mu_L\right) \right] \\ &= -\frac{\hat{r} + \hat{\eta} - \mu_L}{\xi + \mu_L} \end{aligned}$$

Similarly, under the assumption that $\mu_H + \xi > 0$, we get that

$$\lim_{\sigma\to 0} \phi = \infty$$
$$\lim_{\sigma\to 0} \beta_1 = \infty$$
$$\lim_{\sigma\to 0} \beta_2 = -\frac{\hat{r} + \hat{\lambda} - \mu_H}{\mu_H + \xi} = -\psi.$$

The smooth pasting condition for f_H^b can be written as

$$u_1(f_H^b)\left[\frac{\frac{\beta_2}{\beta_1}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = \frac{1}{\beta_1}u_1'(f_H^b)f_H^b + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{1 - \frac{\beta_2}{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.$$

In the limit as $\sigma^2 \to 0$, this equation simplifies to

$$u_1(f_H^b) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\left(\frac{f_\dagger}{f_H^b}\right)^{\psi} = 0$$

Similarly, we can write the smooth pasting condition at $f_\dagger$ as

$$\frac{1}{\beta_1}\big(u_0'(f_\dagger) - u_1'(f_\dagger)\big)f_\dagger + \frac{\phi}{\beta_1}\big(j_H(f_\dagger) - u_0(f_\dagger)\big) =$$
$$u_1(f_H^b)\frac{1 - \frac{\beta_2}{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2} + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \frac{\beta_2}{\beta_1}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}},$$

and taking the limit we get

$$j_H(f_\dagger) = u_0(f_\dagger)$$

Substituting in the smooth pasting condition for f_H^b, we get the following equation for f_H^b

$$u_1(f_H^b) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big)\left(\frac{f_\dagger}{f_H^b}\right)^{\psi} = 0,$$

Substituting the solution for $j_H(f_\dagger)$ in indifference condition

$$j_L(f_\dagger) = \frac{\rho - \hat{r} + \pi\lambda}{\rho + \lambda - \hat{r}} j_H(f_\dagger),$$

we obtain the following equation for $f_\dagger$:

$$\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}\left(1-\frac{(\rho-\hat{r}+\pi\lambda)(\rho+\lambda+\hat{\eta}-\mu_L)}{(\rho+\lambda-\mu_H)(\rho+\lambda-\hat{r})}\right)-\left(1-\frac{(\rho-\hat{r}+\pi\lambda)(\rho+\lambda+\hat{\eta}+\xi)}{(\rho+\lambda-\hat{r})(\rho+\lambda+\xi)}\right)\frac{1}{\hat{r}+\hat{\eta}+\xi}\left(\frac{f_\dagger}{f_L^b}\right)$$
$$+\frac{\hat{\lambda}+(\mu_H-\mu_L)(\gamma-1)-\hat{\eta}}{\rho+\lambda-\hat{r}+(\mu_H-\mu_L)(\gamma-1)-\hat{\eta}}\frac{1}{\gamma-1}\frac{1-\pi}{\hat{r}+\hat{\eta}-\mu_L}\left(\frac{f_\dagger}{f_L^b}\right)^{\gamma}=0$$

Finally, from the limit coefficients (ϕ,β_1,β_2), we obtain that the value function in the H state converges to

$$j_H(f)=\begin{cases}u_0(f) & f\in[0,f_\dagger]\\ u_1(f)+\big(u_0(f_\dagger)-u_1(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{-\psi} & f\in\left(f_\dagger,f_H^b\right],\end{cases}$$

□

Substituting the previous expressions on the equilibrium conditions determining the price, $p_\theta=-j_\theta'$, we get

Proposition 15 (Limit price of long-term debt)**.** *Under the assumptions in Proposition 14, the limit price of long-term debt when $\sigma^2\to 0$ is*

$$p_L(f)=\frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left[1-\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] \tag{A.27}$$

$$p_H(f)=\begin{cases}\frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left[1+\frac{\hat{\eta}}{\rho+\lambda+\xi}-\delta\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] & f\in[0,f_\dagger]\\ \frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi}+\psi\big(u_0(f_\dagger)-u_1(f_\dagger)\big)\frac{1}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{-(\psi+1)} & f\in\left(f_\dagger,f_H^b\right],\end{cases} \tag{A.28}$$

where, as before, the constant δ is given by

$$\delta=\frac{\rho+\lambda-\hat{r}}{\rho+\lambda-\hat{r}+(\mu_H-\mu_L)(\gamma-1)-\hat{\eta}}\in(0,1).$$

Proof. From the solution for the value function, we can obtain the price of the long-term debt. The price of the long-term bond is

$$p_L(f)=\frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left[1-\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] \tag{A.29}$$

and

$$p_H(f) = \begin{cases} \frac{\hat{r}+\xi}{\hat{r}+\hat{\eta}+\xi}\left[1 + \frac{\hat{\eta}}{\rho+\lambda+\xi} - \delta\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] & f \in [0, f_\dagger] \\ \frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi} + \psi\big(u_0(f_\dagger) - u_1(f_\dagger)\big)\frac{1}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{-(\psi+1)} & f \in \left(f_\dagger, f_H^b\right], \end{cases}$$

□

Having computed the price, we obtain the issuance function by substituting p_L, p_H, fp_L', fp_H'. In the low state,

$$g_L(f) = -\frac{\pi(r+\eta)(\xi+\mu_L)}{\hat{r}+\hat{\eta}+\xi} + \frac{\pi\eta\xi(\xi+\mu_L)}{(\hat{r}+\xi)(\hat{r}+\hat{\eta}+\xi)}\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)}.$$

If $\lambda > \bar{\lambda}$, debt issuance policy in the high state

- For $f \in (f_\dagger, f_H^b)$

$$g_H(f) = -\frac{\pi}{\psi+1}\left[r + \lambda + \frac{\xi\lambda}{\hat{r}+\hat{\lambda}+\xi}\frac{f_\dagger}{\big(u_0(f_\dagger) - u_1(f_\dagger)\big)}\left(\frac{f}{f_\dagger}\right)^{\psi+1}\right]$$

- For $f \in [0, f_\dagger]$

$$\begin{aligned} g_H(f) = & \frac{-(\xi+\mu_L)\big(\pi r + (\rho - r)(1-\delta)\big)}{\delta(\hat{r}+\hat{\eta}+\xi)} \\ & + \frac{-(\xi+\mu_L)\hat{\eta}}{\delta}\left[\frac{\pi r}{(\hat{r}+\xi)(\hat{r}+\hat{\eta}+\xi)} + \frac{\rho - r}{(\hat{r}+\hat{\eta}+\xi)(\rho+\lambda+\xi)}\right]\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)} \end{aligned}$$

From here, we get that

$$g_L'(f) = -(\gamma-1)\frac{\pi\eta\xi(\xi+\mu_L)}{(\hat{r}+\xi)(\hat{r}+\hat{\eta}+\xi)f_L^b}\left(\frac{f}{f_L^b}\right)^{-(\gamma-2)} > 0.$$

and

- For $f \in (f_\dagger, f_H^b)$

$$g_H'(f) = -\pi\frac{\xi\lambda}{\hat{r}+\hat{\lambda}+\xi}\frac{f_\dagger^2}{\big(u_0(f_\dagger) - u_1(f_\dagger)\big)}\left(\frac{f}{f_\dagger}\right)^{\psi}.$$

The conditions determining the default boundary implies that

$$u_0(f_\dagger) - u_1(f_\dagger) = -u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{-\psi} = -\left(\frac{1-\pi}{\hat{r}+\hat{\lambda}-\mu_H} - \frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi}f_H^b\right)\left(\frac{f_\dagger}{f_H^b}\right)^{-\psi} > 0,$$

where

$$\frac{1-\pi}{\hat{r}+\hat{\lambda}-\mu_H}-\frac{\hat{r}+\xi}{\hat{r}+\hat{\lambda}+\xi}f_H^b$$

is the value of waiting until the time the regime shifts to default, which must be negative at the default boundary (otherwise, it would not be optimal to default at f_H^b). This means that $g_H'(f)<0$.

- For $f\in[0,f_\dagger]$

$$g_H'(f)=-(\gamma-1)\frac{-(\xi+\mu_L)\hat{\eta}}{\delta}\left[\frac{\pi r}{(\hat{r}+\xi)(\hat{r}+\hat{\eta}+\xi)}+\frac{\rho-r}{(\hat{r}+\hat{\eta}+\xi)(\rho+\lambda+\xi)}\right]f_L^b\left(\frac{f}{f_L^b}\right)^{-(\gamma-2)}<0.$$

No Tax Shield

For the reminder of the analysis, we concentrate our attention on the case with $\pi=0$. In this case, the issuance policy reduces to $g_L(f)=0$, $g_H(f)=0$ for $f>f_\dagger$, and $f<f_\dagger$

$$g_H(f)=\frac{-(\xi+\mu_L)(\rho-r)}{\delta}\left[\frac{(1-\delta)}{r+\eta+\xi}+\frac{\eta}{(r+\eta+\xi)(\rho+\lambda+\xi)}\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)}\right].$$

We can substitute δ and γ to express $g_H(f)$ exclusively in terms of the primitive parameters

$$g_H(f)=\frac{\rho-r}{\rho+\lambda-r}\left[\frac{\eta(\xi+\mu_H)+(\mu_H-\mu_L)(r+\xi)}{r+\eta+\xi}\right.$$
$$\left.+\eta\left(\frac{(\xi+\mu_H)(r+\eta+\xi)-(\xi+\mu_L)(\rho+\lambda+\xi)}{(r+\eta+\xi)(\rho+\lambda+\xi)}\right)\left(\frac{f}{f_L^b}\right)^{\frac{r+\eta+\xi}{\xi+\mu_L}}\right]$$

In the absence of a tax shield, the equation for $f_\dagger$ reduces to

$$\frac{\lambda(\rho+\lambda-\mu_H)-(\rho-r)(\mu_H-\mu_L+\eta)}{(r+\eta-\mu_L)(\rho+\lambda-\mu_H)(\rho+\lambda-r)}-\frac{\lambda(\rho+\lambda+\xi)-(\rho-r)\eta}{(\rho+\lambda-r)(\rho+\lambda+\xi)(r+\eta+\xi)}\left(\frac{f_\dagger}{f_L^b}\right)$$
$$+\frac{\lambda+(\mu_H-\mu_L)(\gamma-1)-\eta}{\rho+\lambda-r+(\mu_H-\mu_L)(\gamma-1)-\eta}\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f_\dagger}{f_L^b}\right)^{\gamma}=0.$$

The proof of Proposition 5 follows similar steps.

Proof of Proposition 7

Comparative Statics g_H: The following comparative statics follow immediately: for any $f\in(0,f_\dagger)$, $g_H(f)$ is decreasing in λ and increasing in ρ, η and μ_H. The effect of μ_L is more difficult to determine. Differentiating the function we get

$$\frac{\partial g_H(f)}{\partial \mu_L} = \frac{\rho - r}{\rho + \lambda - r}\left[-\frac{r+\xi}{r+\eta+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma-1} - \frac{\eta}{r+\eta+\xi} + \frac{\eta}{\xi+\mu_L}\left(\frac{(\gamma-1)(\xi+\mu_H)}{\rho+\lambda+\xi}+1\right)\log\left(\frac{f}{f_L^b}\right)\right]\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)}$$

The sign of the derivative depends on the sign of

$$\Psi(f) \equiv -\frac{r+\xi}{r+\eta+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma-1} - \frac{\eta}{r+\eta+\xi} - \frac{\eta}{-(\xi+\mu_L)}\left(\frac{(\gamma-1)(\xi+\mu_H)}{\rho+\lambda+\xi}+1\right)\log\left(\frac{f}{f_L^b}\right),$$

the function $\Psi(f)$ is decreasing, with $\Psi(f_L^b) < 0$. For any $\eta > 0$, the limit when f goes to zero is $\Psi(f) \to \infty$. Thus, there is $\tilde{f}$ such that $\Psi(f) > 0$ on $[0, \tilde{f})$ and $\Psi(f) < 0$ on $(\tilde{f}, f_L^b]$. If $f_\dagger > \tilde{f}$, then g_H is increasing in μ_L for $f < \tilde{f}$ and decreasing for $f > \tilde{f}$. When $\eta = 0$, the issuance function reduces to $g_H(f) = \frac{(\rho-r)(\mu_H-\mu_L)}{\rho+\lambda-r}$, which is decreasing in μ_L.

Proof of Proposition 8

Sample Path: The ODE describing the evolution of f_t on $(0, f_\dagger)$ can be solved in closed form. Let

$$a_0 = \frac{-(\xi+\mu_L)(\rho-r)(1-\delta)}{(r+\eta+\xi)\delta} - (\xi+\mu_H)$$
$$a_1 = \frac{-(\xi+\mu_L)}{\delta}\frac{(\rho-r)\eta}{(r+\eta+\xi)(\rho+\lambda+\xi)}{f_L^b}^{\gamma-1},$$

so for $f < f_\dagger$, f_t solves

$$\dot{f}_t = a_0 f_t + a_1 f_t^{2-\gamma}.$$

This coefficients can be written as

$$\frac{a_1}{a_0} = \underbrace{-\frac{\eta}{\rho+\lambda+\xi}\left[\left(1+\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{r+\eta+\xi}{\rho-r}\right)\delta - 1\right]^{-1}}_{\equiv\kappa} {f_L^b}^{(\gamma-1)}$$

$$a_0(\gamma-1) = \underbrace{-(r+\eta+\xi)\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)} - \frac{(\rho-r)\Big((\mu_H-\mu_L)(\gamma-1)-\eta\Big)}{(r+\eta+\xi)(\rho+\lambda-r)}\right]}_{\equiv\nu}.$$

Substituting γ and simplifying terms, we get

$$\nu = \frac{(r+\eta+\xi)(\rho-r)}{\rho+\lambda-r}\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r} - \frac{r+\xi}{r+\eta+\xi}\right],$$

which is positive only if

$$\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r} > \frac{r+\xi}{r+\eta+\xi}.$$

In addition, $\kappa \propto \nu$, so it is positive only if ν is positive as well.

Given these definitions, we can can write

$$\frac{\dot{f}_t}{f_t} = -\frac{\nu}{\gamma-1}\left[1-\kappa\left(\frac{f_t}{f_L^b}\right)^{1-\gamma}\right].$$

Moreover, this equation can be solved in closed form. Letting $z_t = \log f_t$, we get the equation

$$\dot{z}_t = -\frac{\nu}{\gamma-1}\left[1-\kappa {f_L^b}^{\gamma-1} e^{(1-\gamma)z_t}\right].$$

The general solution to these equation is given provided in (Zaitsev and Polyanin, 2002, p. 162), and is given by

$$z = \frac{1}{\gamma-1}\log\left(Ce^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\right).$$

From here we get that

$$f = \left[Ce^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\right]^{\frac{1}{\gamma-1}}.$$

The integration constant is determined by the initial condition

$$C = f_0^{\gamma-1} - \kappa {f_L^b}^{\gamma-1},$$

so it follows that

$$f_t = \left[f_0^{\gamma-1} e^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\left(1-e^{-\nu t}\right)\right]^{\frac{1}{\gamma-1}}.$$

Comparative Statics Path Let start considering the speed of adjustment ν

- ξ:

$$\frac{\partial \nu}{\partial \xi} = \frac{\lambda\mu_H\left(\eta-\mu_L+r\right) - \mu_L\left(\eta\lambda + 2\xi(\lambda+\rho) + \mu_L(\rho-r) + r(\lambda-2\xi)\right) + \xi^2(-\lambda-\rho+r)}{\left(\mu_L+\xi\right)^2\left(\rho+\lambda-r\right)}$$

The denominator is positive, and the numerator is positive if and only if

$$\sqrt{\frac{\lambda\left(\mu_H-\mu_L\right)\left(\eta-\mu_L+r\right)}{\lambda+\rho-r}}+\mu_L+\xi>0$$

- μ_L:

$$\frac{\partial \nu}{\partial \mu_L}=\frac{\lambda\left(\mu_H+\xi\right)\left(\eta+\xi+r\right)}{\left(-\mu_L-\xi\right)^2\left(\left(\rho+\lambda-r\right)\right.}>0.$$

- μ_H:

$$\frac{\partial \nu}{\partial \mu_H}=\frac{\lambda\left(\mu_H+\xi\right)\left(\eta+\xi+r\right)}{\left(-\mu_L-\xi\right)^2\left(\left(\rho+\lambda-r\right)\right.}>0.$$

- λ:

$$\frac{\partial \nu}{\partial \lambda}=\frac{\eta\xi\rho+\mu_H(\rho+r)(\eta+\xi+r)-\mu_L(\xi+r)(\rho-r)+\xi r(\eta+2\xi+2r)}{-\left(\mu_L+\xi\right)\left((\rho+\lambda-r)^2\right.}>0.$$

- η:

$$\frac{\partial \nu}{\partial \eta}=\frac{\lambda\left(\mu_H+\xi\right)}{-\left(\mu_L+\xi\right)\left(\left(\rho+\lambda-r\right)\right.}>0.$$

- ρ:

$$\begin{aligned}\nu&=\frac{(r+\eta+\xi)(\rho-r)}{(\rho+\lambda-r)}\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r}-\frac{r+\xi}{r+\eta+\xi}\right]\\&=-\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)(\xi+\mu_L)}-\frac{(\rho-r)(r+\xi)}{(\rho+\lambda-r)}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial \nu}{\partial \rho}&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}-\frac{(r+\xi)(\rho+\lambda-r)-(\rho-r)(r+\xi)}{(\rho+\lambda-r)^2}\\&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}-\frac{(r+\xi)\lambda}{(\rho+\lambda-r)^2}\\&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda-(r+\xi)(\xi+\mu_L)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}<0\end{aligned}$$

A.3 Other Proofs of Section 3

Proof of Corollary 1

Proof. We differentiate the HJB at $\theta=L$, which leads to

$$(r+\eta+\xi)\, j_L^{'}(f)+(r+\xi)+\left(\mu_L+\xi-\sigma^2\right) f j_L^{''}(f)-\frac{1}{2}\sigma^2 f^2 j_L^{'''}(f)=0, \quad \forall f\in\left[0, f_L^b\right].$$

When $f\in[0,f_\dagger]$, we differentiate the HJB at $\theta=H$:

$$(\rho+\lambda+\xi)\, j_H'(f)+(r+\xi)-(\rho+\lambda-r)\, j_L'(f)+\left(\mu_H+\xi-\sigma^2\right) f j_H''(f)-\frac{1}{2}\sigma^2 f^2 j_H'''(f)=0.$$

The difference is

$$\begin{aligned}&\frac{1}{2}\sigma^2 f^2\left[j_L'''(f)-j_H'''(f)\right]-\left(\mu_H+\xi-\sigma^2\right) f\left(j_L''(f)-j_H''(f)\right)-(\rho+\lambda+\xi)\left(j_L'(f)-j_H'(f)\right)\\&=\eta j_L'(f)-(\mu_H-\mu_L)\, f j_L''(f)<0\end{aligned}$$

since $j_L'(f)\le 0$ and $j_L''(f)>0$ from the strict convexity. By the maximum principle, $\triangle(f)=j_L'(f)-j_H'(f)$ can not have a nonpositive minimum in the region $f\in[0,f_\dagger]$.

When $f\in\left(f_\dagger, f_L^b\right]$, we differentiate the HJB at $\theta=H$:

$$(r+\lambda+\xi)\, j_H^{'}(f)+(r+\xi)+\left(\mu_H+\xi-\sigma^2\right) f j_H''(f)-\frac{1}{2}\sigma^2 f^2 j_H'''(f)=0.$$

The difference is

$$\begin{aligned}&\frac{1}{2}\sigma^2 f^2\left(j_L'''(f)-j_H'''(f)\right)-\left(\mu_H+\xi-\sigma^2\right) f\left(j_L''(f)-j_H''(f)\right)\\&-(r+\lambda+\xi)\left(j_L^{'}(f)-j_H^{'}(f)\right)=(\eta-\lambda)\, j_L'(f)-(\mu_H-\mu_L)\, f j_L''(f)<0\end{aligned}$$

where we assume $\eta\ge\lambda$ and $j_L'(f)\le 0$ and $j_L''(f)>0$ from the strict convexity. By the maximum principle, $\triangle(f)\equiv j_L'(f)-j_H'(f)$ can not have a nonpositive minimum in the region $f\in\left(f_\dagger, f_L^b\right]$. Since both $j_L'(f)$ and $j_H'(f)$ are continuous for all $f\in\left[0,f_L^b\right]$, $\triangle(f)$ is continuous for all $f\in\left[0,f_L^b\right]$. It implies that $\triangle(f)$ can not have a nonpositive minimum in the region $f\in\left[0,f_L^b\right]$.

In addition, given that

$$\begin{aligned}\triangle(0)&=-\frac{r+\xi}{r+\eta+\xi}+\frac{(r+\xi)(\eta+\rho+\lambda+\xi)}{(\rho+\lambda+\xi)(r+\eta+\xi)}=\frac{r+\xi}{r+\eta+\xi}\frac{\eta}{\rho+\lambda+\xi}>0,\\ \triangle\left(f_L^b\right)&=-j_H'(f_L^b)>0,\end{aligned}$$

we know $\triangle(f)>0$ for any $f\in\left[0,f_L^b\right]$.

From $p_L(f)=-j_L'(f)$, $p_H(f)=-j_H'(f)$, we know $p_H(f)>p_L(f)$ for any $f\in\left[0,f_L^b\right]$.

Furthermore, when $f \in \left[0, f_L^b\right]$, the firm never repurchases the long-term debt since

$$g(f) = \frac{(\rho - r)(p_H(f) - p_L(f))}{-f p_H'(f)} = \frac{(\rho - r)(p_H(f) - p_L(f))}{f j_H''(f)} > 0.$$

□

B Analysis of Extensions in Section 5

B.1 Section 5.1 with Hedging

As $(\rho - r)(j_H(f) - j_L(f)) > 0$, the solution in Lemma 1 becomes $d_H(f) = j_H(f)$ and $z(f) = j_H(f) - j_L(f)$ for all $f \in [0, f_H^b]$. Given the optimal policy in Lemma 1, we can write the HJB equation in simpler form

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) &= 1 - (r + \xi)\, f + (\rho + \lambda - r)\, j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (0, f_\dagger) \\ (r + \lambda - \mu_H)\, j_H(f) &= 1 - (r + \xi)\, f + q\lambda j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (f_\dagger, f_H^b).\end{aligned} \tag{A.30}$$

The solution to the HJB equation takes the same form given in equation (16). The function u_0 is still given by (17), but the expression for u_1 is different to the one in equation (18) because it includes the term $q\lambda j_L(f)$ capturing the continuation value after the regime switch. The explicit expression for u_1 in this case is provided in the appendix.

The price of debt is now given by the solution to the asset pricing equation.

$$\begin{aligned}(r + \xi + \lambda)\, p_H(f) &= r + \xi + \lambda p_L(f) + \left(g_H(f) + \sigma^2 - \mu_H - \xi\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f \in (0, f_\dagger) \\ (r + \xi + \lambda)\, p_H(f) &= r + \xi + \lambda q p_L(f) + \left(g_H(f) + \sigma^2 - \mu_H - \xi\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f \in (f_\dagger, f_H^b).\end{aligned}$$

From, here, together with the indifference condition $j_H'(f) = -p_H(f)$, we can obtain the equilibrium issuance function. We omit the details, but a similar calculations to the ones in the absence of hedging show that the equilibrium issuance policy is given by (22). Notice that although the form of the issuance function does not change the total issuance of long-term debt does change as the price of long-term debt is now different. The main impact of hedging though is on the value of the threshold $f_\dagger$.

Proof of Lemma 1 and Proposition 12

Proof. In equation (24), d_H and z are chosen to maximize

$$-q\lambda z + (\rho + \lambda - y_H)\, d_H + \lambda q \max\left\{j_L(f) + z - d_H, 0\right\} + \lambda(1-q) \max\left\{j_L(f) - d_H, 0\right\}.$$

There are three situations that we need to consider:

1. If $d_H \leq j_L(f)$, the objective becomes

   $$-q\lambda z + (\rho + \lambda - r)\, d_H + \lambda q \left(j_L(f) + z - d_H\right) + \lambda(1-q) \left(j_L(f) - d_H\right) = (\rho - r)\, d_H + \lambda j_L(f),$$

   which is maximized at $d_H = j_L(f)$ with the maximum value

   $$(\rho + \lambda - r)\, j_L(f).$$

2. If $d_H \in \left(j_L(f), j_L(f) + z\right]$, the objective becomes

   $$-q\lambda z + (\rho + \lambda q - r)\, d_H + \lambda q \left(j_L(f) + z - d_H\right) = (\rho - r)\, d_H + \lambda q j_L(f),$$

   which is maximized at $d_H = j_L(f) + z$ with the maximum value

   $$(\rho - r + \lambda q) j_L(f) + (\rho - r) z.$$

   Given that $d_H = j_L(f) + z \leq j_H(f)$, we know $z \leq j_H(f) - j_L(f)$. The maximized $z = j_H(f) - j_L(f)$, and the maximum value is

   $$(\rho - r) j_H(f) + \lambda q j_L(f).$$

3. If $d_H > j_L(f) + z$, the objective becomes $-q\lambda z + (\rho - r)\, d_H$, which is clearly maximized at $z = 0$ and $d_H = j_H(f)$, with a maximum value

   $$(\rho - r)\, j_H(f).$$

Clearly, the last one is dominated, so the borrower's choice is

- If $(\rho - r) j_H(f) + \lambda q j_L(f) \leq (\rho + \lambda - r) j_L(f)$, then $d_H = j_L(f)$, and z is irrelevant so without loss of generality set as zero.

- Otherwise, then $d_H = j_H(f)$ and $z = j_H(f) - j_L(f)$.

$\square$

Solution HJB Equation

For $f \in (0, f_\dagger)$ there is no change in the differential equation, so the solution remains the same. On $f \in (f_\dagger, f_H^b)$ the HJB equation becomes

$$(r + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f + q\lambda j_L(f) + \mathcal{D}^H j_H(f)$$

If $f \geq f_L^b$, the continuation value $j_L(f)$ and the particular solution is

$$u_1(f) = \frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi} f.$$

When $f < f_L^b$, the particular solution takes the form

$$u_1(f) = \frac{1}{r + \lambda - \mu_H}\left(1 + q\frac{\lambda}{r + \eta - \mu_L}\right) - \frac{r + \xi}{r + \lambda + \xi}\left(1 + q\frac{\lambda}{r + \eta + \xi}\right) f + C\left(\frac{f}{f_L^b}\right)^\gamma,$$

Substituting in the previous the ODE, we find that the constant C is given by

$$C = \frac{\lambda q}{\lambda - \eta + (\mu_H - \mu_L)(\gamma - 1)} \frac{1}{\gamma - 1} \frac{1}{r + \eta - \mu_L}.$$

The solution then to the HJB equation is

$$j_H(f) = \begin{cases} u_0(f) + \big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^\phi & f \in [0, f_\dagger] \\ u_1(f) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big) h_0\left(f \middle| f_\dagger, f_L^b\right) + \big(j_H(f_L^b) - u_1(f_L^b)\big) h_1\left(f \middle| f_\dagger, f_L^b\right) & f \in \left(f_\dagger, f_L^b\right) \\ u_1(f) + \big(j_H(f_L^b) - u_1(f_L^b)\big) h_0\left(f \middle| f_L^b, f_H^b\right) - u_1(f_H^b) h_1\left(f \middle| f_L^b, f_H^b\right) & f \in \left[f_L^b, f_H^b\right] \end{cases}$$

where

$$u_0(f) = \mathcal{A}\frac{1}{\rho + \lambda - \mu_H} - \mathcal{B}\frac{r + \xi}{\rho + \lambda + \xi} f + \delta\frac{1}{\gamma - 1}\frac{1}{r + \eta - \mu_L}\left(\frac{f}{f_L^b}\right)^\gamma$$

$$u_1(f) = \begin{cases} \frac{1}{r+\lambda-\mu_H}\left(1 + \frac{\lambda q}{r+\eta-\mu_L}\right) - \frac{r+\xi}{r+\lambda+\xi}\left(1 + \frac{\lambda q}{r+\eta+\xi}\right) f + C\left(\frac{f}{f_L^b}\right)^\gamma & \text{if } f < f_L^b \\ \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi} f & \text{if } f \geq f_L^b, \end{cases}$$

and the constants $\mathcal{A}, \mathcal{B}, \delta$ are given by

$$\mathcal{A} = \frac{\rho+\lambda+\eta-\mu_L}{r+\eta-\mu_L} \qquad \mathcal{B} = \frac{\rho+\lambda+\eta+\xi}{r+\eta+\xi} \qquad \delta = \frac{\rho+\lambda-r}{\rho+\lambda-r-\eta+(\mu_H-\mu_L)(\gamma-1)}.$$

The functions $h_0(\cdot)$ and $h_1(\cdot)$ are defined in equation (A.13)

To show that $f_\dagger$ is decreasing in q when $f_\dagger > 0$, it suffices to show that $j_H(f)$ is increasing in q.

Lemma 11. *If $(\rho + \lambda(1-q) - r)\, j_L(0) > (\rho - r)\, j_H(0)$, then the value function $j_H(f)$ is strictly increasing in q.*

Proof. For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t}\left(1-(r+\xi)z_t + \nu(z_t, \tilde{j}(z_t)|q)\right) dt \Big| z_0 = f\right]$$

$$\text{subject to} \quad dz_t = -(\xi+\mu_H)z_t dt - \sigma z_t dB_t,$$

where

$$\nu(z, \tilde{j}|q) \equiv \max\{(\rho+\lambda-r)\, j_L(z)\,,\, q\lambda j_L(z) + (\rho - r)\,\tilde{j}\}$$

and $\hat{\rho} \equiv \rho + \lambda - \mu_H$. It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f, \tilde{j}_1|q) \geq \nu(f, \tilde{j}_0|q)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\begin{aligned}\nu(z, \tilde{j}+a|q) &= \max\{(\rho+\lambda-r)\, j_L(z), q\lambda j_L(z) + (\rho-r)(\tilde{j}+a)\} \\ &\leq \max\{(\rho+\lambda-r)\, j_L(z) + (\rho-r)a, q\lambda j_L(z) + (\rho-r)(\tilde{j}+a)\} = (\rho-r)a + \nu(z,\tilde{j}|q),\end{aligned}$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)+a|q)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad + \frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad + \frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f)+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \leq \Phi(\tilde{j})(f)+\frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}$$

As Φ is monotone and satisfies discounting, it follows from Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f)=\Phi(j_H)(f)$.

For any pair of parameters $q_1 \geq q_0$, the inequality $\nu(f,\tilde{j}|q_1) \geq \nu(f,\tilde{j}|q_0)$ implies that the operator Φ is increasing q. It follows from Theorem 1 in Villas-Boas (1997) that the fixed point $j_H(f)=\Phi(j_H)(f)$ increases in q. □

B.2 Section 5.2 with Restructuring

The value function satisfies

$$\begin{aligned}
(\rho+\lambda-\mu_H)\,j_H(f) &= 1-(r+\xi)\,f+(\rho+\lambda-r)j_L(f)-(\mu_H+\xi)fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f), \quad f\in(0,f_\dagger) \\
(r+\lambda-\mu_H)\,j_H(f) &= 1-(r+\xi)\,f+\lambda\alpha j_L(f)-(\mu_H+\xi)fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f), \quad f\in(f_\dagger,f_H^b).
\end{aligned} \tag{A.31}$$

We see that the only difference with the original equation is that now, when the firm is fully levered, there is a term $\lambda\alpha j_L(f)$ capturing the continuation value after the regime shift. Notice that the HJB equation (A.31) takes the same form as the one with hedging in equation (A.30), so hedging and renegotiation serve a similar economic purpose in the model. The asset pricing equation for bond prices becomes

$$\begin{aligned}
(r+\xi+\lambda)\,p_H(f) &= r+\xi+\lambda p_L(f)+\left(g_H(f)+\sigma^2-\mu_H-\xi\right)fp_H'(f)+\frac{1}{2}\sigma^2f^2p_H''(f), \quad f\in(0,f_\dagger) \\
(r+\xi+\lambda)\,p_H(f) &= r+\xi+\lambda\alpha p_L+\left(g_H(f)+\sigma^2-\mu_H-\xi\right)fp_H'(f)+\frac{1}{2}\sigma^2f^2p_H''(f), \quad f\in(f_\dagger,f_H^b).
\end{aligned}$$

Thus, together with the indifference condition $j'_H(f) = -p_H(f)$, we obtain the equilibrium issuance function in the high state is given by the same expression as the one in equation (22). The proof of Proposition 13 follows the one in Proposition 12.

B.3 Section B.3 with Transitory Shocks

In this section, we extend the model to consider some further empirical implications. In the main model, we have assumed that the state $\theta_t = L$, is absorbing. If we interpret the changes in regime as business-cycles, it is natural to assume that these are transitory. We can extend the model to consider this situation. We denote the transition rate from the high state to the low state by λ_{HL}, and the transition rate from the low state to the high state by λ_{LH}. The stationary distribution of the process θ_t is then given by $\Pr(\theta = H) = \lambda_{LH}/(\lambda_{LH} + \lambda_{HL})$.

The equilibrium has the same qualitative features. The only changes is that in the HJB equation (10) for $j_L(f)$ and in the asset pricing equation (20) for $p_L(f)$, we have to add additional terms $\lambda_{LH}\big(j_H(f) - j_L(f)\big)$ and $\lambda_{LH}\big(p_H(f) - p_L(f)\big)$, respectively. We provide the detailed calculations in the appendix. The issuance policy takes the general form provided in equation (22). As in the case where the low state is absorbing, we can obtain a more explicit solution for the equilibrium in the limit when $\sigma \to 0$.

Proposition 16 (Limit long-term debt issuance policy)**.** *Suppose that there is no tax shield (that is, $\pi = 0$), $\mu_L + \xi < 0$, $\mu_H + \xi > 0$, and*

$$(r + \lambda_{HL} - \mu_H)\,(r + \eta - \mu_L) + (r - \mu_H)\,\lambda_{LH} \geq 0.$$

In the limit when $\sigma \to 0$, the issuance policy is

$$g_\theta(f) = \frac{\rho - r}{\gamma - 1}\left[g_0 + g_1\left(\frac{f}{f_\dagger}\right)^{-(\gamma-1)}\right]\mathbb{1}_{\{f<f_\dagger,\,\theta=H\}}$$

where g_0 and g_1 are positive coefficients and $\gamma > 1$ is the unique positive root of

$$\gamma^2 + \left(\frac{\rho + \lambda_{HL} - \mu_H}{\mu_H + \xi} - \frac{r + \lambda_{LH} + \eta - \mu_L}{-(\mu_L + \xi)}\right)\gamma - \frac{(\rho + \lambda_{HL} - \mu_H)\,(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH}}{-(\mu_L + \xi)(\mu_H + \xi)} = 0.$$

When shocks are transitory, the HJB equation for $j_L(f)$ becomes

$$(\rho + \eta - \mu_L)\,j_L\,(f) = 1 - (r + \xi)\,f + (\rho - r)\,j_L(f) + \mathcal{D}^L j_L(f) + \lambda_{LH}\big(j_H(f) - j_L(f)\big).$$

The indifference condition for the issuance of short-term debt in high state remains the same and is given by

$$(\rho + \lambda_{HL} - r)\, j_L(f_\dagger) \geq (\rho - r)\, j_H(f_\dagger).$$

We solve the equation in this region, and the combines the solution using smooth pasting and value matching conditions at the threshold $f_\dagger$. The default boundary are determined using the same value matching and smooth pasting conditions as in the main version of the model.

Solution for $f \in (0, f_\dagger)$. The characteristic equation of the associated homogenous equation is now a quartic equation instead of a quadratic one. Hence the solution takes the general form:

$$\begin{aligned} j_L(f) &= A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2} + A_4 f^{\gamma_3} + A_5 f^{\gamma_4} \\ j_H(f) &= B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2} + B_4 f^{\gamma_3} + B_5 f^{\gamma_4}. \end{aligned}$$

Substituting the conjecture in the ODE, we get the linear system

$$\begin{aligned} (r + \lambda_{LH} + \eta - \mu_L)\, A_0 &= 1 + \lambda_{LH} B_0 \\ (\rho + \lambda_{HL} - \mu_H)\, B_0 &= 1 + (\rho + \lambda_{HL} - r)\, A_0 \\ (r + \xi + \lambda_{LH} + \eta)\, A_1 &= (r + \xi) + \lambda_{LH} B_1 \\ (\rho + \xi + \lambda_{HL})\, B_1 &= (r + \xi) + (\rho + \lambda_{HL} - r)\, A_1. \end{aligned}$$

It follows that

$$\begin{aligned} A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + \lambda_{LH}(r - \mu_H)} \\ A_1 &= \frac{(\rho + \xi + \lambda_{HL} + \lambda_{LH})(r + \xi)}{(\rho + \xi + \lambda_{HL})(r + \xi + \eta) + \lambda_{LH}(r + \xi)} \\ B_0 &= \frac{1 + (\rho + \lambda_{HL} - r)\, A_0}{\rho + \lambda_{HL} - \mu_H} \\ B_1 &= \frac{(r + \xi) + (\rho + \lambda_{HL} - r)\, A_1}{\rho + \xi + \lambda_{HL}} \end{aligned}$$

In addition, for any $i = 1, \ldots, 4$

$$\begin{aligned} (r + \lambda_{LH} + \eta - \mu_L)\, A_{i+1} &= -(\mu_L + \xi)\, A_{i+1}\gamma_i + \lambda_{LH} B_{i+1} + \frac{1}{2}\sigma^2 A_{i+1}\gamma_i(\gamma_i - 1) \\ (\rho + \lambda_{HL} - \mu_H)\, B_{i+1} &= (\rho + \lambda_{HL} - r)\, A_{i+1} - (\mu_H + \xi)\, B_{i+1}\gamma_i + \frac{1}{2}\sigma^2 B_{i+1}\gamma_i(\gamma_i - 1) \end{aligned}$$

If we multiply the equation for A_2 by γ_1, we get

$$(r+\lambda_{LH}+\eta-\mu_L)\gamma_1 A_2 = -(\mu_L+\xi)A_2\gamma_1^2 + \lambda_{LH}B_2\gamma_1 + \frac{1}{2}\sigma^2 A_2\gamma_1^2(\gamma_1-1)$$

when $\lambda_{LH} \neq 0$, from the equation for A_2 we have

$$\lambda_{LH}B_2 = \left[(r+\lambda_{LH}+\eta-\mu_L)+(\mu_L+\xi)\gamma_1 - \frac{1}{2}\sigma^2\gamma_1(\gamma_1-1)\right]A_2.$$

Substituting in the equation for B_2 we obtain an expression for $B_2\gamma_1$ that can be then substituted back in the equation for A_2 (multiplied by γ_1). Canceling A_2, we obtain the characteristic equation for the homogenous equation

$$\begin{aligned}
&\frac{1}{4}\sigma^4\gamma_1^4 + \frac{1}{2}\sigma^2\left(\mu_L+\mu_H+2\xi+\sigma^2\right)\gamma_1^3 \\
&\quad + \left[\frac{1}{4}\sigma^4 - \frac{1}{2}\sigma^2\left(\rho+\lambda_{HL}+r+\lambda_{LH}+\eta-2\left(\mu_L+\mu_H+\xi\right)\right)+(\mu_L+\xi)(\mu_H+\xi)\right]\gamma_1^2 \\
&\quad + \left[\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)(r+\lambda_{LH}+\eta-\mu_L)+(\rho+\lambda_{HL}-\mu_H)\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\right]\gamma_1 \\
&\qquad + (\rho+\lambda_{HL}-\mu_H)(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH} = 0
\end{aligned}$$

This equation has four roots.

Solution for $f \in (f_\dagger, f_H^b)$. In this case, we guess a solution of the form

$$\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4} \\
j_H(f) &= D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.
\end{aligned}$$

From the HJB equation for $j_H(f)$, we get that β_1 and β_2 are the roots for the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)\beta + \mu_H - r - \lambda_{HL} = 0,$$

which are given by

$$\begin{aligned}
\beta_1 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 - 2\sigma^2(\mu_H-r-\lambda_{HL})}}{\sigma^2}, \\
\beta_2 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 - 2\sigma^2(\mu_H-r-\lambda_{HL})}}{\sigma^2}.
\end{aligned}$$

From the equation for j_L, we get that β_3 and β_4 are given by the roots to the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\beta - (r + \lambda_{LH} + \eta - \mu_L) = 0,$$

which are

$$\beta_3 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} > 1,$$

$$\beta_4 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} < 0.$$

Matching coefficients, we get that

$$D_0 = \frac{1}{r + \lambda_{HL} - \mu_H}$$

$$D_1 = \frac{r + \xi}{r + \xi + \lambda_{HL}}$$

$$C_0 = \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L}$$

$$C_1 = \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta}$$

$$C_2 = \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_1 - \frac{1}{2}\sigma^2(\beta_1 - 1)\beta_1}$$

$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_2 - \frac{1}{2}\sigma^2(\beta_2 - 1)\beta_2}.$$

Boundary Conditions. We still need to determine the coefficients (A_i, B_i) for $i = 2, \ldots, 5$, the coefficients D_2, D_3, and C_4, C_5, as well as the thresholds $f_\dagger, f_H^b, f_L^b$.

We start considering $f \in (0, f_\dagger)$. Under reasonable parameters, we have found that all four roots of the quartic characteristic equation are real, and that two of them are positive (let the positive roots be γ_1 and γ_2). If this is the case, the transversality conditions

$$\lim_{f \to 0} j_H(f) < \infty,$$

$$\lim_{f \to 0} j_L(f) < \infty,$$

imply that $A_4 = A_5 = B_4 = B_5 = 0$. Thus, we can write the value function as

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}$$
$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2},$$

where the coefficients A_0, A_1, B_0, B_1 have already been determined. Moreover, from the previous analysis we already have that for $i = 2, 3$

$$\Big[(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\,\gamma_{i-1} - \frac{1}{2}\sigma^2 \gamma_i\,(\gamma_{i-1} - 1) \Big] A_i = \lambda_{LH} B_i,$$

so the coefficients $\{B_2, B_3\}$ are immediately determined by the 2 free coefficients $\{A_2, A_3\}$.

Next, we consider the intervals $(f_\dagger, f_H^b)$ and $(f_\dagger, f_L^b)$. Here we have that $j_\theta(f)$ takes the form

$$j_L(f) = C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4}$$
$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.$$

where we have 4 free coefficients $\{C_2, C_3, C_4, C_5\}$ since $\{D_2, D_3\}$ are fully determined by $\{C_2, C_3\}$. Thus, we have to determine $(A_2, A_3, C_2, C_3, C_4, C_5)$ in addition to the free boundary $(f_\dagger, f_L^b, f_H^b)$; hence, we need 9 boundary conditions. The first boundary condition is the indifference condition

$$(\rho + \lambda_{HL} - r)\, j_L(f_\dagger) = (\rho - r)\, j_H(f_\dagger).$$

The value function must be continuously differentiable at $f_\dagger$ se we have the value matching and smooth pasting conditions at $f_\dagger$

$$j_H(f_\dagger-) = j_H(f_\dagger+)$$
$$j_L(f_\dagger-) = j_L(f_\dagger+)$$
$$j'_H(f_\dagger-) = j'_H(f_\dagger+)$$
$$j'_L(f_\dagger-) = j'_L(f_\dagger+).$$

Finally, we have the value matching and smooth pasting conditions at the default boundary

$$
\begin{aligned}
j_L(f_L^b) &= 0 \\
j_H(f_H^b) &= 0 \\
j_L'(f_L^b) &= 0 \\
j_H'(f_H^b) &= 0.
\end{aligned}
$$

Substituting the value function in these conditions, we get

$$
\begin{aligned}
A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= \frac{\rho - r}{\rho + \lambda_{HL} - r} \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right) \\
B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} &= D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} \\
A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4} \\
-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} &= -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} \\
-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} &= -C_1 + \beta_1 C_1 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1} \\
&\quad + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}
\end{aligned}
$$

and

$$
\begin{aligned}
C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} &= 0 \\
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0 \\
-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} &= 0 \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0.
\end{aligned}
$$

We can simply the above 9 equations into 3 equations and only solve the three unknowns $(f_\dagger, f_L^b, f_H^b)$: From

$$
\begin{aligned}
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0, \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0,
\end{aligned}
$$

we know

$$D_2 = \frac{\frac{\beta_2}{r+\lambda_{HL}-\mu_H} - (\beta_2 - 1)\frac{r+\xi}{r+\xi+\lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_1}(\beta_1 - \beta_2)}$$
$$D_3 = \frac{\frac{\beta_1}{r+\lambda_{HL}-\mu_H} - (\beta_1 - 1)\frac{r+\xi}{r+\xi+\lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_2}(\beta_2 - \beta_1)}.$$

Then we know C_2, C_3 from

$$C_2 = \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_1 - \frac{1}{2}\sigma^2(\beta_1 - 1)\beta_1}$$
$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_2 - \frac{1}{2}\sigma^2(\beta_2 - 1)\beta_2}$$

From

$$B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} = D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2}$$
$$-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} = -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1},$$

we know

$$B_2 = \frac{-D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} + B_1 - \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1}}{\gamma_1 (f_\dagger)^{\gamma_1 - 1}},$$
$$B_3 = \frac{D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} - \left(B_0 - B_1 f_\dagger + \frac{1}{\gamma_1}\left(-D_1 f_\dagger + \beta_1 D_2 (f_\dagger)^{\beta_1} + \beta_2 D_3 (f_\dagger)^{\beta_2} + B_1 f_\dagger\right)\right)}{\left(1 - \frac{\gamma_2}{\gamma_1}\right)(f_\dagger)^{\gamma_2}}$$

Then we know A_2 and A_3 where

$$A_2 = \frac{\lambda_{LH} B_2}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1)}.$$
$$A_3 = \frac{\lambda_{LH} B_3}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma_2 - \frac{1}{2}\sigma^2 \gamma_2 (\gamma_2 - 1)}$$

From

$$C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} = 0$$
$$-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} = 0$$

we know

$$C_4 = \frac{C_1 - \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} - \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}},$$

$$C_5 = \frac{-C_0 + C_1 f_L^b - C_2 \left(f_L^b\right)^{\beta_1} - C_3 \left(f_L^b\right)^{\beta_2} - \frac{C_1 f_L^b - \beta_1 C_2 \left(f_L^b\right)^{\beta_1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2}}{\beta_3}}{\left(1 - \frac{\beta_4}{\beta_3}\right)\left(f_L^b\right)^{\beta_4}}$$

Therefore, we only need to solve $(f_\dagger, f_L^b, f_H^b)$ from the following 3 equations:

$$A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} = \frac{\rho - r}{\rho + \lambda_{HL} - r}\left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right)$$

$$A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} = C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4}$$

$$-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} = -C_1 + \beta_1 C_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1} + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}$$

Limit when $\sigma \to 0$

Consider the case when $\mu_L + \xi < 0 < \mu_H + \xi$. The characteristic equation for γ_i converges to the quadratic equation

$$(\mu_L + \xi)(\mu_H + \xi)\gamma_1^2 + \left[(\mu_H + \xi)(r + \lambda_{LH} + \eta - \mu_L) + (\rho + \lambda_{HL} - \mu_H)(\mu_L + \xi)\right]\gamma_1 + (\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH} = 0$$

The present value of cash flow is finite given the creditors discount rate only if

$$(r + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH} > 0,$$

which implies that quadratic equation has one negative and one positive root. Let γ be the positive root, which can be verified to be always greater than 1. Similarly, the roots β_i converge to

$$\beta_1 = \infty$$

$$\beta_2 = -\frac{r + \lambda_{HL} - \mu_H}{\mu_H + \xi}$$

$$\beta_3 = \frac{r + \lambda_{LH} + \eta - \mu_L}{-(\mu_L + \xi)}$$

$$\beta_4 = -\infty.$$

Thus the solution to the HJB equation on $(0, f_\dagger)$, becomes

$$j_L(f) = A_0 - A_1 f + A_2 f^\gamma$$
$$j_H(f) = B_0 - B_1 f + B_2 f^\gamma,$$

while the solution for $(f_\dagger, f_H^b)$ and $(f_\dagger, f_L^b)$ becomes

$$j_L(f) = C_0 - C_1 f + C_3 f^{\beta_2} + C_4 f^{\beta_3}$$
$$j_H(f) = D_0 - D_1 f + D_3 f^{\beta_2}.$$

The coefficient A_2, B_2 are given by

$$A_2 = \frac{\lambda_{LH}}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma} \frac{B_1 - D_1}{\gamma (f_\dagger)^{\gamma - 1}}.$$

From the continuity and smoothness of $j_H(f)$ at $f_\dagger$, we know

$$B_0 - B_1 f_\dagger + B_2 f_\dagger^\gamma = D_0 - D_1 f_\dagger + D_3 f_\dagger^{\beta_2}$$
$$-B_1 + \gamma B_2 f_\dagger^{\gamma-1} = -D_1 + \beta_2 D_3 f_\dagger^{\beta_2 - 1},$$

which implies that

$$B_2 = \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{(\beta_2 - \gamma) f_\dagger^\gamma}$$
$$D_3 = \frac{(1 - \gamma)(D_1 - B_1) f_\dagger + \gamma (D_0 - B_0)}{(\beta_2 - \gamma) f_\dagger^{\beta_2}}.$$

The coefficients C_3, C_4 are

$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_2}$$
$$C_4 = \frac{C_1 - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}}.$$

Substituting in the HJB equation, we can write the solution for $f \in [0, f_\dagger]$ as

$$j_L(f) = A_0 - A_1 f + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\gamma} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^\gamma$$
$$j_H(f) = B_0 - B_1 f + \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^\gamma .$$

For $f > f_\dagger$, we can write

$$\begin{aligned} j_H(f) &= D_0 - D_1 f + \left[\frac{(1-\gamma)(D_1 - B_1) f_\dagger + \gamma (D_0 - B_0)}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2} \\ &= D_0 \left[1 + \frac{\gamma}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\beta_2}\right] - D_1 f \left[1 - \frac{1-\gamma}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\beta_2 - 1}\right] - \left[\frac{(1-\gamma) B_1 f_\dagger + \gamma B_0}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2} \end{aligned}$$

and

$$\begin{aligned} j_L(f) &= C_0 - C_1 f + C_3 f^{\beta_2} + C_4 f^{\beta_3} \\ &= C_0 - C_1 f \left[1 - \frac{1}{\beta_3} \left(\frac{f}{f_L^b}\right)^{\beta_3 - 1}\right] + C_3 f^{\beta_2} \left[1 - \frac{\beta_2}{\beta_3} \left(\frac{f}{f_L^b}\right)^{\beta_3 - \beta_2}\right]. \end{aligned}$$

where we recollect that the constant $A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1$ are

$$\begin{aligned} A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + \lambda_{LH}(r - \mu_H)} \\ A_1 &= \frac{(\rho + \xi + \lambda_{HL} + \lambda_{LH})(r + \xi)}{(\rho + \xi + \lambda_{HL})(r + \xi + \eta) + \lambda_{LH}(r + \xi)} \\ B_0 &= \frac{1 + (\rho + \lambda_{HL} - r) A_0}{\rho + \lambda_{HL} - \mu_H} \\ B_1 &= \frac{(r + \xi) + (\rho + \lambda_{HL} - r) A_1}{\rho + \xi + \lambda_{HL}} \\ D_0 &= \frac{1}{r + \lambda_{HL} - \mu_H} \\ D_1 &= \frac{r + \xi}{r + \xi + \lambda_{HL}} \\ C_0 &= \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L} \\ C_1 &= \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta}. \end{aligned}$$

Finally, we get the equations determining the thresholds $f_\dagger, f_H^b, f_L^b$ which are given now by

$$\begin{aligned} j_L(f_\dagger -) &= j_L(f_\dagger +) \\ j_L(f_L^b) &= 0 \\ j_H(f_H^b) &= 0 \end{aligned}$$

which can be written as

$$\begin{aligned} A_0 - A_1 f_\dagger + A_2 f_\dagger^\gamma &= C_0 - C_1 f_\dagger + C_3 {f_\dagger}^{\beta_2} + C_4 {f_\dagger}^{\beta_3} \\ C_0 - C_1 f_L^b + C_3 {f_L^b}^{\beta_2} + C_4 {f_L^b}^{\beta_3} &= 0 \\ D_0 - D_1 f_H^b + D_3 {f_H^b}^{\beta_2} &= 0. \end{aligned}$$

Issuance Function: Before deriving the issuance function we need to derive the debt price, which is given by $p_\theta(f) = -j_\theta'(f)$. Taking derivatives for $f \in (0, f_\dagger)$ we get

$$\begin{aligned} p_L(f) &= A_1 + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\gamma} \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1} \\ p_H(f) &= B_1 + \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}. \end{aligned}$$

Finally, we compute the issuance function. We need to find $p_H'(f)$ for $f \in (0, f_\dagger)$. This expression is given by

$$-f p_H'(f) = -\frac{\gamma(\gamma - 1)}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}.$$

Next, we compote $p_H(f) - p_L(f)$ which is

$$p_H(f) - p_L(f) = B_1 - A_1 + \frac{r + \eta - \mu_L + (\mu_L + \xi)\gamma}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\gamma} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \frac{\gamma}{\gamma - \beta_2} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}$$

From here we get that

$$\begin{aligned} g_H(f) &= \frac{(\rho - r)(p_H(f) - p_L(f))}{-f p_H'(f)} \\ &= \frac{\rho - r}{\gamma - 1} \left[g_0 + g_1 \left(\frac{f}{f_\dagger}\right)^{-(\gamma - 1)} \right] \end{aligned}$$

where

$$g_0 = -\frac{r+\eta-\mu_L+(\mu_L+\xi)\gamma}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\gamma}$$
$$g_1 = -\left(1-\frac{\beta_2}{\gamma}\right)\frac{(B_1-A_1)f_\dagger}{\beta_2(D_0-B_0)+(1-\beta_2)(D_1-B_1)f_\dagger}$$

Finally, noting that $p_L(0)=A_1$, $p_H(0)=B_1$, and $B_0=j_H(0)$, an substituting the relations

$$(\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0)=1-(r+\lambda_{HL}-\mu_H)\,j_H(0)$$
$$(\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0)=(r+\xi)-(r+\xi+\lambda_{HL})\,p_H(0),$$

we can write

$$g_1=\left(\frac{1}{\gamma}+\frac{\mu_H+\xi}{r+\lambda_{HL}-\mu_H}\right)\frac{(r+\lambda_{HL}-\mu_H)(p_H(0)-p_L(0))f_\dagger}{(\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0)-\Big[(\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0)\Big]f_\dagger}$$

Letting

$$\varphi(f)\equiv(\rho-r)j_H(f)-(\rho+\lambda_{HL}-r)\,j_L(f)$$

we can write

$$(\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0)-\Big[(\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0)\Big]f_\dagger=\varphi(0)+\varphi'(0)f_\dagger.$$

For $f\in(0,f_\dagger)$,

$$\varphi''(f)=(\rho+\lambda_{HL}-r)\left[\frac{\rho-r}{\rho+\lambda_{HL}-r}-\frac{\lambda_{LH}}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\gamma}\right]\gamma(\gamma-1)\mathcal{Q}f^{\gamma-2},$$

where

$$\mathcal{Q}\equiv\frac{\gamma}{\gamma-\beta_2}\frac{\beta_2(D_0-B_0)+(1-\beta_2)(D_1-B_1)f_\dagger}{f_\dagger}\left(\frac{1}{f_\dagger}\right)^{\gamma-1}.$$

$j_\theta(f)$ is strictly convex only if $\mathcal{Q}>0$. Hence, the sign of φ'' is determined by the sign of the term within the parenthesis. The coefficient γ is the positive root of the quadratic equation

$$\gamma^2+\left(\frac{\rho+\lambda_{HL}-\mu_H}{\mu_H+\xi}-\frac{r+\lambda_{LH}+\eta-\mu_L}{-(\mu_L+\xi)}\right)\gamma-\frac{(\rho+\lambda_{HL}-\mu_H)(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH}}{-(\mu_L+\xi)(\mu_H+\xi)}=0.$$

This equation can be rewritten more conveniently as

$$\frac{\lambda_{LH}}{r+\lambda_{LH}+\eta+(\gamma-1)\mu_L+\gamma\xi}=\frac{\rho+\lambda_{HL}+(\gamma-1)\mu_H+\xi\gamma}{\rho+\lambda_{HL}-r}>\frac{\rho-r}{\rho+\lambda_{HL}-r},$$

which implies that the term within the parenthesis in $\varphi''(f)$ is negative. Thus, we conclude that $\varphi(f)$ is concave on $[0, f_\dagger]$, so $\varphi(f_\dagger) \leq \varphi(0)+\varphi'(0)f_\dagger$. By construction, $\varphi(f_\dagger)=0$, so it follows that $\varphi(0)+\varphi'(0)f_\dagger \geq 0$, which means that $g_1>0$. Moreover, from the previous equation for γ we also get that $r+\lambda_{LH}+\eta+(\gamma-1)\mu_L+\gamma\xi>0$ and

$$r+\eta+(\gamma-1)\mu_L+\gamma\xi=-\lambda_{LH}\frac{r+(\gamma-1)\mu_H+\xi\gamma}{\rho+\lambda_{HL}+(\gamma-1)\mu_H+\xi\gamma}<0,$$

so it follows that $g_0>0$.

B.4 Section 5.3 with Cash Flow Jumps

Suppose that

$$dX_t=\mu X_t dt+\sigma X_t dB_t-(1-\omega^{-1})X_{t-}dN_t,$$

where N_t is a Poisson process with intensity λ and $\omega>1$. Using Ito's Lemma, f_t solves

$$df_t=(g_t-\mu-\xi+\sigma^2)f_t dt-\sigma f_t dB_t+(\omega-1)f_{t-}dN_t$$

Thus, the scaled value function satisfies the delay differential equation

$$\begin{aligned}(\rho+\lambda-\mu)\,j(f)&=1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f)\\&\quad+\max\left\{(\rho-r)\frac{j(\omega f)}{\omega}+\lambda\frac{j(\omega f)}{\omega},(\rho-r)j(f)\right\}\end{aligned}$$

We guess and verify that the optimal short-term debt policy is given by

$$d(f)=\begin{cases}\frac{j(\omega f)}{\omega} & \text{if } f\in[0,f_\dagger]\\ j(f) & \text{if } f\in(f_\dagger,f^b].\end{cases}$$

The HJB equation can be written as

$$(\rho+\lambda-\mu)\,j(f)=1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2f^2j''(f)+(\rho+\lambda-r)\frac{j(\omega f)}{\omega},\ \ f\in(0,f_\dagger)$$
$$(r+\lambda-\mu)\,j(f)=1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2f^2j''(f),\ \ f\in(f_\dagger,f^b).$$

The default boundary solves the value matching and smooth pasting conditions $j(f^b)=j'(f_b)=0$.

Long-term bonds satisfy the asset pricing equation

$$(r+\xi+\lambda)\,p(f)=1-(r+\xi)\,f+\left(g(f)-\mu-\xi+\sigma^2\right)fj'(f)+\frac{1}{2}\sigma^2f^2j''(f)+\lambda p(\omega f),\ \ f\in(0,f_\dagger)$$
$$(r+\xi+\lambda)\,p(f)=1-(r+\xi)\,f+\left(g(f)-\mu-\xi+\sigma^2\right)fj'(f)+\frac{1}{2}\sigma^2f^2j''(f),\ \ f\in(f_\dagger,f^b).$$

Finally, we derive the issuance policy $g(f)$ combining the asset pricing equation with the indifference condition $p(f)=-j'(f)$. This yields

$$g\left(f\right)=\frac{\left(\rho-r\right)\left(p\left(f\right)-p\left(\omega f\right)\right)}{fj''\left(f\right)}.$$

Numerical Computation: For computational purposes, it is easier to work with the the state variable $x=\log(1/f)=-\log f$. Let $\tilde{j}(x)\equiv j(e^{-x})$ and $\delta=\log\omega$. Then, we get

$$\tilde{j}'(x)=-j'(e^{-x})e^{-x}$$
$$\tilde{j}''(x)=j''(e^{-x})e^{-2x}+j'(e^{-x})e^{-x}=j''(e^{-x})e^{-2x}-\tilde{j}'(x)$$

Substituting in the HJB equation we get

$$(r+\lambda-\mu)\,\tilde{j}(x)=1-(r+\xi)\,e^{-x}+\left(\mu+\xi+\frac{1}{2}\sigma^2\right)\tilde{j}'(x)+\frac{1}{2}\sigma^2\tilde{j}''(x)$$
$$-(\rho-r)\min\left\{\tilde{j}(x)-a\frac{\tilde{j}(x-\delta)}{e^{\delta}},0\right\},$$

where

$$a\equiv\frac{\rho+\lambda-r}{\rho-r}.$$

We write this as a system of two first order equations. Letting $y_0(x)=\tilde{j}(x)$ and $y_1(x)=\tilde{j}'(x)$, we can reduce the second order equation to the following system of first order equations

$$\begin{aligned} y_0'(x) &= y_1(x) \\ y_1'(x) &= \frac{2}{\sigma^2}\left[(r+\xi)\, e^{-x} - 1 + (r+\lambda-\mu)\, y_0(x) - \left(\mu+\xi+\frac{1}{2}\sigma^2\right) y_1(x)\right. \\ &\quad \left. +(\rho - r)\min\left\{y_0(x) - a\frac{y_0(x-\delta)}{e^{\delta}}, 0\right\}\right]. \end{aligned}$$

The previous equation is a system of two first order delay differential equations with constant coefficient that can be solved using standard numerical routines. The value matching and smooth pasting conditions at the default boundary $x^b = -\log f^b$ are $y_0(x^b) = y_1(x^b) = 0$. The only remaining step is to specify the transversality condition. From the HJB equation we get $j(0) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}$, so we have the transversality condition

$$\lim_{x\to\infty} y_0(x) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}.$$

To incorporate this transversality condition, we approximate the value function for $f = \epsilon$. This corresponds to a value of x given by $x_\epsilon = -\log \epsilon$. Differentiating the HJB we get

$$(\rho+\lambda+\xi)\, j'(f) = -(r+\xi) - \left(\mu+\xi-\sigma^2\right) f j''(f) + (\rho+\lambda-r) j'(\omega f),$$

and evaluating at $f = 0$, we get

$$j'(0) = -1.$$

Hence, for ϵ close to zero

$$j(\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - \epsilon,$$

which means that

$$y_0(x_\epsilon) = \tilde{j}(x_\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - e^{-x_\epsilon}.$$

Finally, we can write the price and issuance function in terms of the functions $y_0(x), y_1(x)$. The price of long-term bonds is given by

$$p(f) = -j'(f) = \tilde{j}'(x)e^{x} = y_1(x)e^{x}.$$

Letting $x_\dagger = -\log f_\dagger$, we get that on the $[x_\dagger, \infty)$, the issuance policy is

$$g(x) = \frac{(\rho - r)\big(p\left(e^{-x}\right) - p\left(e^{-(x-\delta)}\right)\big)}{-e^{-x} p'\left(e^{-x}\right)} = \frac{(\rho - r)\big(y_1(x) - y_1(x-\delta)e^{-\delta}\big)}{-e^{-x}(y_1'(x) + y_1(x))}$$

B.5 Disaster Shock

Now we show that the disaster shock can be microfounded by a model with three states, high (H), low (L), and disaster (ℓ), where $\mu_H > \mu_L > \mu_\ell$. In other words, the low state can still get worse. As before, let λ be the transition intensity from H to L and η be the one from L to ℓ. We are interested in the condition such that in the low state L, the borrower optimally choose to issue risky short-term debt. In other words, the corresponding $f_\dagger$ is zero in the low state L. To do so, we only need to study the value functions in $\theta = L$ and $\theta = \ell$.

When $\theta = L$, the HJB is

$$\begin{aligned}(\rho+\eta)\, j_L(f) = &\max_{\{0\le d_L\le j_L\}} (1-\pi) - (r(1-\pi)+\xi) f + (\rho+\eta-(1-\pi)y)\, d_\theta(f) + \eta\,(j_\ell(f) - d_L(f))^+ \\ &+\mu_L\left(j_L(f) - j_L'(f) f\right) + \frac{1}{2}\sigma^2 f^2 j_L''(f) - \xi f j_L'(f) \\ \Rightarrow (\rho+\eta-\mu_L)\, j_L(f) = &\max_{\{0\le d_L\le j_L\}} (1-\pi) - (r(1-\pi)+\xi) f + (\rho+\eta-(1-\pi)y_L)\, d_L(f) \\ &+\eta\,(j_\ell(f) - d_L(f))^+ - (\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)\end{aligned}$$

The HJB when $\theta = \ell$ is

$$\begin{aligned}(\rho-\mu_\ell)\, j_\ell(f) &= \max_{\{0\le d_\ell\le j_\ell\}} (1-\pi) - (r(1-\pi)+\xi) f + (\rho-(1-\pi)y_\ell)\, d_\ell(f) - (\mu_\ell+\xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f). \\ &= (1-\pi) - (r(1-\pi)+\xi) f + (\rho-(1-\pi)r)\, j_\ell(f) - (\mu_\ell+\xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f),\end{aligned}$$

which implies that

$$((1-\pi)r - \mu_\ell)\, j_\ell(f) = (1-\pi) - (r(1-\pi)+\xi) f - (\mu_\ell+\xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f)$$

The short rate is

$$y_L(d,f) = r + \eta \mathbb{1}_{d > j_\ell(f)}$$

and

$$y_\ell(d,f) = r.$$

In state ℓ, we have $d = j_\ell(f)$. In state L we have

1. If $d_L = j_\ell(f)$, the flow benefit of issuing short-term debt is

$$(\rho+\eta-(1-\pi)r)\, j_\ell(f)$$

2. If $d_L = j_L(f)$, the flow benefit of issuing short-term debt is

$$(\rho + \eta - (1-\pi)(r+\eta))\, j_L(f)$$

3. Hence, $d = j_\ell$ is optimal if

$$(\rho + \eta - (1-\pi) r)\, j_\ell(f) \geq [\rho + \eta - (1-\pi)(r+\eta)]\, j_L(f)$$

We can conclude that

- If $(\rho + \eta - (1-\pi) r)\, j_\ell(f) \geq [\rho + \eta - (1-\pi)(r+\eta)]\, j_L(f)$ the HJB equation is

$$\begin{aligned}
((1-\pi) r - \mu_\ell)\, j_\ell(f) &= (1-\pi) - (r(1-\pi)+\xi) f - (\mu_\ell + \xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \\
(\rho + \eta - \mu_L)\, j_L(f) &= (1-\pi) - (r(1-\pi)+\xi) f + (\rho + \eta - (1-\pi) r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)
\end{aligned}$$

  which can be reduced to

$$\begin{aligned}
((1-\pi) r - \mu_\ell)\, j_\ell(f) &= (1-\pi) - (r(1-\pi)+\xi) f - (\mu_\ell + \xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \\
(\rho + \eta - \mu_L)\, j_L(f) &= (1-\pi) - (r(1-\pi)+\xi) f + (\rho + \eta - (1-\pi) r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)
\end{aligned}$$

- If $(\rho + \eta - (1-\pi) r)\, j_\ell(f) < [\rho + \eta - (1-\pi)(r+\eta)]\, j_H(f)$ the HJB equation is

$$\begin{aligned}
((1-\pi) r - \mu_\ell)\, j_\ell(f) &= (1-\pi) - (r(1-\pi)+\xi) f - (\mu_\ell + \xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \\
(\rho + \eta - \mu_L)\, j_L(f) &= (1-\pi) - (r(1-\pi)+\xi) f + [\rho + \eta - (1-\pi)(r+\eta)]\, j_L(f) \\
&\quad - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)
\end{aligned}$$

  which can be reduced to

$$\begin{aligned}
((1-\pi)(r+\eta) - \mu_\ell)\, j_\ell(f) &= (1-\pi) - (r(1-\pi)+\xi) f - (\mu_\ell + \xi) f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \\
((1-\pi)(r+\eta) - \mu_L)\, j_L(f) &= (1-\pi) - (r(1-\pi)+\xi) f - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)
\end{aligned}$$

- In state ℓ, the default boundary is f_ℓ^b, satisfying

$$j_\ell(f_\ell^b) = 0$$
$$j_\ell'(f_\ell^b) = 0.$$

Since $j_L(0) = \frac{1-\pi}{\rho+\eta-\mu_L}\frac{\rho+\eta-\mu_\ell}{(1-\pi)r-\mu_\ell}$ and $j_\ell(0) = \frac{1-\pi}{(1-\pi)r-\mu_\ell}$, to let $f_\dagger = 0$ in the low state, it is necessary and sufficient to have

$$(\rho+\eta-(1-\pi)\,r)\,j_\ell(0) \le [\rho+\eta-(1-\pi)\,(r+\eta)]\,j_L(0).$$

That is

$$(\rho+\eta-(1-\pi)\,r)\,\frac{1-\pi}{(1-\pi)\,(r+\eta)-\mu_\ell} \le (\rho+\eta-(1-\pi)\,(r+\eta))\,\frac{1-\pi}{\rho+\eta-\mu_L}\,\frac{\rho+\eta-\mu_\ell}{(1-\pi)\,r-\mu_\ell}.$$

From here, we get

$$(1-\pi)\,\eta^2 + [(1-\pi)\,\rho+\pi\mu_\ell-\mu_L]\,\eta - (\rho-(1-\pi)\,r)\,(\mu_L-\mu_\ell) \le 0.$$

This implies that

$$0 < \eta \le \bar{\eta} = \frac{-[(1-\pi)\,\rho+\pi\mu_\ell-\mu_L] + \sqrt{[(1-\pi)\,\rho+\pi\mu_\ell-\mu_L]^2 + 4\,(1-\pi)\,(\rho-(1-\pi)\,r)\,(\mu_L-\mu_\ell)}}{2\,(1-\pi)}. \tag{A.32}$$