A Theory of Payments-Chain Crises

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Abstract

This paper introduces an endogenous network of payment chains into a business cycle model. Agents order production in bilateral relations. Some payments are executed immediately. Other payments, chained payments, are delayed until other payments are executed. Because production starts only after orders are paid, chained payments induce production delays. In equilibrium, agents choose the amount of chained payments given interest rates and access to internal funds or credit lines. This choice determines the payment-chain network and aggregate total-factor productivity (TFP). The paper characterizes equilibrium dynamics and their innate inefficiencies. Agents internalize the direct costs of their payment delays, but do not internalize the costs provoked on others. This externality produces novel policy insights and rationalizes permanent reductions in TFP under excessive debt.

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Says A: *I could use some of B's goods; but I have no cash to pay for them until someone with cash walks in here!*
Says B: *I could buy some of C's goods, but I've no cash to do it 'till someone with cash walks in here.*

From the book Stamp Scrip, Irving Fisher, 1933

1. Introduction

During financial crises, there are visible statistical declines in credit variables. Though harder to measure, there is also a general perception that the seamless flow of transactions of normal times, slows down during crises. In ways that are yet to be better understood, there is also a concern that productive resources remain idle when agents have to wait longer to be paid and take longer to pay.

This paper formalizes the idea of payments chains and studies their implications in the context of financial crises. Providing a theory of payments-chain crises is important. Since the onset of modern business cycle analysis, economists have argued that TFP should be interpreted as the outcome of credit-market conditions (e.g., Summers, 1986). This view is even more salient in the context of international financial crises. Economic contractions during these crises are predominantly driven by large declines in total factor productivity (TFP). This feature remains puzzling to many models given the muted responses of labor observed during these episodes.[1] The payments-chain crises in this paper offers an alternative theory to explain this co-movement.

To model payments-chain disruptions and their effects on production, I introduce a payments-chain production network, in Section 2. A payments-chain network provides an explicit connection between the timing of economic transactions and the timing of production. In this network, production is organized through random bilateral relations where customers place production orders. Some orders, spot orders, are paid upfront and their production begins immediately. Other orders, chained orders, are paid after other orders are paid. Since funds are transferred with delay, chained orders induce production delays.

[1] See for example Meza and Quintin (2007), Mendoza (2010), Oberfield (2013), or Karabarbounis et al. (2021), which find large TFP declines during the Chilean banking crisis of the early eighties, the Mexican, the East Asian sudden-stop crises of the mid nineties, in the recent Greek crisis. These studies, in turn, cite many other examples. The puzzling co-movement between TFP, capital utilization, and labor flows was first noted by Meza and Quintin (2005).

A payment-chain network is a collection of payment chains that encompass the universe of economic transactions. Each payment chain is a sequence of linked transactions. The first payment of each chain is spot and, thus, executed with funds external to the network. All subsequent payments are chained and, thus, executed with the funds from the previous transaction in its payment chain. An interpretation is that spot orders are paid with internal savings or working-capital lines whereas chained orders represent a form of trade credit.

An important feature is that the later the position in a payment chain, the longer the production delay of a given order. The length-distribution of payment chains in the network is characterized by the ratio of chained orders to total orders. Aggregating across all chains observable average TFP is,

$$\mathcal{A}\left(\mu;\delta\right)=\frac{\delta-\delta\mu}{\mu-\delta\mu}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)<1,$$

where μ is the ratio of chained to total orders and δ is a parameter that captures payment delays.

The payment-chain network of Section 2 is a stand-alone production block portable to other applications. In the second part of the paper, I pursue an application: I study the implications for financial crises. To that end, Section 3 embeds the payment-chain network into a deterministic business-cycle model. In this model, a payment-chain network is formed every period. There is a natural borrower and a natural saver which, for simplicity, I model as households. The saver always has savings to place spot orders. By contrast, the borrower carries outstanding debt. The borrower can obtain funds to place spot order through credit lines. However, her credit lines fall with her level of outstanding debt. When debt limits her access to funds, she can still place chained orders. Placing chained orders is privately costly, as less goods are produced for a chained order given an amount of expenditures. Nevertheless, borrowers may place chained orders to achieve a desired level of consumption. Critically, borrowers internalize the private cost of chained orders, but do not their effects on TFP.

The formula for TFP $\mathcal{A}$ clearly showcases how credit conditions translate into declines in TFP. When credit lines are limited, the fraction of chained orders μ increases. Observable TFP falls because labor inputs remain idle even though the labor supply is inelastic. Hence, observable TFP is entirely explained by non-observable utilization.

In the environment, the evolution of debt impacts TFP through its effects on credit lines. In turn, TFP influences the desire to accumulate debt. For low debt levels, the economy is in a steady state without delays. For moderate debt levels, the economy experiences a *temporary* payment-chain crisis with some production delays. For such moderate debt levels, crises are temporary because borrowers have incentives to repay debts to increase their access to their spot payment facilities in the future and, thereby, order production at lower costs. When indebtedness is excessive, the economy features hysteresis. Hysteresis occurs when the benefits of deleveraging, lower future production costs, require an excessive sacrifice of current consumption. Under this debt overhang, the economy remains permanently disrupted by delays.

I derive policy implications in Section 4. In particular, I study a Ramsey planner that can tax labor income, financial income, and expenditures but cannot distinguish expenditure types. This Ramsey planner implements the solution to a planner problem that directly chooses private debt respecting financial constraints and internalizes the effects of chained orders on TFP.[2] The exercise is meant to show that transitions toward steady states without disruptions are inefficient. The sources of inefficiency are two sided: savers spend too little via spot orders and borrowers spend excessively via chained orders. Because the inefficiency is two-sided, in a transition, debt may be excessively high or low, relative to the social optimum.

In that section, I also revisit fiscal multipliers. When all orders are spot, all forms of government spending are a waste. However, government spending can produce positive multipliers during a payments-chain crisis. A novel insight is that multipliers are positive only if the government purchases goods via spot orders. If government expenditures are chained, they are also a waste. The reason for positive multipliers is not aggregate demand stimulus under nominal rigidities, the conventional motivation behind fiscal policy. Rather, government expenditures stimulate output by speeding up payments, a monetary reinterpretation of fiscal policy.

[2]The planner can implement an efficient transition using a mix of capital-income taxes and labor taxes. Similarly, the planner also designs taxes to reduce debt and abandon the payments-chain hysteresis region.

Literature Review. The literature on financial crises is vast. This paper connects with theories that underscore the sharp declines in aggregate TFP. The link between financial crisis and TFP is not at all obvious because financial crises can manifest through increased marginal costs (e.g., labor wedges), not productivity. One branch of the literature explain declines in aggregate TFP through increased misallocation—see Pratap and Urrutia (2012) or Oberfield (2013). However, a common finding is that heterogeneity can only partially explain TFP declines once models are disciplined with data on input use and heterogeneity. Here, the channel is utilization. Other studies also explain declines in TFP as worse financial conditions increase the cost of utilizing capital—see Meza and Quintin (2007). The mechanism here is different: financial conditions impact TFP through the slow down of payments.

Beyond the focus on financial crises, the paper falls in the cross-roads of several areas. Namely, the monetary-payments literature, the economic-networks literature, and the literature on aggregate-demand externalities. The issue of how payments instruments affect production is a classic theme in monetary economics: Lucas and Stokey (1987) analyzes a stochastic cash-in-advance economy; Kiyotaki and Wright (1989) studies trade with indivisible tokens; Lagos and Wright (2005) a model with divisible money and explicit trading arrangements.[3] Recent work focuses on how the distribution, and not the instruments per se, affect production—see Lippi et al. (2015), Rocheteau et al. (2016), and Brunnermeier and Sannikov (2017). In common with this literature, the distribution of funding affects allocations. The main distinction is that I focus on delays in sequential payments.[4]

Sequential payments appear in many other studies. The payment-chain network is inspired by the credit chain model in Kiyotaki and Moore (1997). The contribution relative to Kiyotaki and Moore (1997) is to present a network of transactions that I then embed into a standard business-cycle model with endogenous payment decisions. Other models of sequential payments include Townsend (1980) which studies sequence of payments with spatial separation, Freeman (1996) and Green (1999) which

[3]See Shi (1997a); Lagos et al. (2011); Lagos and Rocheteau (2009); Li et al. (2012); Nosal and Rocheteau (2011); Rocheteau (2011) for many other directions in that area.

[4]The paper tangentially relates to Bianchi and Bigio (2022) and Piazzesi and Schneider (2018) where implicit payment-flows induce liquidity premia. The payment-chain network here induces explicit payment flows.

study sequential transactions in an overlapping generation environments, La'O (2015) which studies a circular flow of transactions, or Guerrieri and Lorenzoni (2009) which studies sequential transactions in a Lagos and Wright-type environment. Recent work by, Hardy et al. (2022) and Bocola (2022) contrast payments funded externally against trade credit.[5] Relative to these papers, there are two distinctions: here, transactions are formed in a network and the network is endogenous to expenditure-savings decisions. The importance of this body of theoretical work is substantiated with a body of empirical evidence found in a number of recent papers: Boissay and Gropp (2007), Jacobsen (2015), Barrot (2016), and Costello (2020) among many others.

With respect to the economic networks literature, the paper connects with models with economic networks. The contribution with that literature is modest, as network formation is not strategic. By contrast, in Oberfield (2018), a network formed through strategic partnerships. In Kopytov et al. (2022) and Elliott et al. (2022) firms form strategic links, being aware of possible supply-chain break downs. Here, the network is randomly formed, but the distribution of chains is endogenous to financial decisions. Like in Elliott et al. (2014), Alvarez and Barlevy (2021) and Taschereau-Dumouchel (2022), there are also network externalities. In those models, externalities occur when individual defaults provoke subsequent defaults. Here, externalities occurs through payment delays.[6]

Finally, the paper connects with models of aggregate demand externalities. An early model of these externalities is Diamond (1982) where, via search externalities, consumption decisions affect output. An extension, Diamond (1990), deals with credit. In most of the literature, demand externalities result from nominal rigidities. There's been a recent interest in coupling nominal rigidities with financial constraints—for example, Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017). Recent papers, have further introduced sequential transactions into environments with nominal rigidities—for example, Woodford (2022) and Guerrieri et al. (2022). In those models, demand externalities occur when agents cut back any form of expenditures. The nature of demand externalities here is different. In particular, the type of expenditures

[5] See also Biais and Gollier (1997).

[6] Bigio and La'O (2013) considers the propagation of financial shocks through a production network. In that paper, propagation occurs through the misallocation of inputs. Here, there is no such misallocation but production delays.

by the private or public sector matters: spot orders may stimulate output but chained orders depress it. The demand externalities provoked by the slowdown of payments is part of a classic narrative. Almost a century apart, the opening quotation taken from Fisher (1933) and the renowned "baby sitting co-op" analogy of Krugman (1998) belong to that same narrative. The rest of the paper is an attempt to provide an analysis.

2. Payment-Chains and Productivity

This section presents the payment-chain network. I then embed it into a dynamic business-cycle model.

Bilateral Relations. Production is organized through bilateral agreements in which a customer orders a product from an agent that owns a production unit. The agreement is exclusive in that only the agent placing the order can derive utility from its production and production units are exclusively dedicated to producing for a specific client.[7] As examples of such relations, we can think of a home renovation project, a medical service, the manufacturing of a engineered product with a specific blueprint, or the commission of a piece of art.

There are two types of orders, *spot* and *chained* orders. Spot orders are paid immediately. Chained orders are paid after the client receives a payment from another transaction. There are N production units. In turn, there are N^s spot orders and N^x chained orders. In this section, I assume that there are equal amounts of production units and orders, $N = N^s + N^x$. In the section that follows, I work with a limit for $N \to \infty$ and obtain prices that guarantee that this relation holds as a market-clearing condition. Each production unit is assigned an identifier, $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. Likewise, each order is assigned a unique identifier, $i \in \mathcal{N}$. I use $\mathcal{N}^s$ and $\mathcal{N}^x$ to denote the set of identifiers of spot and chained orders respectively.[8] Also, I work with the assumption that each order involves the same paid amount, a condition that is explained in the

[7] The identity of the agents in these agreements is not important at this stage, but it explicit in the following section. Furthermore, although the product is exclusive to the bilateral relation, products ordered by the same agent in different relations may be perfect substitutes.

[8] Naturally, $\mathcal{N}^s$ and $\mathcal{N}^x$ form a partition of $\mathcal{N}$, $\emptyset = \mathcal{N}^s \cap \mathcal{N}^x$ and $\mathcal{N} = \mathcal{N}^s \cup \mathcal{N}^x$, and the number of elements is $N^s = \#\mathcal{N}^s$, $N^x = \#\mathcal{N}^x$.

following section. I discuss the interpretation spot and chained orders in the following section where I introduce financial decisions.

I define two relations that together define the payment-chain network. First, $\mathcal{P} : \mathcal{N} \to \mathcal{N}$, is a one-to-one assignment from an order to a production unit. By assumption $i \neq \mathcal{P}(i)$. The interpretation is that $\mathcal{P}(i)$ is the unit that produces order i. The assignment $\mathcal{P}$ is entirely random: any unit can be assigned to any order with equal probability. Second, a chained income-expenditure relation associates a chained order to a production unit. This relation is the identity function defined on $\mathcal{N}^x$. The interpretation is that order i uses the revenues of production unit i if it happens that $i \in \mathcal{N}^x$. The idea is that although order i is not externally funded, the customer that places order i owns production unit i and hence, will be paid from order $j = \mathcal{P}^{-}(i)$. The assumption that the income-expenditure relation is the identity is innocuous.[9]

To anticipate how the two relations induce a payment-chain network, consider production unit j assigned to order i, $\mathcal{P}(i) = j$. The client placing order i must pay for $j's$ production. This creates a payment link from i to j. In turn, if order j is spot, $j \in \mathcal{N}^s$, the funds paid in order i are not used in further payments. However, if order j is chained, the funds are used again to pay unit $k = \mathcal{P}(j)$. In other words, when, $j \in \mathcal{N}^x$, there is a flow of payments from i to j, and from j to k. If $k \in \mathcal{N}^x$, again, the same funds are used to pay unit $\mathcal{P}(k)$, and so on. The chain of payments goes on until a final order in the chain is placed on some production unit i not associated with a chained order. Since every order is paired with a production unit, the economy features an entire network of transactions, composed of a collection of payment chains.

The payments-chain network determines production. The production of orders occurs within a unit time interval. Each order starts at some time $\tau \in [0, 1]$. Once production starts, it does not stop. Production is linear in time: if production of starts at τ, production is $1 - \tau$ goods.

If every order were to start immediately production would be maximal. This does not occur in general because of two frictions. These frictions make the payments-chain

[9]Formally, we can define the chained income-expenditure relation $\mathcal{X}$ as the identity function on $\mathcal{N}^x$, that is $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}^x$ such that $\mathcal{X}(i) = i$. The idea is that $i \in \mathcal{N}^x$ obtains funds from production unit, $\mathcal{X}(i) = i$. Indeed, the identity function $\mathcal{X}(i)$ can be replaced by any injective function $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}$ so that production units and associated chained orders do not have the same identifiers. Changing identifiers does not alter the equilibrium.

network consequential because production times depend on the location in network. The frictions follow the spirit of Kiyotaki and Moore (1997). The first friction is that production cannot start unless there is a proof of funds. If the order is spot, by definition, the client has funds and can prove that. By contrast, chained orders can prove funds after the payment of the order that is its source of funds. Proof of funds is required to deter fraudulent behavior: without proof of funds, clients could promise payments that they know will never occur. Since production is customized, this would lead to an ex-post renegotiation at disadvantageous terms to the producer.

If funds could be transferred instantaneously, the production of all orders would start immediately. All that would be needed is each spot order to make an initial payment, and funds would reach each chained order instantaneously. In that were the case, the payment-chain network would be inconsequential. The second friction, limited commitment, provokes a delay in the transfer of funds. The idea is that after the customer proofs funds to start an order, the funds are released only after the fraction $1-\delta$ of order's output is inspected. Without inspection, the producer could produce a good for somebody else (or himself) after being paid. Since there is no way to verify in a court the customization of a product, the inspection is necessary to this moral hazard.

Placing these frictions together, the greater the number of chained orders in a chain, the more payment delays, and the longer the times with production units remaining idle. This feature leads to interesting predictions between TFP and the payments chain network. To produce concrete results, I first formalize the definition of the payments-chain network. This definition is convenient to study the distribution of payment-chain lengths in the grand payments-chain network, which I then use to derive TFP.

Payments-Chain Network. A sequence of payments with funds defines a **payments chain**. Obviously, every chain must start with a spot order, followed by chained orders that use the funds that originate with the spot order. The number of subsequent chained orders is the chain length. A **payments-chain network** is the collection of all payment chains, representing the universe of transactions during the production interval. Armed with the production and expenditure-income relations, I employ a formal definition from the networks literature.

Definition 1. *A **payments-chain network** $\mathcal{K}$ is an acyclical directed network with nodes $\mathcal{N} = \{1, 2, ..., N\}$ and links $\mathcal{V} = \{(i, j) \, | \mathcal{P}(i) = j, j \in \mathcal{N}^x\}$, $\mathcal{K} = (\mathcal{N}, \mathcal{V})$. A **payments-chain** of length n is a finite sequence of nodes $\{i_k\}_{k=0}^n$ such that the sequence starts with some $i_0 \in \mathcal{N}^s$ and $\forall k \in \{1, \ldots, n\}$, $\mathcal{P}(i_{k-1}) = i_k$ with $i_k \in \mathcal{N}^x$. By convention, if $i \in \mathcal{N}^s$ and $j = \mathcal{P}(i) \notin \mathcal{N}^x$, then i and j define a chain of length zero.*

The nodes corresponding to the payments-chain network represent both orders and production units. Links represent the sources of funds. Because production is bilateral, there is either one or no links stemming out of each node. A directed link from i to j indicates that i orders from production unit j and that j is a chained order, $\mathcal{P}(i) = j \in \mathcal{N}^x$. Hence the source of funds of order j are the funds paid for in order i. In turn, if a node does not receive a link, it represents a spot order. Furthermore, are directed toward order $j = \mathcal{P}(i)$, order j is also spot in which case i and j for a zero-length chain. The assumption that the network is acyclical guarantees that each transaction has source of funds; there are no cycles composed of only chained links. In general, any (longest) path of links defines a payments chain. The collection of payment chains is encoded in the payments-chain network.

Examples. Let me present an example. Set $\mathcal{N} = \{1, 2, \ldots, 8\}$ and let the subset of spot orders be $\mathcal{N}^s = \{1, 3, 7\}$. Also, define the production relation as follows: let $\{i_n\}_{n \in \mathcal{N}} = \{1, 5, 7, 4, 6, 2, 3, 8\}$ such that $i_{n+1} = \mathcal{P}(i_n)$ and $i_1 = \mathcal{P}(i_N)$.[10]Thus, the links in this payments-chain network are $\mathcal{V} = \{(1, 5), (7, 4), (4, 6), (6, 2), (3, 8)\}$.

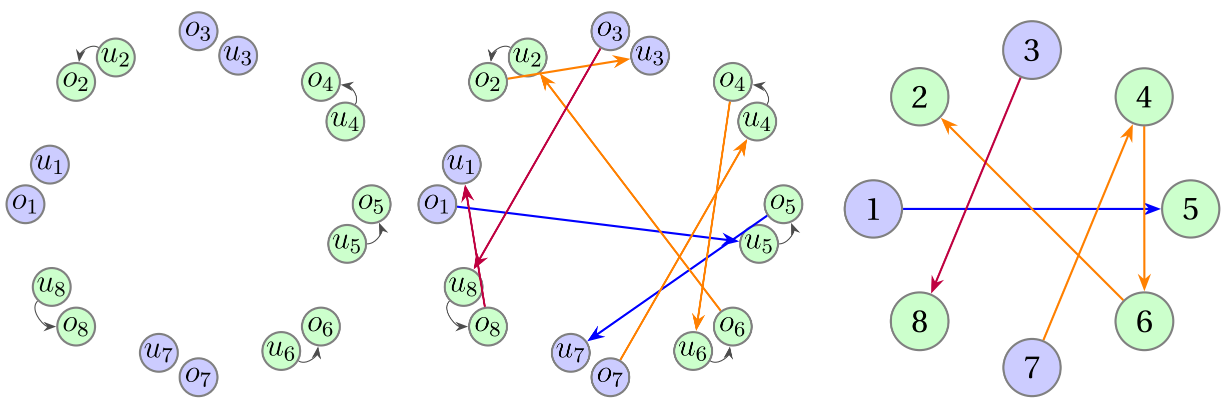
There are several graphs associated to the payment-chain network of this example. The left panel of Figure 1 depicts the chained income-expenditure relation. In that panel, I split each node into counterparts: the production units $\{u_n\}$ and production orders $\{o_n\}$ for nodes, $n \in \{1, 2, ..., 8\}$. The links represent the flow of funds from production units to their corresponding chained orders, defined by the chained income-expenditure relation.

The middle panel adds the links to the flow of payments for production, corresponding to $\mathcal{P}$. That is, the links in that panel add the payments production units. Adding the links of the chained income-expenditure and production relations allows

[10]With this information, we know that order 1 orders from production unit 5, order 5 from unit 7, and so on, ending with 8 ordering from unit 1. We also know that 1 is a spot, 5 is chained, 7 is spot, and so on.

us to trace funds. Notice that the links from orders to production units with the same color, share an original source of funds.

The right panel depicts the actual payments-chain network. The network features three chains. A first chain is from 1 to 5 and is of length of one. A new chain starts at 7, and links nodes 7, 4, 6 and 2. Since 4, 6 and 2 are chained orders, the length is 3 for that chain. The last chain links 3 with 8 and is of length 1. The example does not include chains of length zero, but these would be represented by unlinked nodes.



(a) Income relation (b) Network in detail (c) Payments-Chain Network

Figure 1: Components of the Payments-Chain Network

Average TFP. I now derive the distribution of payments chain lengths, as the number of transactions increases, $N \to \infty$. In doing so, I keep the ratio $\mu \equiv N^x/N$ constant. The fraction plays a critical role: with it, we can obtain a discrete probability distribution $G(n;\mu)$ for the length of payment chains in the network. To derive the distribution, recall that the production relation is random. Hence, as $N \to \infty$, any node has a link directed toward it if it is chained. This occurs with probability μ. A node does not receive a link if it is spot, which happens with probability $1-\mu$. Furthermore, recall that each chained order is funded and thus belongs to specific payment chain which, in turn, starts with a specific spot order. Thus, there is a one-to-one relation between each payment chain and a spot order. Hence, there N^s chains in total, which we can also index by $i \in \mathcal{N}^s$. If chain i forms a chain of length zero, it must be that $\mathcal{P}(i) \in \mathcal{N}^s$. This, happens with probability $1-\mu$ if $\mathcal{P}(i)$ is drawn entirely randomly—each production link indeed occurs with equal probability. Likewise, chain i is of length 1 if $\mathcal{P}(i) \in \mathcal{N}^x$ but $\mathcal{P}(\mathcal{P}(i)) \notin \mathcal{N}^x$. This happens with probability $\mu \times (1-\mu)$. In a chain of length two, there are two consecutive chained orders followed by a spot order. This occurs with probability $\mu^2(1-\mu)$. Proceeding by induction, we arrive at the following:

Proposition 1. Let $n \in \{0, 1, 2, \ldots\}$ be the length of a payment chain in the payment-chain network. Then, n is a random variable with probability mass function (p.m.f.) $G(n;\mu)$ where $G(n;\mu)$ is the geometric p.m.f. with parameter μ, that is, $G(n;\mu) = (1-\mu)\mu^n$.

We use this distribution to solve for production in the network. For that, we first derive the production vector of a chain of arbitrary length. In a chain of length zero, there is one order whose production begins immediately.

In a chain of length 1, the production of the first order begins immediately, but there is a delay in the second order. For the second order, the funds are received after $1-\delta$ of the production of the first order is finished. This happens at time $\tau = 1-\delta$. Hence, this leaves only δ time to produce the second order in the chain. For the second order production is δ of which the $1-\delta$ fraction must be inspected. If there is a third order in the chain, the transfer of funds occurs $(1-\delta)\delta$ time after the first transfer at $1-\delta$. Adding these consecutive delays, the production of the third order can only start by $(1-\delta)+\delta(1-\delta) = 1-\delta^2$. This leaves δ^2 time to produce that third order.

We can deduce a pattern of delays by forward induction.[11] In a chain of length n, the corresponding production in the $n+1$ consecutive orders is $\{1, \delta, \delta^2, \ldots, \delta^n\}$. Using the distribution of chain lengths, we obtain the following result.

Of course, if the chain is of length 0, production begins immediately.

Proposition 2. (Output per worker): *Given μ and δ, the average output corresponding to chained orders converges to:*

$$\mathcal{A}(\mu;\delta) = \frac{1-\mu}{\mu}\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) < 1. \tag{1}$$

[11] If the k-th node initiates production at time $1-\delta^{k-1}$, the delay from its inspection is $(1-\delta)\delta^{k-1}$, which added to previous delays leads to a transfer of funds only by time $1-\delta^{k-1}+(1-\delta)\delta^{k-1} = 1-\delta^k$. This, leaves δ^k time for production to the subsequent unit. Since we computed the delay for $k = 1, 2, \ldots$. the productions in a chain of length n are $\{1, \delta, \delta^2, \ldots, \delta^n\}$.

as $N \to \infty$. $\mathcal{A}$ is strictly concave, decreasing, and satisfies

$$\lim_{\mu \to 0} \mathcal{A}(\mu;\delta) = \delta \textit{ and } \lim_{\mu \to 1} \mathcal{A}(\mu;\delta) = 0 \textit{ and } \lim_{\delta \to 0} \mathcal{A}(\mu;\delta) = 0 \textit{ and } \lim_{\delta \to 1} \mathcal{A}(\mu;\delta) = 1.$$

Average output is $\mathcal{Y}(\mu) = (1-\mu) + \mu\mathcal{A}(\mu) \le 1$.

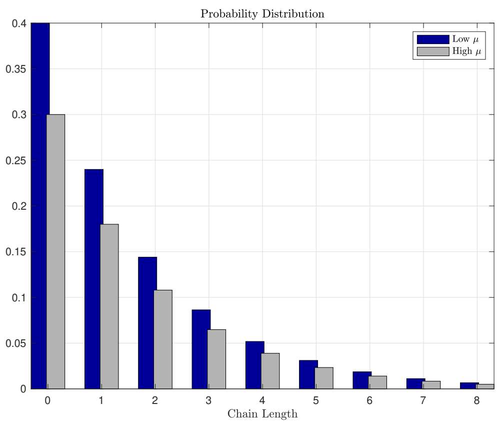
This theorem is key as it presents a formula for output. To obtain output, I first compute the average production of chained orders, $\mathcal{A}(\mu;\delta)$.[12] We can think of $\mathcal{A}$ as a measure of average of total factor productivity (TFP) of production in charge of chained orders. Hence, with abuse of terminology, I refer to $\mathcal{A}$ simply as TFP. $\mathcal{A}$ is strictly concave and decreasing in μ and has well behaved limits.[13] Total output $\mathcal{Y}(\mu;\delta)$ is constructed by noting there are $(1-\mu)$ spot orders for which production is 1 and μ for chained orders whose average production is $\mathcal{A}$. Output inherits the properties of TFP.

Clearly, payment delays enter in the formula for $\mathcal{A}$ because they make resources idle when payments are delayed. Hence, TFP here is entirely driven by utilization and is purely observable productivity. As an illustration, the left panel of Figure 2 plots the distributions of chain lengths, corresponding to two values of μ. As μ increases, the distribution shifts mass to chains of greater length. The right panel graphs $\mathcal{A}$ as a function of μ, for two values of δ. Notice how productivity falls as the fraction μ increases. Also, the lower δ, the greater the delays and, hence, the lower TFP.

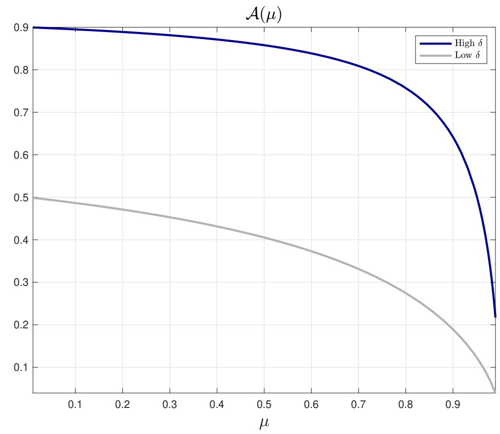
The main insight so far is that TFP losses result from the delay in payments. It is

[12]To obtain output, I first compute the average production of chained orders in a n-length chain, $\left(n^{-1}\sum_{i=1}^{n}\delta^{i}\right)$. With the average production of each chained orders, I can compute the expected value of production in chains with at least one chained order. Integrating across all possible lengths, we obtain $\mathcal{A}(\mu;\delta)$, the average production for chained orders. For that, I use the discrete probability $(1-\mu)\,\mu^{n}/\mu$. This is the distribution of chain length conditioned on $n>0$, obtained from $G(n;\mu)$. Integrating across all possible lengths, we obtain $\mathcal{A}(\mu;\delta)$, the average production for chained orders.

[13]With respect to the limits of $\mathcal{A}$, as $\mu \to 1$ the chains of chained orders become larger and larger but the per-order production decreases to zero, since the additional production of chained order decreases exponentially. On the other hand, when $\mu \to 0$ the chain length tends to 1 and thus its per-worker productivity tends to δ because production will be delayed by at most one period. In turn, when $\delta \to 0$ the required amount of production inspection to transfer funds tends to 1 and the production related to the chained order has no time left to build so the per-order production is zero. Conversely if $\delta \to 1$, one could think chained orders as spot ones since they can obtain funds immediately, naturally per-order production tends to one since time left to build does so for each worker. Interestingly, $\mathcal{A}(\mu;\delta)$ resembles an entropy function, but I am unaware of any connections between a geometric distribution, a discounted sequence and entropy measures.



(a) Chain length distribution



(b) TFP

Figure 2: Chain-Length Distributions and TFP

Note: Left panel graphs $G(n)$ for $\mu = \{0.6, 0.7\}$. Right panel $\mathcal{A}(\mu)$ for $\delta = \{0.5, 0.9\}$.

worth clarifying the sources of these losses. Unlike search models, there is no congestion externalities: orders do not crowd out each other because the assignment is one to one. Furthermore, the fact prices do not vary with the chain length also has nothing to do with the TFP losses: payments are transfers without implication for production. The source of TFP losses the delays in production caused by the deferment of payments. The economy is at full capacity when if all transactions are spot. However, TFP losses are magnified by the random assignment of orders. Holding fixed the number of spot orders, if payment chains could be reorganized to be of the same length, the economy would still feature TFP losses, but less so than with under random assignment because of Jensen's inequality.

Discussion: Reduction of Economic Complexity. The payments-chain network here is simplified to convey the main message. In practice, economies involve much more complexity than exhibited here: Transactions differ by size and are coupled in more intricate production networks. Moreover, I deliberately made assumptions so that production ends in finite time, so that the payment-chain network can be introduced into a business cycle model that I can characterize. Furthermore, in reality, agents may find ways to negotiate transactions prices, features I do not allow. Studying these dimensions makes the problem more realistic, but would complicate the analysis unnecessarily.[14] Despite its simplicity and crude assumptions, the payments-chain network illustrates the effects of delays in payment chains. A virtue is that with this, we can obtain a mapping from a financial quantity, μ, to observable TFP. Because μ depends on agent decisions in the following section, this allows us to study policy exercises immune to the Lucas critique.

A final note is that I do not allow for order withdrawals. To allow withdrawals, we could find an endogenous threshold production that would trigger the withdrawal provided that we obtain a relative price for funds and production, as I do in the following section. In that case, the production in a chain would drop to zero for orders above a given position. An analogue, but more complicated, formula to TFP (1) can be derived for the case where endogenous withdrawals are allowed.

3. Payment-Chains in a Business-Cycle Model

I now incorporate the payments-chain network into a business cycle setting. I take the previous results as inputs.

3.1 Environment

Timing. The horizon is infinite. Expenditure and savings decisions are programmed at integer dates, $t = \{0, 1, 2, ...\}$. Production happens within the time interval between integer dates, through the payments-chain network of the previous section. It is no longer necessary to refer to the time interval of production; it is understood to happen within dates. There is perfect foresight, but I study transitions that can be reinterpreted as an unanticipated shock. The numeraire is units of production, which in this application are labor units.

Demographics. The economy is populated by two representative households. One is a working-class household (workers) that inelastically supplies labor, but has neg-

[14] Statistical physics handles similar levels of complexity when studying the dynamics of interacting particles, so possibly, such methods can be used in this environment. In particular, statistical physics offers tools to calculate statistical properties of complex phenemona. I believe these tools will be eventually used in economics.

ative financial wealth. The other is a financial wealthy household (savers) that does not produce. Each household consumes goods by placing production orders in the payments-chain network. I use superscripts s and w to distinguish saver and worker decisions when necessary.

Income, Expenditures, Transactions, and Prices. Production and consumption depend on the expenditure mix between spot and chained orders by both households. The worker household is endowed with N labor units. As in the previous section, a labor unit is assigned to a single order. I normalize payment per labor unit to $1/N$ so that total labor income is normalized to one.

I work with the limit as $N \to \infty$. As I increase the number of orders, I let the production of each labor unit scale with $1/N$, keeping the maximal feasible production equal to 1. Implicitly, households place a large number of orders. Aggregating across all those orders, the risk in the quantity of goods obtained in different orders is diversified as $N \to \infty$. This assumption is akin to the classic "big-family" assumption that appears in, for example, Lucas and Stokey (1987) and Shi (1997b).

In the previous section, I implicitly assumed that all production units are paid the same, regardless of their production. A motivation for this assumption is pairwise stability—see **?** for a definition. Pairwise stability requires agents to accept the links in a network at the moment of placing orders—without knowledge of the location in the network. To accept an order, labor units must earn the same because, otherwise, they would sever links and and a new network would have be form. In tun, to place an order, households must be happy to do so, anticipating the average amount of goods obtained by placing placing spot and chained orders respectively. This is the case in this section.

At any t, both households choose amounts of spot orders. To place spot orders, households must posses funds. Savers do not have production units, so they can only make spot payments. Hence, for the rest of the paper, it is understood that the saver's period consumption C^s is purchased by making spot payments. By contrast, the worker household chooses amounts of spot and chained orders. The total of goods purchased by workers is C^w

$$C^w = S^w + X^w, \tag{2}$$

where S^w and X^w are her purchased amounts of spot and chained goods. As above, each chained order must be backed by a specific labor income unit.

By placing spot and chained orders, summing across households, there is a total of S spot goods and X chained goods produced. Total (adding the expenditures of both households) expenditures in spot and chained orders are E^s and E^x, respectively. In equilibrium, total income must equal total expenditures so $E^x + E^s = 1$. Therefore, the ratio of chained expenditures to total expenditures is $\mu = E^x$ and, thus, $E^s = 1 - \mu$.

In the previous section, I worked with integer amounts of production units and orders, and imposed the countability condition, $N = N^s + N^x$. Here, expenditures are chosen freely by households, so this condition is no necessarily met. To reconcile both approaches, we can count the number of spot and chained orders with a floor and ceiling function, $N^s = \lfloor E_t^s/N \rfloor$, , $N^x = \lceil E_t^x/N \rceil$, given expenditure choices. As long as $E^x + E^s = 1,$ the countability condition is satisfied. Moreover, any inconsistencies between the expenditure choices and the amount of goods bought froun rounding errors vanish as $N \rightarrow \infty$.

Now, recall that total output depends on μ. Since for each spot order there is one unit of output, $E^s = S = 1 - \mu$. Thus, using Theorem 2 we have that $\mu \mathcal{A}(\mu) = X$. If we substitute $\mu = E^x$ into this relation, we find that $E^x = \mathcal{A}^{-1}(\mu) X$. Because the number of orders tends to infinity, we can treat $q(\mu) \equiv \mathcal{A}^{-1}(\mu)$ as a price of chained goods per unit of chained expenditure. I use this this auxiliary price, to define a worker expenditure bundle:

$$S^w + q(\mu) X^w = E^w. \tag{3}$$

where E^w are the workers expenditures.

Savers. The saver's period utility is $\log(\cdot)$. He maximizes discounted lifetime utility over the sequence $\{C_t^s\}$. Savers begin each t with real deposits, D_t, their only source of wealth. Deposits earn an equilibrium return R_t. Given D_t, savers choose future savings, D_{t+1}, and current expenditures.

Problem 1. (Saver's Problem): *Given* D_0 *and* $\{R_{t+1}\}_{t \geq 0}$,

$$\max_{\{C_t^s\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C_t^s\right),$$

subject to the budget constraint, $R_{t+1}^{-1} D_{t+1} + C_t^s = D_t,\ \forall t \geq$*0.*

Workers. Worker and saver preferences are identical. Different from savers, workers begin each t with debt, B_t, and choose between current expenditures and future debt B_{t+1}. The choice B_{t+1} is limited by a natural debt limit, $\bar{B} = 1/(1-\beta)$. Because they are in debt, , to make spot expenditures, the worker must borrow intra-period. Namely, she comes with B_t, but at the start of the period her debt increases to $B_t + S_t^w$. Intra-period debt carries no interest and its repayment is always feasible—labor income always exceeds worker expenditures.[15] By the end of the period, intra-period is either paid or add to the balance B_{t+1}, depending on her expenditures. Critically, intra-period debt is limited by a time-varying spot-borrowing line (SBL), $\tilde{B}_t$ which caps spot transactions:

$$S_t^w \leq \bar{S}_t \equiv \max\left\{\tilde{B}_t - B_t, 0\right\}. \tag{4}$$

An interpretation of $\tilde{B}_t$ is that it is a credit line that caps the amount of intra-period borrowing by $\bar{S}_t$, as further discussed below.

To circumvent the SBL, the worker can place chained orders. However, chained orders are costlier because $q \geq 1$. Obviously, if the worker has little intra-period borrowing capacity, she has to make these costly expenditures.

Problem 2. (Workers's Problem): *Given* B_0 *and* $\left\{R_{t+1}, \tilde{B}_t\right\}_{t\geq 0}$*,*

$$\max_{\{S_t^w, X_t^w\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C_t^w\right),$$

subject to the budget constraint, B_t*+*E_t^w*=*$R_{t+1}^{-1}B_{t+1}$*+1,* $\forall t \geq$*0, to the expenditure mix (3) and total consumption (2), to the intra- and inter-period constraints (4), and to the natural debt limit* $B_t \leq \bar{B}$*.*

[15] In equilibrium given the presence of savers, $S_t^w < E$.

Market Clearing. Clearing in the asset market requires:

$$D_t = B_t, \tag{5}$$

Since savers do not possess production units, they always maintain positive savings, $D_t > 0$. Thus, without loss of generality, I work under the assumption that the worker is always in debt. Hence, and from now on, I no longer make reference to D_t, and use B_t to represent both saver deposit and worker debt. Given the consumption choices of both households, $\{X_t^w, S_t^w, C_t^s\}$, the goods-market clearing condition is:

$$C_t^s + S_t^w + X_t^w = \mathcal{Y}(\mu_t). \tag{6}$$

Adding both household's budget constraints yields an income-expenditure identity:

$$C_t^s + S_t^w + q_t X_t^w = 1, \tag{7}$$

that I exploit below.

Definition 2. *Given a sequence of* $\left\{\tilde{B}_t\right\}_{t\geq 0}$, *a sequence* $\{B_t, C_t^s, S_t^w, X_t^w, R_t, q_t\}_{t\geq 0}$ *is a* ***symmetric competitive equilibrium*** *if:*

1. *Given* $\{R_t, q_t\}_{t\geq 0}$, $\{B_t, S_t^w, X_t^w\}_{t\geq 0}$ *solves the worker's problem and* $\{B_t, C_t^s\}_{t\geq 0}$ *solves the saver's problem.*

2. *Markets clear; (5) and (6) are satisfied.*

In equilibrium,the ratio of chained expenditures satisfies the following payments identity:

$$\mu_t = q(\mu_t) \cdot X_t^w. \tag{8}$$

3.2 Characterization

Solution to household problems. The solution to the saver's problem is typical of log utility and goes without proof.

Proposition 3. *Given B_0, the solution to the saver's problem is:*

$$C_t^s = (1-\beta) B_t \text{ and } B_{t+1} = R_{t+1}\beta B_t \;\; \forall t \geq 0. \tag{9}$$

The worker's problem is more complicated as she must decide between spot and chained expenditures. Given total expenditures E_t^w, since $q_t \geq 1$, cost minimization requires her to spend spot as much as the SBL allows. Thus, given E_t^w, spot and chained expenditures are respectively:

$$S_t^w = \min\left\{\bar{S}_t, E_t^w\right\} \tag{10}$$

and

$$X_t^w = \left(E_t^w - \min\left\{\bar{S}_t, E_t^w\right\}\right)/q_t. \tag{11}$$

Given this optimal split, I invoke the principal of optimality to cast the worker's problem into a Bellman equation:

Problem 3. (Workers's Problem): *Given B_0 and $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$, workers choose a sequence of debt holdings $\{B_{t+1}\}_{t\geq 0}$ which follow from the solution to:*

$$V_t^w(B) = \max_{B' \leq \tilde{B}} \log\left(S^w + X^w\right) + \beta V_{t+1}^w(B') \tag{12}$$

where S^w and X^w are given by (10) and (11), respectively, and total expenditures by $E^w = B' R_{t+1}^{-1} + 1 - B$.

The Bellman equation reformulates the worker's problem as an expenditure-savings problem. The solution yields expenditures which, together with the optimal expenditure split, determine spot and chained consumption. The time index in the the value function reflects the time dependence on R_{t+1}, q_t, and $\tilde{B}_t$. The Bellman equation representation allows to understand the implications of payment-chain networks for business-cycle dynamics.

The following Lemma identifies threshold points that are key to the characterization.

Lemma 1. ***Threshold points.*** *Define the* ***efficiency threshold****,* $B^{\star}_{t+1} \equiv R_{t+1}\left(\tilde{B}_t - 1\right)$*. Then,* $S^w_{t+1} = 0$ *if and only if* $B_{t+1} > \tilde{B}_{t+1}$*. In addition,* $X^w_t > 0$ *if and only if* $B_{t+1} > B^{\star}_{t+1}$*.*

Proof. The Lemma is immediate. If $B_{t+1} > \tilde{B}_{t+1}$, the worker cannot spend spot. In turn, $B_{t+1} > R_{t+1}\left(\tilde{B}_t - 1\right)$ happens if and only if $E^w_t = B_{t+1}/R_{t+1} - B_t + 1 > \tilde{B}_t - B_t$. In that case, the worker spends sufficiently high that his chosen expenditure E^w_t requires him to spend on some chained orders. □

It is convenient to define some relevant objects that result from these threshold points.

Definition 3. (Marginal Prices and Marginal inflation):

I. ***Average Price.*** *The average goods price per unit of worker expenditure is* $Q_t \equiv E^w_t / C^w_t$*.*

II. ***Marginal Prices and Marginal Inflation.*** *Given* B'*, the price of the good at* t *bought with a marginal increase in* B' *is:*

$$\tilde{q}^E_t\left(B'\right) \equiv 1 + \left(q_t - 1\right) \cdot \mathbb{I}_{\left[B' \geq B^{\star}_{t+1}\right]},$$

the price of the good purchased at $t+1$ *after a marginal decrease in* B' *is*

$$\tilde{q}^B_{t+1}\left(B'\right) \equiv 1 + \left(q_{t+1} - 1\right) \cdot \mathbb{I}_{\left[B' \geq \tilde{B}_{t+1}\right]}.$$

Marginal inflation is $\Pi_{t+1}\left(B'\right) \equiv \tilde{q}^B_{t+1}\left(B'\right) / \tilde{q}^E_t\left(B'\right)$*, a continuous function of* B' *except at discontinuity points* $\left\{B^{\star}_{t+1}, \tilde{B}_{t+1}\right\}$*.*

The average price of goods for the worker is the ratio of her expenditures to the quantity of goods bought with those expenditures. Marginal prices have the following interpretation: If at t the worker spends in chained goods, the reduction in her debt is financed with a reduction in chained expenditures. Otherwise, if she does not spend on chained goods, a reduction in her debt is financed with a reduction in spot goods. Since the price of chained goods is q_t and the price of spot goods is 1, the worker sacrifices $1/\tilde{q}^E_t$ units of consumption per unit of debt reduction. Likewise, if she spends spot at $t+1$, any past savings can translate into spot expenditure at $t+1$. Otherwise, if she only spends in chained goods at $t+1$, any past savings translate into chained expenditures.

Hence, the worker can buy $1/\tilde{q}^{B}_{t+1}$ additional goods by reducing her debt at t on the margin. Marginal inflation is the ratio of marginal prices, a definition that enters in the following generalized Euler equation.

Proposition 4. (Workers's First-Order Condition): *Fix a sequence* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$ *such that* $\tilde{B}_t$ *is an increasing and* $\beta R_{t+1} \leq 1$. *Then, any solution* $\{B_{t+1}\}_{t\geq 0}$ *to the worker's problem satisfies the following generalized Euler equation:*

$$\frac{E^w_{t+1}}{E^w_t}\frac{Q_t}{Q_{t+1}} = \beta \frac{R_{t+1}}{\Pi_{t+1}\left(B_{t+1}\right)} \quad \textit{if} \quad B_{t+1} \neq B^{\star}_{t+1} \tag{13}$$

and

$$\beta q_t R_{t+1} \geq \frac{E^w_{t+1}}{E^w_t} \geq \beta R_{t+1} \quad \textit{if} \quad B_{t+1} = B^{\star}_{t+1}.$$

The Euler equation has a clear interpretation: The left-hand side is the usual marginal rates of substitution between t and $t+1$ consumption.[16] The right-hand side captures the usual relation between discounting and rate of return, βR_{t+1}, but deflated by marginal inflation. Marginal inflation is the ratio of the relevant prices that the deliver marginal utilities at t and $t+1$, respectively $\tilde{q}^e_t$ and $\tilde{q}^s_{t+1}$. In an optimal solution, under the assumptions presented in the proposition, this generalized Euler equation holds exactly except at the discontinuity point $B^{\star}_{t+1}$. The inequalities that are satisfied when $B_{t+1} = B^{\star}_{t+1}$ correspond to an optimality condition based of a sub-differential: the condition says that increasing debt is optimal to the left of $B^{\star}_{t+1}$ but not to the right.[17]

The Euler equation is necessary, but not sufficient in this model. The reason is that the objective function in the worker's problem, (12), is not concave in B'. This is introduces further challenges to characterize a solution because multiple (finite) sequences of B_{t+1} may satisfy the worker's Euler equation.

The stationary solution to the worker's problem reveals interesting properties.

Proposition 5. (Stationary Worker's Problem): *Fix* $\tilde{B}_t = B_{ss}$ *and* $R_t = \beta^{-1}$ *and let* $\tilde{B}_{ss} > 1$. *Then, a solution to the worker's problem satisfies:*

I. *If* $B_0 \leq B^{\star}\left(1/\beta, \tilde{B}_{ss}\right)$, *then* $B_t = B_0\ \forall t$.

[16]The ratio of expenditures to average prices, is the ratio of marginal utilities under log preferences.

[17]There is also a discontinuity in the Euler equation at $B_{t+1} = \tilde{B}_{t+1}$, but the corresponding sub-differential does not yield an optimality condition.

II. *There exists a threshold $B^h > B_{ss}$ such that:*

◇ *If $B_0 > B^h$, then $B_t \to B^\star\left(1/\beta, \tilde{B}_{ss}\right)$ in finite time*

◇ *If $B_0 \leq B^h$, then $B_t = B_0\ \forall t$.*

The threshold B^h and the convergent sequence is given in the proof.

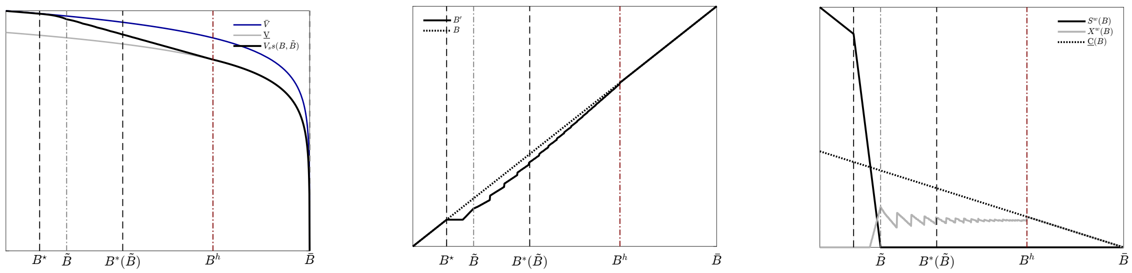
The proposition showcases a property of the worker's problem that has important implication in general equilibrium. To illustrate the Proposition, Figure 3 plots a numerical solution of the stationary version of the worker's Bellman equation, under the case where $R_{t+1} = \beta^{-1}$ and a that $\tilde{B}_t = \tilde{B}_{ss} > 0$. In the figure, I also includes the value functions when $\tilde{B}_t = 0$ and $\tilde{B}_t = \infty$, which I denote by $\bar{V}$ and $\underline{V}$ respectively.

Notice that for debt levels below $B^\star$, V is exactly on top of $\bar{V}$. Once debt is below that threshold, the worker consumes a constant amount of spot consumption permanently and this gives the same value as if the SBL was irrelevant. When debt is above a threshold B^h, the value function overlaps $\underline{V}$. Once debt is above that threshold, the worker consumes constant a amount of chained consumption permanently and this gives the same value as if the SBL was zero. Thus, for high debt levels, even though the worker can save to eventually enjoy better prices, he has no incentives to do so, because the benefits of deleveraging come to fare in the future. We can verify this through the middle panel of the figure that depicts the savings function.[18] Debt stays put when debt is really high or very low. For moderate debt levels, there is deleveraging $B_{t+1} < B_t$ and, furthermore, we can observe that fore a rang of values, B_t reaches $B^\star$. In this model saving away from financial constraints happens if constraints are not too tight. I return to this figure property after I characterize the equilibrium dynamics: it leads to important normative implications.

Equilibrium Dynamics: From Sequential to a Functional Representation. From now, I analyze the equilibrium dynamics of B_t. Despite that $\tilde{B}_t$ is an arbitrary sequence, the equilibrium is recursive with state variable $B_t \times \tilde{B}_t \times \tilde{B}_{t+1} \in [0, \bar{B}]^3$.[19] Let me present

[18]The right panel presents the optimal expenditures. As we can observe from the figure, the expenditure mix changes as debt levels cross threshold.

[19]This means that we have a relationship between a sequence and a recursive formulation: a variable m_t, can be expressed as an equilibrium function, $m : [0, \bar{B}]^3 \to \mathbb{R}_+$ such that $m_t = m\left(B_t, \tilde{B}_t, \tilde{B}_{t+1}\right)$.



(a) Value function (b) Debt Policy (c) Consumption Policy

Figure 3

Note: Figures are calculated using value function iteration: $\beta = 0.9$, $q = 2.5$and $\tilde{B} = 0.2 \cdot \bar{B}$.

some of the equilibrium functions we encountered by this point. From (9), (7), and (10), we have:

$$C^{s}\left(B\right) = \left(1-\beta\right)B, \quad E^{w}\left(B\right) = 1-\left(1-\beta\right)B, \tag{14}$$

and

$$\text{and} \quad S^{w}\left(B,\tilde{B}\right) = \min\left\{\bar{S}\left(B,\tilde{B}\right), E^{w}\left(B\right)\right\}.$$

Several other equilibrium functions are deduced from these functions directly, $\mu = E^{w} - S^{w}$, $q \equiv \mathcal{A}^{-1}\left(\mu\right)$, $X^{w} \equiv \mu/q$, $Q \equiv E^{w}/C^{w}$, and, finally, $C^{w} \equiv S^{w} + X^{w}$.

The only endogenous argument of the state is B_t. Thus, we need a map $\mathcal{B}$ from the current state to its future value, $B' = \mathcal{B}\left(B,\tilde{B},\tilde{B}'\right)$. If we obtain that map, the equilibrium rate will satisfy $R_{t+1} = \mathcal{R}\left(B_t,\tilde{B}_t,\tilde{B}_{t+1}\right)$ where $\mathcal{R}\left(B,\tilde{B},\tilde{B}'\right) \equiv \beta^{-1}\mathcal{B}\left(B,\tilde{B},\tilde{B}'\right)/B$, following the saver's optimal rule. With that equilibrium function, we can define the threshold function $B^{\star}\left(B,\tilde{B},\tilde{B}'\right) \equiv \mathcal{R}\left(B,\tilde{B},\tilde{B}'\right)\cdot\left(\tilde{B}-1\right)$. I solve for $\mathcal{B}$ below, after defining a functional representation for the equilibrium prices:

$$\Pi\left(B';B,\tilde{B},\tilde{B}'\right) \equiv \tilde{q}^{B}\left(B';\tilde{B}'\right)/\tilde{q}^{E}\left(B';B,\tilde{B},\tilde{B}'\right),$$

where

$$\tilde{q}^{E}\left(B';B,\tilde{B},\tilde{B}'\right) \equiv 1+\left(q\left(B,\tilde{B}\right)-1\right)\cdot\mathbb{I}_{\left[B'\geq B^{\star}\left(B,\tilde{B},\tilde{B}'\right)\right]}$$

Hence, from now I use m to represent the function that maps the state into its equilibrium value m_t. I also adopt the convention of using m' to refer to m_{t+1}.

and

$$\tilde{q}^B\left(B';\tilde{B}'\right)\equiv 1+\left(q\left(B',\tilde{B}'\right)-1\right)\cdot\mathbb{I}_{\left[B'\geq\tilde{B}'\right]}.$$

To obtain an equilibrium, we need to obtain $\mathcal{B}$. To that end, I combine the worker and saver Euler equations, (13) and (9), and substitute out R_{t+1} and using the equilibrium worker expenditures (14), to obtain an equilibrium condition B':

$$\underbrace{\frac{B}{1-(1-\beta)B}\cdot Q\left(B,\tilde{B}\right)}_{\equiv\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right)}=\underbrace{\frac{B'}{1-(1-\beta)B'}\frac{Q\left(B',\tilde{B}'\right)}{\Pi\left(B';B,\tilde{B},\tilde{B}'\right)}}_{\equiv\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)}. \tag{15}$$

Thus, any B_{t+1} must be a solution B' the equation above. The left hand side and right hand sides define functions equations $\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right)$ and $\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)$ that, at equality, define an aggregate Euler equation. Because Q is discontinuous, there may be multiple solutions B' to that equation. The following result tells us which solution B' is consistent with the symmetric competitive equilibrium and, thus, defines $\mathcal{B}$.

Proposition 6. (Equilibrium Rates and Expenditures): *Consider a weakly monotone increasing sequence of spot borrowing lines $\tilde{B}_t\to\tilde{B}_{ss}$. For any $B_0<B^h\left(\tilde{B}_{ss}\right)$, if an equilibrium exits, then $B_{t+1}=\mathcal{B}\left(B_t,\tilde{B}_t,\tilde{B}_{t+1}\right)$ where*

$$\mathcal{B}\left(B,\tilde{B},\tilde{B}'\right)=\max\left\{B^{\star}\left(\tilde{B}\right),\ \arg\min_{B'}\left\{\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right)=\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)\right\}\right\}.$$

Proposition 6 is key. With it, we can describe the equilibrium dynamics through a sequence of corollaries. Implicitly, the proposition yields an algorithm to compute equilibria.[20]

Steady States. I use the ss subscript to denote steady states. In principle, workers could make both spot and chained expenditures in steady state.However, from the saver's problem we know that in steady state, $R=\beta^{-1}$. Coupled with the worker's

[20] Starting from, B_0, the sequence of debt generated in equilibrium is given by $\mathcal{B}\left(B,\tilde{B},\tilde{B}'\right)$, which is the smallest solution B' to the equation $\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right)=\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)$. For each B, we obtain B' and update the state accordingly.

marginal inflation must also equals one which can only occur if the worker makes one type of expenditure—recall Proposition 4. Thus, in steady state the worker makes either only spot expenditures or only chained expenditures. I define an **undisrupted steady state** as a steady state with only spot expenditures, and hence, $\mathcal{Y} = 1$. In turn, a steady states with below-capacity output, $\mathcal{Y} < 1$ are **disrupted steady state**. The following corollary presents a condition that guarantees that the economy is in an undisrupted steady state.

Corollary 1. *Fix $\tilde{B}_t = \tilde{B}_{ss}$. For any $\tilde{B}_{ss} > 0$, the economy is in an undisrupted steady state for any $B_t \leq B^{\star}\left(1/\beta, \tilde{B}_{ss}\right)$.*

If $B_t \leq B^{\star}\left(1/\beta, \tilde{B}_{ss}\right)$ the worker can borrow (intra-is that period) more than his current income net of interests (at the steady state rate β^{-1}). If that is is the case, the economy is in an undisrupted steady state. Observe that if $\tilde{B}_{ss} \leq 1$, then $B^{\star}\left(1/\beta, \tilde{B}_{ss}\right) > 0$ in which case, no undisrupted steady state exists.

Convergence Toward an Undisrupted Steady State. Next, I describe domain of attraction toward undisrupted steady states. I turn to its counterpart, conditions under which it remains in a disrupted steady state after. The domain of attraction toward undisrupted steady states is the region for which the equation $\mathcal{E}\left(B'; B, \tilde{B}, \tilde{B}'\right) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)$ has a solution where $B' < B$. This region has an upper bound:

$$B^{*}\left(\tilde{B}\right) \equiv \tilde{B} / \left(C^{w}\left(\tilde{B}, \tilde{B}\right) + C^{s}\left(\tilde{B}\right)\right) \geq \tilde{B}.$$

This inequality follows from total consumption being less than total expenditures. We have the following.

Corollary 2. *Let $\tilde{B}_t = \tilde{B}_{ss}\, \forall t$. For any $B_0 < B^{*}\left(\tilde{B}_{ss}\right)$, $B_{t+1} < B_t$ if $B_t \in \left(B^{\star}(\beta^{-1}, \tilde{B}_{ss}),\ B^{*}(\tilde{B}_{ss})\right)$ and $B_{t+1} = B_t$ if $B_t \leq B^{\star}(\beta^{-1}, \tilde{B}_{ss}, \tilde{B}_{ss})$. If, in addition, $\tilde{B}_{ss} > 1$ the economy converges toward an undisrupted steady state in finite time (and approaches zero debt if $\tilde{B}_{ss} \leq 1$).*

Corollary 2 describes the domain of attraction toward undisrupted steady states. If workers hold debt between $\tilde{B}$ and $B^{*}\left(\tilde{B}\right)$ they repay debt their debts.[21] The deleverage

[21] Indeed, they will delever at the rate $\mathcal{R} < \beta^{-1}$ consistent with the condition in Proposition 4. In this region, equation (15) may have two solutions, but only the lowest value of B' is optimal.

will continue until she reaches some steady state debt once $B' \leq B^\star\left(\tilde{B}_{ss}\right)$. An implication is that in the domain of attraction of undisrupted steady states, production delays are temporary.

To illustrate the implications from Corollary 2, Figure 4 describes a transition toward an undisrupted steady state. The figure plots different debt levels in the x-axis, holding $\tilde{B}$ fixed. The solid blue and dashed gray curves plot the functions $\mathcal{E}$ and $\mathcal{E}'$ correspondingly. The arrows in the figure trace the path of debt generated in equilibrium, following the sequence of solutions to B' implicit in Proposition 6. The left panel shows that for initial conditions where $B_0 > B^*\left(\tilde{B}_{ss}\right)$, the economy fails to converge. In that region, the only solution to $\mathcal{E} = \mathcal{E}'$ happens at $B' = B$. The middle panel plots the equilibrium interest rate, $R\left(B, \tilde{B}, \tilde{B}'\right)$.[22] The right panel highlights a gray area in the phase diagram, this is the area of hysteresis that I describe below.

Whereas Figure 4 presents the phase diagram for a fixed value of $\tilde{B}_{ss}$, we can combine Proposition 6 and Corollary 2 to describe transitions when $\tilde{B}_t$ is not at steady state. We can interpret this path as the dynamics that follow a credit crunch. As long as $B_0 \leq B^*\left(\tilde{B}_{ss}\right)$ and $\tilde{B}_{ss} > 1$, the economy converges to an undisrupted steady state as explained in the following proposition.

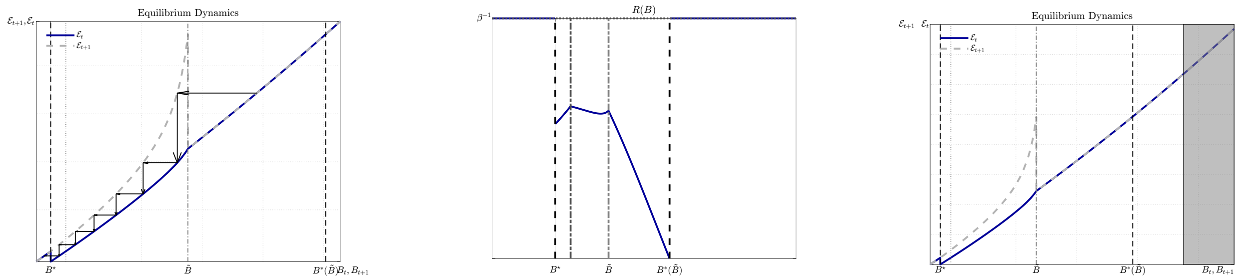
Proposition 7. (Credit-Crunch Transtions): *Consider an increasing sequence $\left\{\tilde{B}_t\right\}$ corresponding to a credit crunch with three phases: (a) for $t \leq T_e$, the economy is in the extreme phase where $\tilde{B}_t \leq B_0$; (b) for $t \leq T_r$ the economy is in a recovery phase where $\tilde{B}_t \in \left[B_0, \tilde{B}_{ss}\right]$; (c) for $t > T_r$, the economy is in its normal phase where $\tilde{B}_t = \tilde{B}_{ss}$. We have the following:*

I. *Extreme Phase, $t \leq T_e - 1$: $B_t = B_0$, and $R_t = 1/\beta$ and the worker consumes only chained goods.*

II. *Smoothing Phase, $t = T_e$: the worker accumulates debt, $R_t > 1/\beta$, $S_t = 0$ and X_t is higher than the previous value.*

III. *Recovery Phase, $t \in \{T_e, \ldots, T_r - 1\}$: the path of debt and interest rates is ambiguous, but there is consumption of both goods $S_t, X_t > 0$.*

[22] Note that there are regions of relevance that depend on the cases in the function. The discontinuity arises due to the indicators. The solution is obtained by replacing the market clearing condition into the the worker's Euler equation.



(a) Transition Plot (b) Equilibrium Rates (c) Hysteresis Region

Figure 4

Note: Figures are calculated using the formulas in proposition 6 for $\beta = 0.9$, and $\tilde{B} = 0.2 \cdot \bar{B}$.

IV. *Repayment Phase, $t = T_r$: the worker repays debt, $R_t < 1/\beta$ and X_t is lower than its previous value.*

V. *Steady-State Phase, $t \geq T_r + 1$: $B_t = B_{ss}$, $R_t = 1/\beta$ and the worker consumes only spot goods.*

Along the transition $B_0 > B_{ss}$.

There are different phases in a transition. In its extreme phase, TFP is at its lowest because workers only spend by making chained orders and debt remains constant. In the smoothing phase, workers anticipate that they will be able to spend on spot goods in the subsequent period. To smooth consumption, they increase current chained expenditures taking in more debt. During the recovery phase, workers trade-off consumption smoothing and against paying off debt to increase their credit lines and spend spot. The repayment phase is the last period with positive chained expenditures. Eventually, the economy converges to a steady state with less debt than at the start of the transition. Figure 5 presents an example of a transition where the different phases that correspond to Proposition 7 are evident.

Debt-Overhang and Hysteresis of Payment-Chain Crises. Up to this point, I only considered undisrupted steady states. In now turn to the study of hysteresis in payments-chain crises. Hysteresis is provoked when there is debt overhang.

Corollary 3. (Hysteresis): *Let $\tilde{B}_t = \tilde{B}_{ss}\ \forall t$. For $B_t \geq B^h\left(\tilde{B}_{ss}\right)$, the economy is permanently in a disrupted steady state.*

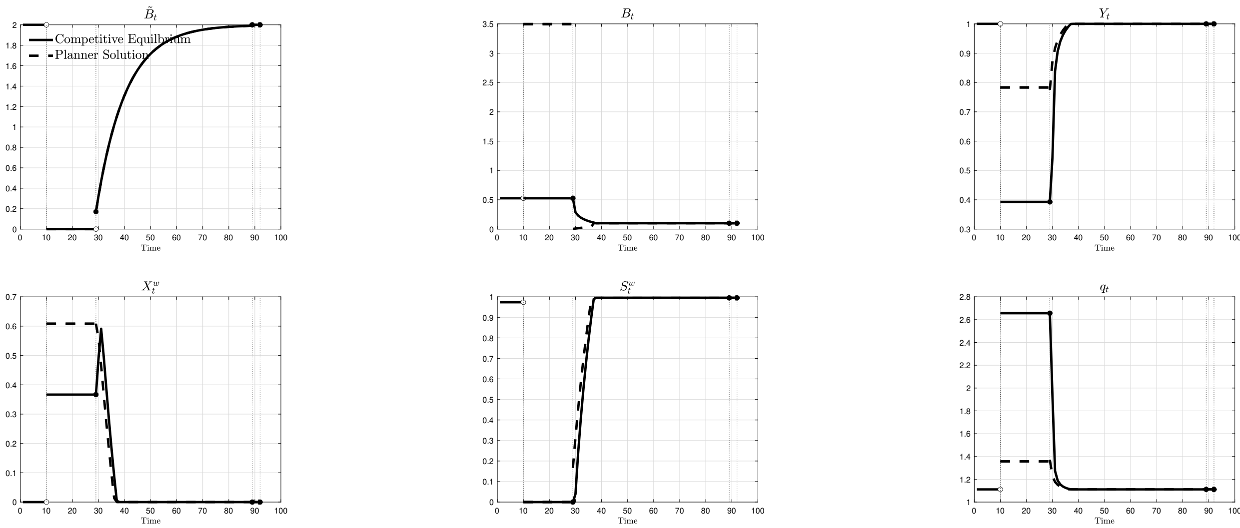


Figure 5: Efficient and Competitive Transition after a Smooth Credit Crunch.

Note: This figure reports a numerical example of a credit crunch episode. I consider two possibilities, I violent and a smooth transition.

Hysteresis occurs when workers have so much debt that the required sacrifice in terms of current consumption does not compensate the benefits relaxing constraints in the future to make spot orders.[23] Recall from Figure 3 that for sufficiently high debt levels, the worker's value function yields the same value as if $\tilde{B}_t = 0$ forever. This means that the economy experiences a debt overhang, she could save to make spot consumption, assuming the rate is $R_{t+1} = \beta^{-1}$. Of course, in general equilibrium, the rate is endogenous. Figure 4 depicts the region of hysteresis in the gray area, and shows that the hysteresis region, equilibrium rates are indeed, $R_{t+1} = \beta^{-1}$.[24]

If debt starts in hysteresis region, it never falls. Excessive debt hampers the ability to make payments quickly, and the economy may ends up in a permanent state of below capacity production. With deterministic dynamics, hysteresis cannot be reached from other states. In a more general setting, hysteresis can be induced by shocks that provoke excess pptimism or low discount factors. This debt overhang rationalizes several verbal descriptions of the Japanese lost decades or Europe after the Great Financial Crisis. The scenario showcases an extreme episode where there are clear productivity losses that could be avoided. In the following section, I formalize this sense of social

[23] An obvious case occurs when $\tilde{B}_{ss} = 0$. In that case, there are no incentives to deleverage. But even for $\tilde{B}_{ss} > 0$, we have the following proposition.

[24] For that rate, the only solution to the equilibrium condition $\mathcal{E} = \mathcal{E}'$ happens at $B' = B$.

inefficiency and analyze policy remedies.

I should note that the the domain of attraction of undisrupted steady states and hysteresis are disconnected. This implies that between $B^*\left(\tilde{B}_{ss}\right)$ and $B^h\left(\tilde{B}_{ss}\right)$, a symmetric competitive equilibrium does not exist. I do not explore other forms of equilibria.[25]

Discussion: Interpretation of Financial Frictions in the Model. For savers, it is clear that their spot payments represent the use of cash. For workers, an interpretation of $\max\left\{\tilde{B}_t - B_t, 0\right\}$ is: funds drawn from a "business credit line." This is because S_t^w are business related payments that can be made only if we have a liquid liability line available. Hence, spot payments by the worker can be thought of as credit card lines, overdraft facility programs or directly as supply chain finance facilities.[26] In turn, chained payments can be though of as account receivables that remain outstanding. In practice, for many account receivables, the product is not yet delivered or produced, until a fraction of the payment is anticipated. This is the sense in which they provoke delays.

Related to the point above, the SBL here is different from the standard hard debt limits studied in most macro finance models. There's a economic motivation behind it: if a bank wants to cut back on credit, it can tighten credit lines, but will necessarily force client to repay debt principals. If loan repayment is suddenly forced, it can trigger default in which case it may lead to costly underwritings for banks. In such cases, it is more convenient for the bank to role over debt than to force repayments.

Finally, it is worth discussing the misallocation of funding. As a result of the big family assumption, the worker will receive labor income flows while there are still pending chained orders. This happens because the worker may receive payment on orders that are not chained before it receives payments that are pledged to chained expenditures that have not cleared. The big family assumption is present to avoid a distribuiton of

[25] Note that in this middle region, at an individual level, workers would want to delever as seen in Figure 3. In general equilibrium, this would imply a rate below β^{-1}, for which there are no solutions to equation (15), as shown in Figure 4. The economy may possibly feature sunspot equilibria; a situation I do not consider in this paper.

[26] Other examples of facilities are Business Credit Lines, Standby Letters of Credit and Supplier Finance Programs. Under a Supplier Finance Program, the buyer wants to pay later whereas suppliers request cash. Supplier Finance enables suppliers to be paid by banks against the receivables. Descriptions of these programs are offered by some of the largest financial institutions: J.P. Morgan Supplier Finance Facility or Citibank Supplier Finance Facility.

ex-post outcomes, while keeping the simplicity of the TFP function derived earlier. I should note that chained orders would still induce delays even if within the worker could not pledged income to accelerate the payment of chained orders.[27] Hence, the assumption that some funds remain idle, change the exact functional form of the TFP function, but not it's essence.

4. Policy Implications

4.1 Constrained Inefficiency

In Section 3, we expressed the chained expenditure ratio as a function of debt levels. Earlier, in Section 2 we found that higher chained expenditure ratios lead to longer payments chains which provoke declines in observed TFP. Those losses result from the poor organization of payments chains, given a level of chained expenditures. Ideally, a planner would reorganize payments so that each chain is of equal length, but governments do not have that power. What governments can do is influence the decisions to spend by different agents. This section studies policies that alter expenditure-savings decisions, respecting the transactions technology and the financial constraint.[28] The spirit of the exercise is not normative. Rather, the purpose is to clarify the sources of constrained inefficiency of the competitive equilibrium .

Ideal Pareto Weights. To study constrained inefficiency, I consider a transition of a competitive equilibrium that reaches an undisrupted steady-state debt-level B_{ss} starting from B_0. I study a Ramsey Planner problem with Pareto weights on workers θ which delivers the same steady-state debt level of the competitive equilibrium:

$$\frac{1-(1-\beta)\,B_{ss}}{(1-\beta)\,B_{ss}}=\frac{\theta}{1-\theta}. \tag{16}$$

[27] For the case where chained orders can be paid with any incoming payment, the chain length distribution can again be derived analytically. In that case, for $N^x > N^s$, the minimal chain length would be zero, the maximal chain length $\lceil N^x/N^s \rceil$, and the probability distribution uniform among chains with $n \leq \lfloor N^x/N^s \rfloor$. For $N^x \leq N^s$, the minimal chain is zero with probability $N^s - N^x$ and the maximal chain length 1.

[28] Traffic regulation is a useful analogy: government's cannot assign drivers into different lanes, but they can tax vehicles.

Since the economy is efficient in an undisrupted steady state, any difference between the transition path of debt in a planner and competitive equilibrium solutions uncovers an inefficiency only along transitions.

The Ramsey Problem. Consider a sequence of debt taxes $\{\tau_t^k\}$, labor taxes $\{\tau_t^\ell\}$, and expenditure taxes $\{\tau_t^c\}$. The Ramsey problem is:

Problem 4. (Ramsey Problem): *Given* B_0 *and* $\left\{\tilde{B}_t\right\}$*:*

$$\max_{\left\{\tau_{t+1}^k,\tau_t^c,\tau_{t+1}^\ell\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right)+\theta\log\left(C_t^w\right)\right],$$

subject to the (modified) saver budget constraint and optimality:

$$\left(1+\tau_{t+1}^k\right) R_{t+1}^{-1} B_{t+1} + \left(1+\tau_t^c\right) C_t^s = B_t,\ \forall t \geq \boldsymbol{0},$$

to the (modified) worker budget constraint and optimality:

$$B_t + \left(1+\tau_t^c\right) E_t^w = R_{t+1}^{-1} B_{t+1} + 1 - \tau_t^\ell,\ \forall t \geq \boldsymbol{0},$$

to the government's budget constraint:

$$\tau_{t+1}^k R_{t+1}^{-1} B_{t+1} + \tau_t^c \left(C_t^s + E_t^w\right) + \tau_{t+1}^\ell = 0,\ \forall t \geq \boldsymbol{0},$$

and the structure of transactions: (i) optimal expenditures (10-11), (ii) total consumption (2), (iii) the optimality conditions of the worker and saver problems, (iv) the payments constraints (8), and (v) the shadow price, $q_t = \mathcal{A}\left(\mu_t\right)^{-1}$*.*

This Ramsey planner distorts the competitive equilibrium by using credit, labor, and expenditure taxes. This planner cannot distinguish between the two expenditure forms but takes into account the agents' optimal behavior, their constraints, market clearing, respects the transactions technology, and satisfies the budget balance condition.

The Primal Problem. I solve the Ramsey problem by solving an equivalent primal problem.

Proposition 8. (Solution to Ramsey): *The allocations induced by a solution to the Ramsey Problem are the same allocations as the solution to the following Primal Planner Problem:*

Problem 5. (Primal Planner):*Taking* $\left\{\tilde{B}_t\right\}$ *as given:*

$$\max_{\{B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \mathcal{P}\left(B_t, \tilde{B}_t\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) \equiv (1-\theta)\log\left(C^s\left(B\right)\right) + \theta\log\left(S^w\left(B, \tilde{B}\right) + X^w\left(B, \tilde{B}\right)\right).$$

Let $\left\{B^p_{t+1}\right\}_{t\geq 0}$ *be a solution to the Primal Planner Problem. The solution to the Ramsey Problem can be implemented setting* $(1+\tau^c_0) = B^p_0/B_0$ *and a sequence of debt taxes that satisfies a formula given in the proof. Any sequence of labor and subsequent expenditure taxes that satisfies the budget solvency condition implements the Primal problem.*

The main insight from this proposition is that the Ramsey problem can be solved by solving a more relaxed Primal Problem where a planner directly chooses the path of debt. Indeed, the constraint set in the Primal Problem includes the constraint set of the original Ramsey problem.[29] Hence, if a solution to the primal can be implemented with taxes, it solves the optimal Ramsey program. The proposition shows that an implementation exists. In the implementation, capital taxes and a period-zero expenditure tax are needed by the Ramsey planner to distort the evolution of debt and replicate the Primal solution. This implementation is possible because saver expenditures are invariant to capital taxes so net-of-tax interest rates determines the evolution of debt.[30] Understanding the path of debt chosen by the Planner is critical to under-

[29]This is immediate since market clearing in the asset market and the budget balance, implies, by Walras's law, that the resource constraint holds.

[30]Under log utility, debt determines the savers' expenditures through the $(1-\beta)$ rule of log utility. Since worker income is one, and income equals expenditures, once saver expenditures are given, worker expenditures are given. But once worker expenditures are known, the composition of their expenditures is known, through expenditure minimization. When labor taxes are used to balance the budget, expenditure taxes are redundant. The expenditure tax is needed at time zero to control B^p_0. If I were to restrict $\tau^c_0 = 0$, the Primal Planner would be modified only in that B_0 would be taken as given.

stand the inefficiencies of this environment.

Proposition 9. (Solution of the Primal Problem): *The solution to the Primal Planner Problem $B_t = B^p\left(\tilde{B}_t\right)$ where B^p is the solution to:*

$$\mathcal{P}^o\left(\tilde{B}\right) = \max_{B\in[0,\bar{B}]} \mathcal{P}\left(B, \tilde{B}\right).$$

B^p and $\mathcal{P}^o$ satisfy the following properties:

I. Efficient Allocation. *For $\tilde{B} \geq \frac{1+\theta\beta}{1-\beta}$, the planner's problem is unconstrained: $\mathcal{P}^o\left(\tilde{B}\right) = \mathcal{P}^o\left(\bar{B}\right)$, $B^p = B_{ss}$, and there are no chained expenditures and TFP is maximal, $X^w = 0,\ \mathcal{A} = 1$.*

II. Inefficient Allocations. *For $\tilde{B} < \frac{1+\theta\beta}{1-\beta}$, the planner always distorts social insurance. The planner may or may not distort TFP and may feature $B^p > B_{ss}$ or $B^p < B_{ss}$ depending on the thresholds $\left\{\tilde{B}^i, \tilde{B}^s\right\}$ characterized in the Appendix.*

II.a. Inefficient Insurance | Productive Efficiency. *For $\tilde{B} \in \left[\tilde{B}^i, \frac{1+\theta\beta}{1-\beta}\right)$, the planner's problem is constrained: $\mathcal{P}^o\left(\tilde{B}\right) < \mathcal{P}^o\left(\bar{B}\right)$. The planner distorts expenditures $B^p = B^\star\left(\tilde{B}\right) < B_{ss}$. However, there are no chained expenditures and TFP is maximal, $X^w = 0,\ \mathcal{A} = 1$.*

II.b. Inefficient Insurance and Inefficient Production | Complementary Case. *If $\tilde{B}^s < \tilde{B}^i$, then $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right)$, the planner's problem is constrained: $\mathcal{P}^o\left(\tilde{B}\right) < \mathcal{P}^o\left(\bar{B}\right)$. The planner distorts expenditures and $B^p < B^\star\left(\tilde{B}\right)$ is the unique solution to:*

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{1-\theta}\frac{Q\left(B,\tilde{B}\right)}{q\left(B,\tilde{B}\right)}\left(\frac{q\left(B,\tilde{B}\right) - \beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B,\tilde{B}\right)\right)\right)}{1-\beta}\right). \tag{17}$$

Moreover, the solution satisfies $X^w, S^w > 0$ and TFP is inefficient, $\mathcal{A} < 1$. A marginal decrease in debt at B^p increases efficiency.

II.c. Inefficient Insurance and Inefficient Production | Conflicting Case. *For $\tilde{B} \in \left[0, \tilde{B}^s\right]$, we have that B^p is the unique solution to:*

$$\frac{1-(1-\beta)\,B}{(1-\beta)\,B} = \frac{\theta}{(1-\theta)}\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-(1-\beta)\,B\right)\right). \tag{18}$$

Moreover, for this debt level $S^w\left(B, \tilde{B}\right) = 0$ and the solution is the same as when $\tilde{B} = 0$. A marginal decrease in debt at B^p decreases efficiency.

The solution to the Primal is interesting in-itself because it reveals novel economic insights. First, the Planner can choose a sequence of debt on behalf of agents. Once the planner chooses a level of debt, because of log utility, saver expenditures are determined. Since total income is 1, and total income equals total expenditures, the choice of B determines the worker expenditures. Since production is static and only depends on the expenditure mix, the Primal Problem is solved as a sequence of static problems. Thus, the problem can be entirely formulated in terms of a choice of B, given $\tilde{B}$.

If the SBL is above the threshold, $\tilde{B} \geq \frac{1+\theta\beta}{1-\beta}$, the planner sets $B = B_{ss}$, and production is efficient. Furthermore, condition (16) is satisfied: meaning that the ratio of marginal utilities equals the ratio of Pareto weights, a notion of unconstrained efficient social insurance.

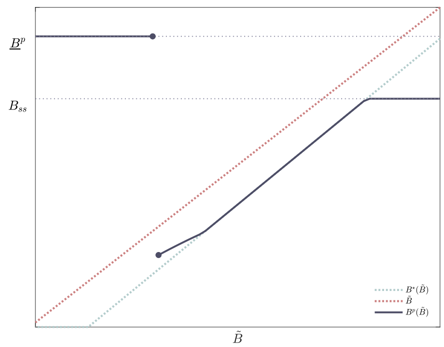
If the SBL is below the efficiency threshold, the planner faces a trade-off between social insurance and productive efficiency, a common trade-off in macroeconomics. The novel aspect of the theory is that the planner has two ways to increase productive efficiency: one option is for the planner to choose debt below the SBL, $B < \tilde{B}$. Below the SBL, any reduction in debt translates, on the margin, into more spot consumption by the worker and, thus, increases efficiency. Alternatively, the planner can choose above the SBL, $B < \tilde{B}$. Above the SBL, any increase in debt, translates on the margin, into more spot consumption by the saver. All in all, the planner can increase efficiency on the margin, either by distributing wealth toward the worker if $B < \tilde{B}$ or by distributing wealth toward the saver if $B > \tilde{B}$. Because of this ambivalent nature of efficiency, the Planner's objective function $\mathcal{P}\left(B, \tilde{B}\right)$ is not concave in B. The nature of the Planner's solution changes as the SBL becomes tighter.

I use Figure 6 to describe the economics behind the planner's solution. The left panel plots B^p for different values of $\tilde{B}$. When the SBL is moderately tight, $\tilde{B} \in [\tilde{B}^i, \frac{1+\theta\beta}{1-\beta})$, the Planner does not provoke productive inefficiency—TFP is 1. However, since $B^\star\left(\tilde{B}\right) < B_{ss}$, productive efficiency can only be achieved setting $B^p = B^\star\left(\tilde{B}\right)$. The planner trades-off social insurance and redistributes wealth toward the worker to guarantee productive efficiency. The Planner's solution is at a corner because the derivative of his objective is discontinuous at $B^\star\left(\tilde{B}\right)$. This discontinuity follows from $\mathcal{A}(0) < 1$, a feature of the payments-chain network structure: even when the chained expenditure ratio is zero, and individual chained expenditure experiences a delay.

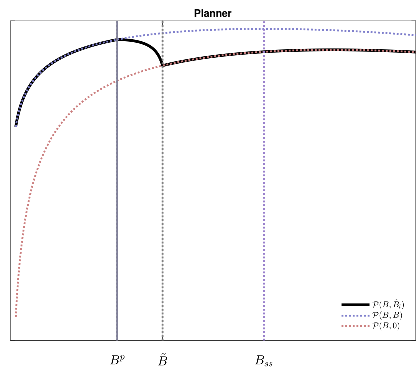
When the SBL is further tight, $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right)$, the Planner is willing to accept some productive inefficiency. To guarantee productive efficiency, for those levels of the SBL, the planner would have to redistribute even more wealth to the worker, at the expense of saver's consumption. In that region, that trade-off is not worth it and thus $B^\star\left(\tilde{B}\right) < B^p < \tilde{B}$. However, still in this area redistributing wealth to the agent facing the constraint complements productive efficiency. Recall that the Planner has to balance productive efficiency with social insurance, as given by equation (17).

When the SBL below falls to an extreme value, $\tilde{B} < \tilde{B}^s$, the nature of the planner's solution changes dramatically. Increasing productive efficiency by redistributing wealth to the worker would require debt levels below $\tilde{B}$, but at those low levels of the SBL, that would require an even greater sacrifice of saver consumption. At that point, the planner switches strategy: it begins to redistribute wealth away from the worker, the agent that is constrained. The planner gives up on generating productive efficiency inducing spot expenditures by the worker. Rather, it makes the worker only spend in chained goods, but takes wealth away from her to make the saver spend spot. Once the worker only spends spot, the SBL becomes irrelevant, so the planner chooses a constant debt level in this region. This debt level is higher than the unconstrained ideal debt level B_{ss} given by Equation (18).

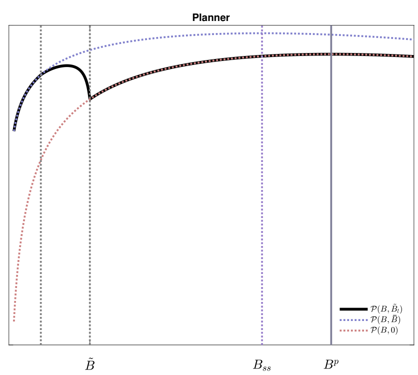
The ambivalent nature of the Planner's problem is germane to nature the payment-chain crises. In typical models with pecuniary externalities, a planner will want to rebalance wealth toward the financially constrained agent to induce productive efficiency. Thus, social insurance and productive efficiency reinforce each other. Here, for extremely low values of the SBL, the planner may switch to a policy where social insurance and efficiency are in conflict. The middle and right panels of Figure 6 illustrate how the planner changes strategy. In the middle panel, the planner prefers a value of debt where social insurance and efficiency compliment. The right panels shows how he switches strategy as the SBL is tighter.



(a) Planner Debt Choice



(b) $\mathcal{P}\left(B, \tilde{B}\right)$ (moderate $\tilde{B}$)



(c) $\mathcal{P}\left(B, \tilde{B}\right)$ (tight $\tilde{B}$)

Figure 6: Primal Planner Solution

Note: Figures are calculated using value function iteration: $\beta = 0.9$, $q = 2.5$and $\tilde{B} = 0.2 \cdot \bar{B}$.

Competitive Equilibrium vs. Efficient Transitions: An illustration. To illustrate how transitions are inefficient, Figure 5 overlays the planner solution to the competitive equilibrium described earlier. During the extreme phase of the crisis, the planner redistributes wealth toward savers to induce more spot expenditures. To do so, the planner must subsidize savings. From the outset, the policy seems draconian: the planner taxes the agent is suffering most, the worker, to subsidize savers. However, by making savers wealthier, the planner induces them to make more spot expenditures. This increases TFP, and actually, leads to an increase in worker consumption, despite the increase in labor taxes. This increase comes about through the reduction of the price of chained consumption, which more than offsets the increment in labor taxes. A regressive policy from the outset, increases the welfare of both agents. During the recovery phase the planner reverses the policy: it taxes savings to reduce debt. In doing so, the planner allows the worker to make some spot expenditures. As the credit standards are further relaxed, the planner increase debts slowly reaching the efficient steady state.

Though I considered the planner solution along a transition toward an undisrupted steady state, we also know that the planner solution described in Proposition 9 would not allow hysteresis in payments-chain crises.[31] The exercise above showcases that exiting the hysteresis region, may require policies that seem draconian from the outset.

4.2 Fiscal Policy and the Bocola Effect

Government Spending: Pay for stuff vs. Spending. I now consider government expenditures. I distinguish between government expenditures paid spot or paid chained to tax receipts. It turns out that the type of government expenditures matters dramatically. I call this effect, the Bocola effect of government expenditures, because economist Luigi Bocola suggested this distinction. To my knowledge, this is the first formal description of this effect.

To explain the Bocola effect in a simple way, I treat government expenditures as isomorphic to household expenditures. I label by G^s the spot government expenditures. To spend spot, implicitly I assume that the government must also borrow intra-period. In turn, the government can make chained expenditures, G^x. For that, I treat governments taxes as income units isomorphic to the labor income of households. For simplicity, I assume that the government raises labor taxes and satisfies a balanced budget at the end of the period. Furthermore, the resources used by both forms of expenditures are wasted resources. I consider the following problem.

[31] Indeed, in the hysteresis region, $\frac{Q_t}{q_t} = 1$, so $B_t = B_{t+1}$ is the only solution to the debt accumulation equation of the competitive equilibrium, (15). In the planner's solution, that equation is altered by the term, $\left(1 - \epsilon^{q}_{\mu,t}\right) > 1$.

Problem 6. (Government Problem with Expenditures): *Given $B_0 = B_{ss}$ and $\left\{\tilde{B}_t\right\}$:*

$$\mathcal{P}_0^g = \max_{\left\{\tau_t^k, G_t^x, G_t^s\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the saver's optimal behavior (9), a worker budget constraint:

$$B_t + E_t^w = R_{t+1}^{-1} B_{t+1} + 1 - \tau_t^{\ell},\ \forall t \geq 0$$

the worker's optimal behavior (13), (10-11), and (2), a government budget-balance constraint with expenditures:

$$\tau_t^{\ell} = G_t^x + G_t^s,\ \forall t \geq 0,$$

and the ratio of chained expenditures relative to total expenditures:

$$\mu_t = q_t X_t^w + G_t^x \text{ and } q_t = \mathcal{A}\left(\mu_t\right)^{-1}.$$

The problem is similar to the Ramsey problem other than for the government expenditures and the exclusion of credit taxes. Instead of solving this problem, I compute government multipliers, near no expenditures. These multipliers transparently showcase how the payment time of government expenditures matters:

Proposition 10. (Infinitesimal Government Multiplies): *Fix $\{G^x, G^s\} = (0,0)$. Consider an unexpected marginal increase in government expenditures of type $g \in \{x, s\}$ at time t. We have that:*

$$\frac{\partial \mathcal{P}_t^g}{\partial G^g} = \underbrace{\frac{\theta}{C^w}}_{\text{marginal ind. utility}} \times \underbrace{\frac{\partial C^w}{\partial G^g}}_{\text{multiplier}} \text{ for } g \in \{x, s\}.$$

The consumption responsesare:

$$\frac{\partial C^w}{\partial G^x} = \begin{cases} -1 & B < B^{\star}\left(\tilde{B}\right) \\ -\mathcal{A}\left(\mu\right) & B > B^{\star}\left(\tilde{B}\right) \end{cases}, \qquad \frac{\partial C^w}{\partial G^s} = \begin{cases} -1 & B < B^{\star}\left(\tilde{B}\right) \\ -\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right) & B > B^{\star}\left(\tilde{B}\right) \end{cases}.$$

Finally, the output multipliers are:

$$\frac{\partial \mathcal{Y}}{\partial G^{x}}=\begin{cases}\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)-1 & B<B^{\star}\left(\tilde{B}\right)\\ \\ 0 & B>B^{\star}\left(\tilde{B}\right)\end{cases},\quad \frac{\partial \mathcal{Y}}{\partial G^{s}}=\begin{cases}0 & B<B^{\star}\left(\tilde{B}\right)\\ \\ 1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right) & B>B^{\star}\left(\tilde{B}\right).\end{cases}$$

When $B \leq B^{\star}$, production is efficient. For such low levels of debt, Proposition 10 shows that any form of government expenditures are a waste to society. First, either form of government expenditure reduces worker expenditures one-for-one, leading to a reduction in welfare.[32] In terms of output, if the expenditure is spot, the multiplier is zero as transfers spot expenditures from the private to the public sector. If in turn the expenditures are chained, the income multiplier is negative because the government is increasing productive inefficiency, $\left(1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\right)>0$.

In a payments-chain crisis, when $B > B^{\star}$, the multipliers behave differently. Chained expenditures, are again detrimental for welfare.[33] They carry a zero output multiplier because they transfer an inefficient source of expenditures from the private sector to the public sector. By contrast, if the government spends spot, it provokes a positive externality. This externality is captured by the elasticity $\epsilon_{\mu}^{\mathcal{A}}$. Spot government expenditures also crowd-out worker chained expenditures but the income extracted from workers are spent upfront. Ultimately, this reduces the average chain length and increases TFP. In an deep crisis, spot government expenditures may even increase worker consumption. In particular, this occurs when $\epsilon_{\mu}^{\mathcal{A}}<-1$, a condition that is shown to be possible in the Appendix. In terms the output multiplier of spot expenditures, it is always positive in a payments chain crisis. This is because the government substitutes inefficient private expenditures for efficient public expenditures. Welfare increases precisely when the multiplier is above 1.

We learn a new lesson. In a payments-chain crisis, government expenditures can have a positive benefits but only if paid upfront. The mechanics are different from the arguments regarding aggregate demand stimulus. In this setting, a government

[32]The social cost of government expenditures are entirely born by workers because worker pay these taxes, the evolution of debt is not distorted.

[33]Government expenditures crowd-out worker chained expenditures one-for-one so divided by $1/q$, this gives us the reduction in chained consumption of chained government expenditures.

that subscribes to the idea that it can stimulate demand, simply by spending, would unwillingly reduce welfare and have no effects on output. To have positive effects, the government must spend by paying for things upfront during a payments-chain crisis.

Discussion: Fiscal Transfers, and Ricardian Equivalence. Because the labor is inelastic to labor taxes and saver expenditures are inelastic to capital taxation, the Ramsey planner problem is equivalent to one where we allow for transfers. The equivalence holds, only if transfers cannot be used immediately for spot payments.

Regarding transfers, there are some distinctions with the taxes I considers. For example, announcements of future transfers can have pervasive effectsin the midst of a payments-chain crisis. The reason is that it may lead workers to spend more. If they only make chain expenditures, this can reduce TFP. The model lacks Ricardian equivalence.

5. Conclusion

The contribution of this paper is to propose a payments interpretation of financial crises. The economic problem here is the inefficient timing of payments. This inefficiency causes delays in production and is a coordination failure aggravated by limited funding. These inefficiencies are encoded in $\mathcal{A}$. Whereas the policy recommendations have a similar flavor to those in environments with demand externalities, the paper shows that policies should be directed at accelerating payments.

In building the theory, I made two shortcuts. First, I assumed that all transactions are bilateral and for the same amounts. In practice, payments are much more complex, as is the nature of economic production. Second, I assumed that households produce. In practice, payment chains are more relevant for firms. Developing payment-chain networks with a richer variety of transactions and a business cycle model with firm production is important to bring the model closer to reality. Nonetheless, I expect the lessons here to hold in more general settings. I hope the paper prompts new ways to think of financial crises and their remedies.

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Appendix (Not intended for publication)

Contents

A. Proofs of Section 2

A.1 Proof of Proposition 2

Part 1. Derivation of TFP. In the body of the paper, I showed that for a chain of three orders, of which two are chained, the vector of production is $\{1, \delta, \delta^2\}$. The following induction argument generalizes: If the k-th node initiates production at time $1-\delta^{k-1}$, the delay from its inspection is $(1-\delta)\,\delta^{k-1}$, which added to previous delays leads to a transfer of funds only by time $1-\delta^{k-1}+(1-\delta)\,\delta^{k-1} = 1-\delta^k$. This, leaves δ^k time for production to the subsequent unit. Since we computed the delay for $k=1,2,\ldots$. the productions in a chain of length n are $\{1, \delta, \delta^2, \ldots, \delta^n\}$.

It follows that the average output for chained orders, that is excluding the output of the spot order in the chain, in a payments-chain of length n is

$$\bar{y}^x_n = \frac{1}{n}\sum_{m=1}^{n}\delta^m = \frac{\delta}{n}\left(\frac{1-\delta^n}{1-\delta}\right).$$

Recall that a chained order will necessarily fall in a chain with length $n \geq 1$. Thus, the p.m.f of lengths conditional on this event is

$$G^x(n;\mu) = \frac{(1-\mu)\,\mu^n}{\mu}.$$

Next, we turn to our goal of finding the expected output of a chained order:

$$\begin{aligned}
\mathbb{E}\left[\bar{y}^x\right] &= \sum_{n=1}^{\infty}\bar{y}^x_n G^x(n;\mu),\\
&= \sum_{n=1}^{\infty}\frac{(1-\mu)\,\mu^n}{\mu}\cdot\frac{\delta}{n}\left(\frac{1-\delta^n}{1-\delta}\right),\\
&= \frac{(1-\mu)}{\mu}\cdot\frac{\delta}{(1-\delta)}\cdot\sum_{n=1}^{\infty}\left(\frac{\mu^n}{n}-\frac{(\delta\mu)^n}{n}\right),\\
&= \frac{1-\mu}{\mu}\cdot\frac{\delta}{1-\delta}\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right),
\end{aligned}$$

The last equality follows from:

$$\sum_{n=1}^{\infty} a^{n-1} = \frac{1}{1-a} \leftrightarrow$$
$$\sum_{n=1}^{\infty} \frac{a^n}{n} = \ln\left(\frac{1}{1-a}\right)$$

for $|a| < 1$, which can be shown by simply taking derivatives to both sides. A simple weak law of large numbers yields the desired result. Recall that we can have spot identities $i \in \mathcal{N}^s$ as unique identifiers for payment chains (spot orders for a one-to-one map with chains). Let $\ell(i)$ be the length of the payment chain that starts with spot order i. Given that for each $i \in \mathcal{N}^s$, $\ell(i) \sim i.i.d.\, G(\mu)$, then

$$\mathcal{A}(\mu;\delta) \equiv \underset{N\to\infty}{\text{plim}} \frac{1}{N^s} \sum_{i\in\mathcal{N}^s} \overline{y}^x_{\ell(i)} = \mathbb{E}\left[\overline{y}^x\right].$$

In words, the average output among chained orders converges to $\mathbb{E}\left[\overline{y}^x\right]$ as the network gets larger ($N \to \infty$). Next, we derive expected output—the limit as $N \to \infty$ is implicitly. The fraction of spot orders is $(1-\mu)$. Production is 1 in their case. The fraction of chained orders is μ, and they produce on average $\mathcal{A}(\mu)$. Thus, total output is:

$$\mathcal{Y}(\mu) = (1-\mu) + \mu\mathcal{A}(\mu).$$

Next, we obtain the derivative and limits of $\mathcal{Y}(\mu), \mathcal{A}(\mu)$.

Part 2. Limits. We first consider the limit as $\mu \to 0$:

$$\lim_{\mu\to 0} \mathcal{A}(\mu;\delta) = \frac{\delta}{(1-\delta)} \lim_{\mu\to 0}\left(\frac{1}{\mu}-1\right)\cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right) = \lim_{\mu\to 0} \frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu}.$$

The last term is the ratio of two variables that converge to zero. Using L'Hospital's rule:

$$\lim_{\mu\to 0} \frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu} = \frac{\delta}{(1-\delta)} \frac{\lim_{\mu\to 0}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu}\right)}{1} = \delta.$$

where I used

$$\frac{\partial \ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\partial \mu} = \frac{1-\mu}{1-\delta\mu}\left(\frac{1-\delta\mu}{1-\mu}\right)\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right) = \left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right).$$

For output, the limit is:

$$\lim_{\mu\to 0}\mathcal{Y}\left(\mu;\delta\right) = \lim_{\mu\to 0}\left(1-\mu\right)\lim_{\mu\to 0}\left(1+\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right) = 1.$$

Next, we consider the limit as $\mu \to 1$:

$$\lim_{\mu\to 1}\mathcal{A}\left(\mu;\delta\right) = \frac{\delta}{(1-\delta)}\lim_{\mu\to 1}\left(\frac{1}{\mu}-1\right)\lim_{\mu\to 1}\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This is the product of numbers that go to 0 and infinity. Using L'Hospital's rule:

$$\lim_{\mu\to 1}\mathcal{A}\left(\mu;\delta\right) = \frac{\lim_{\mu\to 1}\left(-\frac{1}{\mu^2}\right)}{\lim_{\mu\to 1}\left(\frac{1}{1-\mu}-\frac{\delta}{1-\delta\mu}\right)} = 0.$$

For output, the limit is:

$$\lim_{\mu\to 1}\mathcal{Y}\left(\mu\right) = \lim_{\mu\to 1}\left(1-\mu\right) + \lim_{\mu\to 1}\mu\lim_{\mu\to 1}\mathcal{Y}^{x}\left(\mu\right) = 0.$$

Next, we consider the limit as limit as $\delta \to 0$:

$$\lim_{\delta\to 0}\mathcal{A}\left(\mu;\delta\right) = \left(\frac{1}{\mu}-1\right)\lim_{\delta\to 0}\frac{\delta}{(1-\delta)}\cdot\lim_{\delta\to 0}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) = 0.$$

For output,

$$\lim_{\delta\to 0}\mathcal{Y}\left(\mu\right) = \left(1-\mu\right) + \mu\lim_{\delta\to 0}\mathcal{Y}^{x}\mathcal{A}\left(\mu;\delta\right) = \left(1-\mu\right).$$

Finally, we consider the limit as $\delta \to 1$:

$$\lim_{\delta\to 1}\mathcal{A}\left(\mu;\delta\right) = \left(\frac{1}{\mu}-1\right)\lim_{\delta\to 1}\delta\cdot\lim_{\delta\to 1}\frac{1}{(1-\delta)}\cdot\lim_{\delta\to 1}\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This derivative is of the ratio of two numbers that go to zero. Using L'Hospital's rule:

$$\lim_{\delta\to1}\mathcal{A}(\mu;\delta)=\left(\frac{1}{\mu}-1\right)\frac{\lim_{\delta\to1}\frac{1-\mu}{1-\delta\mu}\left(\frac{-\mu}{1-\mu}\right)}{-1}=\left(\frac{1-\mu}{\mu}\right)\left(\frac{\mu}{1-\mu}\right)=1.$$

For output,

$$\lim_{\delta\to1}\mathcal{Y}(\mu)=(1-\mu)+\mu\lim_{\delta\to1}\mathcal{Y}^{x}\mathcal{A}(\mu;\delta)=1.$$

This concludes the derivation of the limits of interest.

Part 3. Monotonicity. Next, we investigate the derivatives of $\mathcal{A}$ and $\mathcal{Y}$. We can write:

$$\mathcal{A}(\mu;\delta)=\left(\frac{1}{\mu}-1\right)\cdot\frac{\delta}{(1-\delta)}\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

Thus,

$$\mathcal{A}_{\mu}=\frac{\delta}{(1-\delta)}\left(\left(-\frac{1}{\mu^{2}}\right)\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right)+\left(\frac{1}{\mu}-1\right)\left(\frac{-\delta}{1-\delta\mu}+\frac{1}{1-\mu}\right)\right).$$

Factoring out $-1/\mu^2$:

$$\begin{aligned}\mathcal{A}_{\mu}&=-\frac{\delta}{(1-\delta)}\frac{1}{\mu^{2}}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right)-\mu(1-\mu)\left(\frac{-\delta}{1-\delta\mu}+\frac{1}{1-\mu}\right)\right)\\&=-\frac{\delta}{(1-\delta)}\frac{1}{\mu^{2}}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right)-\mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right).\end{aligned}$$

To show that the derivative is negative, we need to show that the term in the parenthesis is positive. Or likewise that

$$\ln(1-\delta\mu)-\left(\frac{\mu-\delta\mu}{1-\delta\mu}\right)>\ln(1-\mu).$$

A concave function $f(x)\equiv\log(1-x)$ satisfies:

$$f(x)+f'(x)|y-x|>f(y).$$

Let $x = \delta\mu$ and $y = \mu$. Because $\{\delta, \mu\} < 1$,

$$|y - x| = \mu - \delta\mu.$$

Thus, we have:

$$\ln(1 - \delta\mu) + \underbrace{\left(-\frac{1}{1 - \delta\mu}\right)(\mu - \delta\mu)}_{f'(x)|y-x|} > \ln(1 - \mu).$$

which proves the desired inequality. Hence, $\mathcal{A}_\mu < 0$ for any $\mu > 0$. At $\mu = 0$, the derivative is zero.

We also obtain that:

$$\mathcal{Y}_\mu = -1 + \mu\mathcal{A}_\mu + \mathcal{A} = -\underbrace{(1 - \mathcal{A})}_{>0} - \frac{\delta}{(1 - \delta)}\frac{1}{\mu}\left(\underbrace{\left(\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\right)}_{>0}\right) \text{ for } \mu > 0.$$

The derivative is also zero at $\mu = 0$.

Part 4. Concavity. Next we perform the convexity analysis. $\mathcal{A}_{\mu\mu}$ is

$$\frac{\delta}{(1 - \delta)}\left[2\frac{1}{\mu^3}\left(\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\right) - \frac{1}{\mu^2}\left(\frac{1}{1 - \mu} - \frac{\delta}{1 - \delta\mu} - \frac{1 - \delta}{1 - \delta\mu} - \mu\delta\frac{1 - \delta}{(1 - \delta\mu)^2}\right)\right]$$

After some algebraic manipulations, the second term in parenthesis simplifies to:

$$\frac{1}{1 - \mu} - \frac{\delta}{1 - \delta\mu} - \frac{1 - \delta}{1 - \delta\mu} - \mu\delta\frac{1 - \delta}{(1 - \delta\mu)^2} = \mu\frac{(\delta - 1)^2}{(1 - \mu)(1 - \delta\mu)^2}.$$

Thus:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1 - \delta)}\frac{1}{\mu^3}\left[2\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - 2\mu\left(\frac{1 - \delta}{1 - \delta\mu}\right) - \frac{\mu^2(1 - \delta)^2}{(1 - \mu)(1 - \delta\mu)^2}\right].$$

We can add the second and third terms to obtain:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1 - \delta)}\frac{1}{\mu^3}\left[\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right)^2 - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\left(\frac{2 - \mu - 3\delta\mu + 2\delta\mu^2}{(1 - \mu)(1 - \delta\mu)}\right)\right].$$

This function is strictly concave if:

$$\ln(1-\delta\mu)^2 < \ln(1-\mu)^2 + \frac{1}{(1-\mu)^2}\left(\frac{\mu(1-\mu)(1-\delta)}{(1-\delta\mu)^2}\left(2-\mu+2\delta\mu^2-3\delta\mu\right)\right) \quad (19)$$

Let $F(x) = \ln(x)$. Set $x_0 = (1-\mu)^2$ and $x_1 = (1-\delta\mu)^2$ so that $x_0 - x_1 = -\mu(1-\delta)(2-\mu(1+\delta))$. By strict concavity of $\ln(x)$ we have

$$\ln(1-\delta\mu)^2 < \ln(1-\mu)^2 - \frac{1}{(1-\mu)^2}\mu(1-\delta)(2-\mu(1+\delta))$$

so to prove that $\mathcal{A}_{\mu\mu}$ is strictly negative, we need to prove that the right hand side of the expression above is smaller than the condition needed for concavity, condition (19),

$$-\frac{1}{(1-\mu)^2}\mu(1-\delta)(2-\mu(1+\delta)) \leq \frac{1}{(1-\mu)^2}\left(\frac{\mu(1-\mu)(1-\delta)}{(1-\delta\mu)^2}\left(2-\mu+2\delta\mu^2-3\delta\mu\right)\right).$$

Cancelling common terms and rearranging, this condition is equivalent to:

$$-(1-\delta\mu)^2(2-\mu-\mu\delta) \leq (1-\mu)\left(2-\mu+2\delta\mu^2-3\delta\mu\right). \quad (20)$$

The term on the left is negative—and strictly negative for $\delta, \mu < 1$.[34] Hence, the inequality above is verified as long as:

$$2 \geq \alpha(\mu,\delta) \equiv \mu - 2\delta\mu^2 + 3\delta\mu.$$

Hence, as long as

$$2 \geq \alpha^* = \max_{\{\mu,\delta\}\in[0,1]^2} \alpha(\mu,\delta)$$

the condition for concavity holds for all $\{\mu,\delta\} \in [0,1]^2$. We study this max function. Fix any μ. Since

$$\delta\left(3\mu - 2\mu^2\right) \geq 0,$$

[34] This follows immediately because μ and δ are fractions.

the maximal value of α is achieved when $\delta = 1$. Hence, the objective is

$$\alpha^* = \max_{\{\mu,\delta\}\in[0,1]^2} \alpha(\mu,\delta) = \max_{\{\mu\}\in[0,1]} \alpha(\mu,1) = \max_{\{\mu\}\in[0,1]} 4\mu - 2\mu^2.$$

Maximizing the last expression over μ yields $\mu = 1$ as a solution and $\alpha^* = 2$ as the value. Hence, the inequality holds and guarantees (20). This suffices to prove concavity.

Next, we verify the concavity of total output. We have that $\mathcal{Y}_{\mu\mu}$ is

$$\begin{aligned}
=&\mathcal{A}_\mu + \mu\mathcal{A}_{\mu\mu}\\
=&-\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) + \frac{\delta}{(1-\delta)}\frac{2}{\mu^2}\mu\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)\\
&-\frac{\mu}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right)\\
=&\mathcal{A}_{\mu\mu} - \frac{\delta}{(1-\delta)}\frac{\mu}{\mu^3}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)\\
<&0.
\end{aligned}$$

A.2 Related Results Used Elsewhere

Part 5. Inverse productivity. Now, we study the inverse of productivity. Let

$$q(\mu;\delta) = \mathcal{A}^{-1}(\mu;\delta).$$

Clearly, the function has the limits:

$$\lim_{\mu\to 0} q(\mu;\delta) = \delta^{-1} \text{ and } \lim_{\mu\to 1} q(\mu;\delta) = \infty \text{ and } \lim_{\delta\to 0} q(\mu;\delta) = \infty \text{ and } \lim_{\delta\to 1} q(\mu;\delta) = 1.$$

We also have that:

$$q_\mu = -\frac{\mathcal{A}_\mu}{\mathcal{A}^2} > 0.$$

We use the limit of the derivative of this function:

$$q_\mu(\mu) = \frac{\frac{\delta}{(1-\delta)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)}{\left(\frac{(1-\mu)}{\mu}\cdot\frac{\delta}{(1-\delta)}\cdot\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right)^2}$$

Next, we check the convexity of function:

$$q_{\mu\mu} = -\frac{\mathcal{A}_{\mu\mu}}{\mathcal{A}^2} + \frac{\mathcal{A}_{\mu}}{\mathcal{A}^3} > 0.$$

Hence, q is convex in μ.

Part 6. Elasticity of $\mathcal{A}$. A useful object in later derivations is the elasticity of $\mathcal{A}$. Consider the derivative of

$$\mathcal{A}(\mu)\,\mu.$$

We have:

$$\mathcal{A}'(\mu)\,\mu + \mathcal{A}(\mu) = \mathcal{A}(\mu)\left[1+\epsilon_{\mu}^{\mathcal{A}}\right].$$

Recall that,

$$\mathcal{A}(\mu)\,\mu = (1-\mu)\left(\frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right).$$

Hence,

$$\begin{aligned}\mathcal{A}'(\mu)\,\mu + \mathcal{A}(\mu) &= \frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + (1-\mu)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right)\right)\\ &= \frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1-\delta}{1-\delta\mu}\right)\right).\end{aligned}$$

Hence, we obtain that:

$$\begin{aligned}\left[1+\epsilon_{\mu}^{\mathcal{A}}\right] &= \frac{\frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1-\delta}{1-\delta\mu}\right)\right)}{\mathcal{A}(\mu)}\\ &= \frac{\mu}{1-\mu}\left(\frac{\frac{1-\delta}{1-\delta\mu}}{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)} - 1\right).\end{aligned}$$

We are interested in the sign of $1+\epsilon_{\mu}^{\mathcal{A}}$ and its limits. We know $\epsilon_{\mu}^{\mathcal{A}} < 0$. Thus, the sign of $1+\epsilon_{\mu}^{\mathcal{A}}$ is the sign of:

$$\left(\frac{1-\delta}{1-\delta\mu}\right) - \ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

The limits of the function that governs the sign are:

$$\lim_{\mu\to 0}\left(\frac{1-\delta}{1-\delta\mu}\right)-\ln\left(\frac{1-\delta\mu}{1-\mu}\right)=(1-\delta)>0,$$

and

$$\lim_{\mu\to 1}=\left(\frac{1-\delta}{1-\delta\mu}\right)-\ln\left(\frac{1-\delta\mu}{1-\mu}\right)=-\infty.$$

Since the function is continuous in $\mu,$ the sign is ambiguous.

Next, to establish monotonicity, notice that

$$\begin{aligned}\delta\frac{1-\delta}{(1-\delta\mu)^2}-\frac{1-\delta}{(1-\delta\mu)(1-\mu)} &= (1-\delta)\left(\frac{-\delta-1+\delta(1-\delta\mu)}{(1-\delta\mu)^2(1-\mu)}\right)\\ &= -(1-\delta)\left(\frac{1+\delta^2\mu}{(1-\delta\mu)^2(1-\mu)}\right)\\ &< 0.\end{aligned}$$

Hence, there's a unique crossing point where the function $1+\epsilon_\mu^{\mathcal{A}}$ is negative.

Finally, I compute relevant limits. Firs, we compute:

$$\lim_{\mu\to 0}1+\epsilon_\mu^{\mathcal{A}}=(1-\delta)\lim_{\mu\to 0}\frac{\mu}{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}=\frac{1-\delta}{\lim_{\mu\to 0}\frac{1-\mu}{1-\delta\mu}\frac{1-\delta\mu}{1-\mu}\left(\frac{-\delta}{1-\delta\mu}-\frac{-1}{1-\mu}\right)}=1.$$

where the first equality are the surviving terms after taking limits, the second equality follows from L'Hospital's rule. For the limit $\mu\to 1$, by L'Hospital's rule:

$$\lim_{\mu\to 1}\frac{1}{1-\mu}\left(\frac{1}{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}\right)=\lim_{\mu\to 1}\frac{-\frac{1}{1-\mu^2}}{\frac{1-\delta}{(1-\delta\mu)(1-\mu)}}=\lim_{\mu\to 1}-\frac{(1-\delta\mu)}{(1-\mu)(1-\delta)}.$$

Adding terms the missing term:

$$\begin{aligned}\lim_{\mu\to 1}\left[1+\epsilon_\mu^{\mathcal{A}}\right] &= \lim_{\mu\to 1}-\frac{(1-\delta\mu)}{(1-\mu)(1-\delta)}-\frac{\mu}{1-\mu}\\ &= \lim_{\mu\to 1}\frac{-1+2\delta\mu-\mu}{(1-\mu)(1-\delta)}\\ &= -2\lim_{\mu\to 1}\frac{1}{(1-\mu)}\\ &= -\infty.\end{aligned}$$

We finally compute the elasticity. From

$$\frac{\mathcal{A}'(\mu)}{\mathcal{A}(\mu)}\mu < 0,$$

we know that the elasticity $\epsilon_{\mu}^{\mathcal{A}}$ is decreasing. Hence, $1+\epsilon_{\mu}^{\mathcal{A}}$ starts at zero and falls continuously until diverging at $\mu \to 1$.

B. Proofs of Section 3

B.1 Preliminary Observations

We begin with a set of identities that are convenient to proof the main results.

Average Price: Definitions and Identities. Recall that the average price of a worker as it's expenditures relative to total consumption: $Q \equiv E^w/C^w$. I show below that the saver's expenditure is $E^s = (1-\beta)\,B$. By the income expenditure identity, we have that:

$$E^w = 1 - (1-\beta)\,B. \tag{21}$$

Moreover, it is show that total consumption by the worker, given its optimal expenditures, is given by:

$$C^w = \frac{E^w - S^w}{q} + S^w = \frac{E^w}{q} - \left(\frac{1}{q} - 1\right) S^w.$$

where

$$S^w = \min\left\{\max\left\{0, \tilde{B} - B\right\}, 1 - (1-\beta)\,B\right\},$$

$X^w = (E^w - S^w)/q$.

Thus, we have that:

$$Q = \left(\frac{1}{q} - \left(\frac{1}{q} - 1\right)\frac{S^w}{E^w}\right)^{-1} = \left(\frac{1}{q}\left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}\right)^{-1}.$$

This implies that Q is the harmonic mean of the price of goods, weighted by the expenditure share.

Also, we have that:

$$Q = \frac{E^w - S^w}{C^w} + \frac{S}{C^w} = q\frac{X^w}{C^w} + \frac{S}{C^w}.$$

Hence, Q is as well, the average price, weighted by the consumption shares.

Marginal Expenditure and Borrowing Prices. Next, I define two prices that enter in marginal decisions. Namely, the prices at which the worker is trading-off consumption between periods due to an increase in borrowing. First, I define the *marginal expenditure price*:

$$\tilde{q}_t^e \equiv q_t \mathbb{I}_{[B_t \geq B^\star(\tilde{B}_t)]} + \left(1 - \mathbb{I}_{[B_t \geq B^\star(\tilde{B}_t)]}\right).$$

The interpretation is that it is the price of the good that would be consumed on the margin with an additional unit of expenditures (increasing B_{t+1}/R_{t+1} in the budget constraint). If $B \geq B^\star$, it must be that all spot consumption possible, given the financial resources, is spent[35]. If that is the case, any marginal expenditure is spent on chained goods. Thus, price per marginal unit of consumption is q_t. Otherwise (if $B < B^\star$), only spot goods are consumed and the relevant price to evaluate marginal decisions is 1.

Next, I define the *marginal borrowing price*:

$$\tilde{q}_{t+1}^b \equiv q_{t+1} \mathbb{I}_{[\tilde{B}_{t+1} \leq B_{t+1}]} + \left(1 - \mathbb{I}_{[\tilde{B}_{t+1} \leq B_{t+1}]}\right).$$

If a marginal unit of borrowing is taken today, this translates to a marginal reduction in expenditures in the future. Any marginal income expenditures in the future will first be spent in spot goods, unless the agent is so constrained ($\tilde{B} \leq B$) that spot consumption is not possible in which case marginal consumption is based on chained goods.

Analysis of the Marginal Expenditure Price. Next, we describe the behavior of $\tilde{q}_t^e$ in equilibrium. We have that $\tilde{q}_t^e = \tilde{q}^e\left(B_t, \tilde{B}_t\right)$ where:

$$\tilde{q}^e\left(B, \tilde{B}\right) \equiv q\left(\mu\left(B, \tilde{B}\right)\right) \mathbb{I}_{[B \geq B^\star(\tilde{B})]} + \left(1 - \mathbb{I}_{[B \geq B^\star(\tilde{B})]}\right).$$

In this expression, I am using the fact that:

$$\mu\left(B, \tilde{B}\right) = 1 - (1-\beta) B - \min\left\{\max\left\{0, \tilde{B} - B\right\}, 1 - (1-\beta) B\right\}.$$

[35]From log utility we know that at $B = B^\star$ the worker will not want to consume an additional unit but nonetheless we define the price he would pay.

Observe that

$$\mu\left(B,\tilde{B}\right)=\begin{cases}0 & B<B^{\star}\left(\tilde{B}\right),\\ 1+\beta B-\tilde{B} & B\in\left[B^{\star}\left(\tilde{B}\right),\tilde{B}\right],\\ 1-(1-\beta)B & B>\tilde{B}.\end{cases}$$

The function $\mu\left(B,\tilde{B}\right)$ is continuous, starts at zero and increases up to $B=\tilde{B}$, starting from that point, the function is decreasing. The intuition is that as debt increases in the middle region, any reduction in wealth reduces spot expenditures, but in net, gives an increase in chained expenditures by market clearing. That is, the reduction in spot expenditures is one for one with the reduction in total expenditures. Since q_t is monotone in μ_t, the function $\tilde{q}^e\left(B,\tilde{B}\right)$ must follow the same pattern in the interior of middle region and in the last one. Also notice that when $B<B^{\star}\left(\tilde{B}\right)$, $q^e=1$. Next, we have that since μ is continuous, μ goes to 0 from above as $B\downarrow B^{\star}\left(\tilde{B}\right)$ so we can write,

$$\lim_{B\downarrow B^{\star}\left(\tilde{B}\right)}\tilde{q}^e\left(B,\tilde{B}\right)=\lim_{\mu\downarrow 0}q(\mu)\cdot 1=\delta^{-1}$$

where I used the fact that $\lim_{\mu\downarrow 0}\mathcal{A}(\mu)=\delta$. Hence, the function $\tilde{q}^e$ is discontinuous at $B=B^{\star}$ because $\lim_{B\uparrow B^{\star}\left(\tilde{B}\right)}\tilde{q}^e=1\neq\delta^{-1}$, the function is also not monotonic. Then[36]

$$\tilde{q}^e\left(B,\tilde{B}\right)=\begin{cases}1 & B<B^{\star}\left(\tilde{B}\right)\\ q & B\in\left[B^{\star}\left(\tilde{B}\right),\tilde{B}\right]\\ q & B>\tilde{B}.\end{cases}$$

Next, we are interested in the behavior of $\frac{Q}{q^e}$, for reasons that become clear in the main text. We have that for $B<B^{\star}\left(\tilde{B}\right)$, since $q=1$, it must be that $\frac{Q}{q^e}=1$. For $B\geq\tilde{B}$ also $q^e=Q=q$. Therefore, $\frac{Q}{q^e}=1$. In the middle range of values, we have that:

$$\frac{Q}{q^e}=\frac{1}{q}\cdot\frac{q}{\left(1-\frac{S^w}{E^w}\right)+q\frac{S^w}{E^w}}=\frac{1}{\left(1-\frac{S^w}{E^w}\right)+q\frac{S^w}{E^w}}.$$

[36]In fact, the value of q^e at $B=B^{\star}$ will not be used since the function is discontinuous at that point and optimality conditions will be characterized via the right and left limit. However, I choose to define it according to the intuition presented lines above.

Thus, we have the following formula:

$$\frac{Q}{q^e} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right) \\ \frac{1}{1-\frac{\min\left\{\max\left\{0,\tilde{B}-B\right\},1-(1-\beta)B\right\}}{1-(1-\beta)B}+q\left(\mu\left(B,\tilde{B}\right)\right)\frac{\min\left\{\max\left\{0,\tilde{B}-B\right\},1-(1-\beta)B\right\}}{1-(1-\beta)B}} & B \in \left[B^\star\left(\tilde{B}\right),\tilde{B}\right] \\ 1 & B > \tilde{B}. \end{cases}$$

Since at $B = \tilde{B}$ we have $\frac{S^w}{E^w} = 0$, the function is continuous at that point. However,

$$B \downarrow B^\star\left(\tilde{B}\right) \implies \frac{S^w}{E^w} \downarrow 1, \text{ and } q \downarrow \delta^{-1} > 1,$$

at that point. Thus, $\frac{Q}{q^e} = \delta$ at $B = B^\star\left(\tilde{B}\right)$ (and at its right limit) and $Q/q^e = 1$ at the left limit of this point. Namely, the function Q/q^e is discontinuous at $B^\star$.

Analysis of the Marginal Borrowing Price. Next, we investigate the behavior of the marginal borrowing price. Recall that this is the price of consumption at which the worker trades-off future consumption when he borrows marginally. Let B_{t+1} be the debt level the worker chooses today for next period and $\tilde{B}_{t+1}$ the next period's SBL. We have that $\tilde{q}^b_{t+1} = \tilde{q}^b\left(B_{t+1},\tilde{B}_{t+1}\right)$ where:

$$\tilde{q}^b\left(B,\tilde{B}\right) \equiv q\left(\mu\left(B,\tilde{B}\right)\right)\mathbb{I}_{[B\geq\tilde{B}]} + \left(1-\mathbb{I}_{[B\geq\tilde{B}]}\right).$$

Then,

$$\tilde{q}^b\left(B,\tilde{B}\right) = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right), \\ 1 & B \in \left[B^\star\left(\tilde{B}\right),\tilde{B}\right], \\ q & B > \tilde{B}. \end{cases}$$

We have observed that for $B = B^\star\left(\tilde{B}\right)$, all consumption is spot and thus $q = 1$. However, at $B = \tilde{B}$, the function features a discontinuity since:

$$\lim_{B\downarrow\tilde{B}} \tilde{q}^b\left(B,\tilde{B}\right) = q\left(\mu\left(\tilde{B},\tilde{B}\right)\right) = q\left(1-(1-\beta)\tilde{B}\right) > 1.$$

Next we investigate the behavior of $\frac{Q}{q^b}$.

For any $B < B^\star\left(\tilde{B}\right)$, both q^b and Q must equal 1, thus, $\frac{Q}{q^b} = 1$. Then, we have that for $B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]$, it must the case that $\frac{Q}{q^b} = Q$ because $q^b = 1$. Finally, when $B > \tilde{B}$, $\tilde{q}^b\left(B, \tilde{B}\right) = Q = q$. Thus, we have:

$$\frac{Q}{\tilde{q}^b} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right), \\ Q & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right], \\ 1 & B > \tilde{B}, \end{cases}$$

where

$$Q = \left(\frac{1}{q}\left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}\right)^{-1}$$

and $\lim_{B \downarrow B^\star\left(\tilde{B}\right)} Q = 1$.

Average Price Elasticity. Next, I derive the elasticity of the average price with respect to total debt, in equilibrium—i.e., after replacing $E^w = 1 - (1-\beta)B$. This elasticity is critical for the Ramsey policy analysis.

We have that the average price is the harmonic mean:

$$Q = \frac{1}{\frac{1}{q}\left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}}.$$

where both the price q, E^w, and S^w are functions of B.

Thus, we have that:

$$\frac{\partial Q}{\partial B} = -Q \cdot \frac{\left(1 - \frac{S^w}{E^w}\right)\frac{\partial}{\partial B}\left[\frac{1}{q}\right] - \left(\frac{1}{q} - 1\right)\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]}{\left(\frac{1}{q}\left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}\right)}.$$

The numerator has two additional derivatives. The first one is:

$$\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]=\begin{cases}0 & B<B^\star\left(\tilde{B}\right)\\ \left[\frac{\tilde{B}-B}{1-(1-\beta)B}\right]\left(\frac{(1-\beta)}{1-(1-\beta)B}-\frac{1}{\tilde{B}-B}\right) & B\in[B^\star\left(\tilde{B}\right),\tilde{B})\\ 0 & B\geq\tilde{B}\end{cases}$$

where the term in the intermediate region follows from:

$$\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]=\frac{S^w}{E^w}\left(\frac{1}{S^w}\frac{\partial}{\partial B}\left[S^w\right]-\frac{1}{E^w}\frac{\partial}{\partial B}\left[E^w\right]\right),$$

but then for $B\in[B^\star\left(\tilde{B}\right),\tilde{B})$

$$\frac{\partial}{\partial B}\left[S^w\right]=-1,$$

and

$$\frac{\partial}{\partial B}\left[E^w\right]=-\left(1-\beta\right).$$

Hence:

$$\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]=\left[\frac{\tilde{B}-B}{1-\left(1-\beta\right)B}\right]\left(\frac{\left(1-\beta\right)}{1-\left(1-\beta\right)B}-\frac{1}{\tilde{B}-B}\right)<0.$$

We evaluate the limits of this function. Clearly, at

$$\left.\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]\right|_{B\uparrow\tilde{B}}=-\frac{1}{1-\left(1-\beta\right)\tilde{B}}<-1,$$

and at

$$\begin{aligned}\left.\frac{\partial}{\partial B}\left[\frac{S^w}{E^w}\right]\right|_{B\downarrow B^\star} &= \frac{\left(1-\beta\right)\left(\tilde{B}-B^\star\left(\tilde{B}\right)\right)}{\left(1-\left(1-\beta\right)B^\star\left(\tilde{B}\right)\right)^2}-\frac{1}{\left(1-\left(1-\beta\right)B^\star\left(\tilde{B}\right)\right)}\\ &= \frac{\left(1-\beta\right)\left(\tilde{B}-B^\star\left(\tilde{B}\right)\right)-\left(1-\left(1-\beta\right)B^\star\left(\tilde{B}\right)\right)}{\left(1-\left(1-\beta\right)B^\star\left(\tilde{B}\right)\right)^2}\\ &= \frac{\left(1-\beta\right)\tilde{B}-1}{\left(1-\left(1-\beta\right)B^\star\left(\tilde{B}\right)\right)^2}\\ &= -1.\end{aligned}$$

The last steps follows from the definition of $B^{\star}$. Hence, this derivative is discontinuous.

For the second derivative of interest, recall that:

$$\frac{\partial}{\partial B}\left[q\right]=\frac{\partial}{\partial \mu}\left[q\right]\cdot\frac{\partial \mu}{\partial B},$$

and

$$\frac{\partial \mu}{\partial B}=\frac{\partial}{\partial B}\left[E^{w}\right]-\frac{\partial}{\partial B}\left[S^{w}\right]$$

since $\mu=qX=E^{w}-S^{w}$. Then, we have that:

$$\frac{\partial}{\partial B}\left[q\right]=q_{\mu}\mu\frac{\frac{\partial}{\partial B}\left[E^{w}\right]-\frac{\partial}{\partial B}\left[S^{w}\right]}{E^{w}}.$$

We can express this as:

$$\frac{\partial}{\partial B}\left[q^{-1}\right]=\frac{1}{q}\epsilon_{\mu}^{q}\frac{\frac{\partial}{\partial B}\left[E^{w}\right]-\frac{\partial}{\partial B}\left[S^{w}\right]}{E^{w}-S^{w}}.$$

Hence:

$$\frac{\partial Q}{\partial B}\frac{1}{Q}=-Q\cdot\left\{\left(1-\frac{S^{w}}{E^{w}}\right)\frac{1}{q}\epsilon_{\mu}^{q}\frac{\frac{\partial}{\partial B}\left[E^{w}\right]-\frac{\partial}{\partial B}\left[S^{w}\right]}{E^{w}-S^{w}}-\left(\frac{1}{q}-1\right)\frac{S^{w}}{E^{w}}\left[\frac{\frac{\partial}{\partial B}\left[S^{w}\right]}{S^{w}}-\frac{\frac{\partial}{\partial B}\left[E^{w}\right]}{E^{w}}\right]\right\}.$$

B.2 Proof of Proposition 4 (Worker's Euler equation)

Recall also the relation between the $\tilde{B}_t$ and $B^{\star}_{t+1}$:

$$B^{\star}_{t+1}=R_{t+1}\left(\tilde{B}_t-1\right).$$

The following Lemma is used to reduce the set of cases we have to deal with. Let $\tilde{B}_t$ be an increasing sequence and $\beta R_{t+1}\leq 1\;\forall t$. Then, $B^{\star}_{t+1}\geq\tilde{B}_{t+1}$. Assume by contradiction that $B^{\star}_{t+1}\geq\tilde{B}_{t+1}$. Substituting the expression for $B^{\star}_{t+1}$, we have that

$$R_{t+1}\left(\tilde{B}_t-1\right)=B^{\star}_{t+1}\geq\tilde{B}_{t+1}\geq\tilde{B}_t.$$

Hence,

$$(1-1/R_{t+1})\,\tilde{B}_t \geq 1.$$

If indeed $\beta \leq R_{t+1}^{-1}$, the condition above implies that:

$$(1-\beta)\,\tilde{B}_t \geq 1.$$

However, this last inequality implies that:

$$\tilde{B}_t \geq \frac{1}{1-\beta} = \bar{B}.$$

This is a contradiction. I now derive the worker's Euler equation, a necessary but not sufficient condition for optimality. Recall that we can write the worker's total expenditures as a function of B_{t+1}:

$$E_t^w = 1 - B_t + \frac{B_{t+1}}{R_{t+1}}.$$

Given her total expenditures, spot expenditures are:

$$S^w\left(B_t, \tilde{B}_t, B_{t+1}\right) = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}}\right\}.$$

and her chained expenditures are:

$$q_t X_t^w = E_t^w - S^w\left(B_t, \tilde{B}_t, B_{t+1}\right).$$

Adding both types of expenditures dividing by the price, the worker's consumption is:

$$\begin{aligned} C_t &= \frac{1 - B_t + \frac{B_{t+1}}{R_{t+1}} - S^w\left(B_t, \tilde{B}_t\right)}{q_t} + S^w\left(B_t, \tilde{B}_t, B_{t+1}\right) \\ &= \frac{1 - B_t + \frac{B_{t+1}}{R_{t+1}}}{q_t} + \left(1 - \frac{1}{q_t}\right)\min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}}\right\}. \end{aligned}$$

Now consider a sequence $\{B_{t+1}\}_{t\geq 0}$. We obtain that the worker's problem can be written entirely in terms of the worker's debt level, without reference to his expenditures:

$$\sum_{t\geq 0}\beta^t\log\left(C_t\right)=\sum_{t\geq 0}\beta^t\log\left(\frac{1-B_t+\frac{B_{t+1}}{R_{t+1}}}{q_t}+\left(1-\frac{1}{q_t}\right)\underbrace{\min\left\{\max\left\{\tilde{B}_t-B_t,0\right\},1-B_t+\frac{B_{t+1}}{R_{t+1}}\right\}}_{S^w\left(B_t,\tilde{B}_t,B_{t+1}\right)}\right).$$

There are two kinks in the term $S^w\left(B_t,\tilde{B}_t,B_{t+1}\right)$. These kinks occur at the threshold points given in Lemma 1, the points $\left\{\tilde{B}_t,B^\star_{t+1}\right\}$. Since the control variable in this problem is B_{t+1}, we have to consider the kinks. The first kink, $\tilde{B}_{t+1}$, corresponds to the value of B_{t+1} to the left of that value, consumption in time $t+1$ features zero spot consumption. The second kink, $B^\star_{t+1}$, corresponds to the level of debt B_{t+1} from which to its right, there is some chained consumption in time t .

Consider two consecutive periods in the worker's optimal sequence:

$$\log\left(C_t\right)+\beta\log\left(C_{t+1}\right)$$

Define the function

$$\begin{gathered}\Upsilon_t\left(B_{t+1};B^o_t,B^o_{t+2},\tilde{B}_t,\tilde{B}_{t+1},R_{t+1},q_t\right)\equiv\\ \log\left(\frac{1-B^o_t+\frac{B_{t+1}}{R_{t+1}}}{q_t}+\left(1-\frac{1}{q_t}\right)\min\left\{\max\left\{\tilde{B}_t-B^o_t,0\right\},1-B^o_t+\frac{B_{t+1}}{R_{t+1}}\right\}\right)+\\ \beta\log\left(\frac{1-B_{t+1}+\frac{B^o_{t+2}}{R_{t+2}}}{q_{t+1}}+\left(1-\frac{1}{q_{t+1}}\right)\min\left\{\max\left\{\tilde{B}_{t+1}-B_{t+1},0\right\},1-B_{t+1}+\frac{B^o_{t+2}}{R_{t+2}}\right\}\right).\end{gathered}$$

The function Υ_t represents the value of utility at t and $t+1$, considering the optimal choices $\left\{B^o_t,B^o_{t+2}\right\}$, for an arbitrary level of debt B_{t+1}. An optimal solution must satisfy:

$$\log\left(C_t\right)+\beta\log\left(C_{t+1}\right)=\max_{B_{t+1}}\Upsilon_t\left(B_{t+1};B^o_t,B^o_{t+2},\tilde{B}_t,\tilde{B}_{t+1},R_{t+1},q_t\right).$$

Thus, we use a perturbation argument, with respect to B_{t+1}, to derive a generalized Euler equation. In all points $B_{t+1}\in\left(0,\overline{B}\right)$ other than the threshold points, the objective is continuous, locally concave and differentiable in B_{t+1}. The kinks are in fact points of no differentiability—because of a discontinuity of the derivative of Υ_t with respect to

B_{t+1}.

Let's consider the differentiability points first and then deal with the kinks. The objective of the first terms is increasing in B_{t+1}. The objective of the second term is decreasing. Thus, in the intervals determined by the kinks, marginal benefits and costs of increasing B_{t+1} cross at most at a single point. At the kinks, the derivatives feature discontinuities, hence, multiple critical points may arise. I present the analysis of the critical points.

I break the analysis into each of the following cases.

I. Let $B_t^o \geq \tilde{B}_t$.

I.a $B_{t+1} < \tilde{B}_{t+1}$, the derivative is:

$$\Upsilon_t'\left(B_{t+1}\right) = \frac{1}{C_t}\frac{1}{q_t R_{t+1}} - \beta\frac{1}{C_{t+1}},$$

regardless of whether $B_{t+1} > B_{t+1}^{\star}$.

I.b $B_{t+1} > \tilde{B}_{t+1}$, the derivative of Υ_t with respect to B_{t+1} is:

$$\Upsilon_t'\left(B_{t+1}\right) = \frac{1}{C_t}\frac{1}{q_t R_{t+1}} - \beta\frac{1}{C_{t+1}}\frac{1}{q_{t+1}}$$

I.a-I.b. Combining both case, observe that at $\tilde{B}_{t+1}$ the following strict inequality holds

$$\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon_t'\left(B'\right) < \lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon_t'\left(B'\right) \tag{22}$$

From the right of $\tilde{B}_{t+1}$, the value of forgone consumption at $t+1$ given an increase in B_{t+1} at t, is lower due to the higher price of consumption at $t+1$. As a result, if limit form the right of $\Upsilon_t' \leq 0$ there is no critical point to the right of $\tilde{B}_{t+1}$ and if $\Upsilon_t' \geq 0$ from the left then there is no critical point to the left of $\tilde{B}_{t+1}$.

In summary, when $B_t^o \geq \tilde{B}_t$, and there is only consumption of chained goods:

- If $B' \in \left(0, \tilde{B}_{t+1}\right)$ is a local maximum, then $\Upsilon_t' = 0$. If furthermore $\lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon_t'\left(B'\right) \leq 0$ then only one possible value of B_{t+1} satisfies the Euler equation.

- If $B' \in \left(\tilde{B}_{t+1}, \overline{B}\right)$ is a local maximum, then $\Upsilon_t' = 0$. If furthermore $\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon_t'(B') \geq 0$ then only one possible value of B_{t+1} satisfies the Euler equation.

- Since the objective is concave on both intervals, there's only one possible solution to B'.

- $B = \tilde{B}_{t+1}$ is not a solution since this requires $\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon_t'(B') \geq 0 \geq \lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon_t'(B')$, which contradicts (22).

Hence, we have shown the following Lemma. When $B_t \geq \tilde{B}_t$, B_{t+1} satisfies the Euler equation with equality:

$$\Upsilon_t'(B_{t+1}) = 0.$$

and $B_{t+t} = \tilde{B}_{t+1}$ is not a solution.

II. Let $B_t^o < \tilde{B}_t$ in a solution to the worker's problem. We know by the Lemma above that $B_{t+1}^\star < \tilde{B}_{t+1}$. Hence, we ave the following cases:

II.a $B_{t+1} < B_{t+1}^\star < \tilde{B}_{t+1}$, there is only spot consumption at t and some spot consumption at $t+1$, hence the derivative of the objective is:

$$\Upsilon'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}.$$

II.b For $B_{t+1} \in \left(B_{t+1}^\star, \tilde{B}_{t+1}\right)$ there is some chained consumption at t and some spot consumption at $t+1$, hence the derivative of the objective is:

$$\Upsilon_t'(B_{t+1}) = \frac{1}{q_t}\frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}.$$

II.c. For $B_{t+1}^\star < \tilde{B}_{t+1} < B_{t+1}$ there is some chained consumption at t and no spot consumption at $t+1$, hence the derivative of the objective is:

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{q_t}\frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}\frac{1}{q_{t+1}}.$$

II.a.-II.c At then not differentiable points the following strict inequalities hold

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t(B') > \lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t(B')$$

and

$$\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon'_t(B') < \lim_{B' \downarrow \tilde{B}_{t+1}} \Upsilon'_t(B').$$

where the inequalities follow the same arguments as in case I.

- If $B \in \left(0, B^\star\left(R_{t+1}, \tilde{B}_t\right)\right)$ is a solution then $\Upsilon'_t(B) = 0$. If furthermore $\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t(B') \leq 0$, then only possible value of B_{t+1} satisfies the Euler equation.
- If $B = B^\star\left(R_{t+1}, \tilde{B}_t\right)$ is a solution then

  $$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t(B') \geq 0 \geq 0 \lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t(B')$$

  where at most one inequality is strict.
- If $B \in \left(B^\star\left(R_{t+1}, \tilde{B}_t\right), \tilde{B}_{t+1}\right)$ is a solution then $\Upsilon'_t(B) = 0$. If furthermore $\lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t(B') \leq 0$, then only possible value of B_{t+1} satisfies the Euler equation.
- $\tilde{B}_{t+1}$ is not a solution as it yields a contradiction.
- If $B \in \left(\tilde{B}_{t+1}, \overline{B}\right)$ is a solution then $\Upsilon'_t(B) = 0$. If furthermore $\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon'_t(B')$, then only possible value of B_{t+1} satisfies the Euler equation.
- Again, by concavity, we have a unique path in each case.

We shown the following When $B_t < \tilde{B}_t$, B_{t+1} either satisfies the Euler equation with equality:

$$\Upsilon'_t(B_{t+1}) = 0.$$

or $B' = B^\star_{t+1}$ if:

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t(B') \geq 0 \geq \lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t(B').$$

Necessity. Using the definition of Q_t, we have:

$$\frac{1}{C_t}\cdot\frac{\Pi_{t+1}\left(B_{t+1}\right)}{R_{t+1}}-\beta\frac{1}{C_{t+1}}=\frac{Q_t}{E_t}\cdot\frac{\Pi_{t+1}\left(B_{t+1}\right)}{R_{t+1}}-\beta\frac{Q_{t+1}}{E_{t+1}}.$$

The cases above, are captured by the term $\Pi_{t+1}\left(B_{t+1}\right)$. Hence, the equation above yields the sign of the derivative of the change in B_{t+1}. Moreover,

$$\Upsilon_t'\left(B'\right)\geq 0\rightarrow\frac{Q_t}{E_t}\cdot\frac{1}{R_{t+1}}-\beta\frac{Q_t}{C_{t+1}}\geq 0\rightarrow\frac{E_{t+1}}{E_t}\cdot\frac{Q_t}{Q_{t+1}}\geq\beta\frac{R_{t+1}}{\Pi_{t+1}\left(B_{t+1}\right)}.$$

and viceversa. Thus:

$$\lim_{B'\uparrow\tilde{B}_{t+1}}\Upsilon_t'\left(B'\right)\geq 0\rightarrow\frac{E_{t+1}}{E_t}\geq\beta R_{t+1}$$

and

$$\lim_{B'\downarrow B^{\star}_{t+1}}\Upsilon_t'\left(B'\right)\leq 0\rightarrow q_t\beta R_{t+1}\geq\frac{E_{t+1}}{E_t}$$

Collecting all the cases above and using the definition of Π_t, we arrive at the I show in the text.

Proposition 11. (Workers's First-Order Condition): *Fix a sequence* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$ *such that* $\tilde{B}_t$ *is an increasing and* $\beta R_{t+1}\leq 1$. *Then, any solution* $\{B_{t+1}\}_{t\geq 0}$ *to the worker's problem satisfies the following generalized Euler equation:*

$$\frac{E^w_{t+1}}{E^w_t}\frac{Q_t}{Q_{t+1}}=\beta\frac{R_{t+1}}{\Pi_{t+1}\left(B_{t+1}\right)}\quad\textit{if}\quad B_{t+1}\neq B^{\star}_{t+1}\tag{23}$$

and

$$q_t\beta R_{t+1}\geq\frac{E_{t+1}}{E_t}\geq\beta R_{t+1}\quad\textit{if}\quad B_{t+1}=B^{\star}_{t+1}.$$

B.3 Euler Equation in the Stationary Problem

Consider the stationary problem of the worker where $\tilde{B}_t=\tilde{B}$.

B.4 Proof of Proposition 6

Let's recall the main equation of the Proposition

$$\underbrace{\frac{B}{1-(1-\beta)B}\cdot Q\left(B,\tilde{B}\right)}_{\equiv\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right)}=\underbrace{\frac{B'}{1-(1-\beta)B'}\frac{Q\left(B',\tilde{B}'\right)}{\Pi\left(B';B,\tilde{B},\tilde{B}'\right)}}_{\equiv\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)}.$$

First, I will show that for a subset of $B>\tilde{B}'$ there are two roots for the equation $\mathcal{E}=\mathcal{E}'$. The roots satisfy $B_1'<\tilde{B}'<B_2'$ and $B_2'=B$ thus $R=\beta^{-1}$ and $q=q'$. Then the proof finishes noticing that because of the hysteresis result, for each $B\in\left(\tilde{B}',B^h\right)$ the larger root B_2' cannot be an individual optimum.

Let's fix $\tilde{B},\tilde{B}'$. Define $B^*>\tilde{B}$ such that

$$\mathcal{E}'\left(B^*;\tilde{B},\tilde{B},\tilde{B}'\right)=\lim_{B'\uparrow\tilde{B}'}\mathcal{E}'\left(B';\tilde{B},\tilde{B},\tilde{B}'\right)$$

$$\frac{B^*}{1-(1-\beta)B^*}\cdot q\left(1-(1-\beta)B^*\right)=\lim_{B'\uparrow\tilde{B}'}\frac{B'}{1-(1-\beta)B'}\cdot\frac{Q\left(B',\tilde{B}'\right)}{\Pi\left(B';B^*,\tilde{B},\tilde{B}'\right)}$$

$$B^*=\frac{\tilde{B}'}{\frac{1-(1-\beta)\tilde{B}'}{q\left(1-(1-\beta)\tilde{B}'\right)}+(1-\beta)\tilde{B}'}=\frac{\tilde{B}'}{C^w\left(\tilde{B}',\tilde{B}'\right)+C^s\left(\tilde{B}'\right)}>\tilde{B}'$$

This result comes because

$$\lim_{B'\uparrow\tilde{B}'}\Pi\left(B';B^*,\tilde{B},\tilde{B}'\right)=\left(q\left(1-(1-\beta)B^*\right)\right)^{-1}\quad\lim_{B'\uparrow\tilde{B}'}Q\left(B',\tilde{B}'\right)=q\left(1-(1-\beta)\tilde{B}'\right).$$

The graphical interpration of B^* is found in the left panel of Figure 4. The inequality holds because $q>1$. Furthermore, this satisfies $B^*>\tilde{B}$ as required since the SBL sequence is weakly increasing.[37] Now I will argue that for $B\in\left(\tilde{B}',B^*\right)$ the equation $\mathcal{E}=\mathcal{E}'$ has two roots: one above $\tilde{B}'$ and one below. The one above is trivial since $B=B'>\tilde{B}'\geq\tilde{B}$ satisfies it. For the one below I use a simple continuity and monotonicity

[37] This is important because the relevant points to study are those for which the all chained equilibrium is possible. These are the cases where $B^*>\tilde{B}'\geq\tilde{B}$.

argument. I will show that there is a unique $B' < \tilde{B}'$ such that

$$\mathcal{E}'\left(B';\tilde{B},\tilde{B},\tilde{B}'\right) = \lim_{B'\downarrow\tilde{B}'}\mathcal{E}'\left(B';\tilde{B},\tilde{B},\tilde{B}'\right)$$

$$\frac{B'}{1-(1-\beta)B'}\cdot Q\left(B',\tilde{B}'\right) = \frac{B'}{1-(1-\beta)B'}\cdot\frac{q\left(1-(1-\beta)\tilde{B}'\right)}{\lim_{B'\uparrow\tilde{B}'}\Pi\left(B';\tilde{B},\tilde{B},\tilde{B}'\right)}$$

$$\frac{B'}{1-(1-\beta)B'}\cdot Q\left(B',\tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\tilde{B}'}\cdot\frac{q\left(1-(1-\beta)\tilde{B}'\right)}{\frac{q\left(1-(1-\beta)\tilde{B}'\right)}{q\left(1-(1-\beta)\tilde{B}\right)}}$$

$$\frac{B'}{1-(1-\beta)B'}\cdot Q\left(B',\tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\tilde{B}'}\cdot q\left(1-(1-\beta)\tilde{B}\right) > 0$$

Lets call this B', $\underline{B}$ and think of the interval $\left(\tilde{B}',B^*\right)$. The interpretation of $\underline{B}$ is that it is the small root for the debt level "$\tilde{B}'+\varepsilon$". Meaning, if $\tilde{B}'$ is the small root for B^*(the end of the interval) then $\underline{B}$ is the small root for the start of the interval. Now, notice that the RHS is a constant with respect to $\tilde{B}'$. To establish existence of $\underline{B}$ further notice that the LHS tends to $\lim_{B'\uparrow\tilde{B}'}\mathcal{E}'\left(B';B^*,\tilde{B},\tilde{B}'\right)$ as $B'\uparrow\tilde{B}'$ (this was shown in the previous step) and this is larger than the RHS just by comparing magnitudes ($\tilde{B}'\geq\tilde{B}$ and $q\left(1-(1-\beta)B^*\right)>1$). Then notice that as $B'\downarrow 0$ the LHS goes to zero which is lower than the RHS. By continuity of the LHS we can apply the intermediate value theorem for existence. Uniqueness is granted since the LHS is increasing in B'. The fraction is clearly increasing in B', the average price too because as $B'\uparrow\tilde{B}'$ the share of chained expenditure increases and also its price does so. This statement also uses the fact that Q is the weighted harmonic mean of prices with expenditure weights. Since the LHS is continuous and increasing it maps the interval $\left(\underline{B},\tilde{B}'\right)$ onto

$$\left(\lim_{B'\downarrow\tilde{B}'}\mathcal{E}'\left(B';\tilde{B},\tilde{B},\tilde{B}'\right),\lim_{B'\uparrow B^*}\mathcal{E}'\left(B';\tilde{B},\tilde{B},\tilde{B}'\right)\right).$$

So (since $\mathcal{E}'$ is increasing and continuous) it covers all the image of $\left(\tilde{B}',B^*\right)$. This proves that for $B\in\left(\tilde{B}',B^*\right)$ there exist two roots that solve $\mathcal{E}=\mathcal{E}'$. One is $B_2'=B>\tilde{B}'$ and the other is a $B_1'\in\left(\underline{B},\tilde{B}'\right)$.

To make affairs clearer, suppose that $B^*<B^h$ then for all $B\in\left(\tilde{B}',B^*\right)$ the larger

root $B' = B$ is not an equilibrium and there is a region (B^{*}, B^{h}) that does not have a symmetric competitive equilibrium because the only root that solves equation (15) provides allocations and prices that are not an equilibrium. Now suppose that $B^{*} > B^{h}$, then for $B \in \left(\tilde{B}', B^{h}\right)$ the larger root $B' = B$ is not an equilibrium and for $B \in (B^{h}, \overline{B})$.

So summarizing both cases. For $B_0 < B^{h}\left(\tilde{B}_0\right)$, if a (symmetric competitive) equilibrium exists then it is given by the smaller root of equation 15.

B.4.1 Proof of Corollary 1

At $B_t \leq B^{\star}\left(\beta^{-1}, \tilde{B}_{ss}\right)$ if $R = \beta^{-1}$ then marginal and average prices are equal to 1. Then it is evident that $B_{t+1} = B_t$ solves equation 15 and the expenditure is $1 - (1-\beta) B_t \leq \tilde{B}_{ss} - B_t$ by assumption. As a consequence, the steady state is non-disrupted.

B.4.2 Proof of Corollary 2

That $B_{t+1} < B_t$ if $B_t \in \left(\tilde{B}, B^{*}\left(\tilde{B}\right)\right)$ is immediate from the result proved in Proposition 6 since we are choosing the smaller root and the larger root is $B_{t+1} = B_t$. For $B_t > B^{\star}$ it is enough to show that $\beta R_{t+1} < 1$. This was done in step 3 of the proof of Proposition 6.

B.5 Proof of Proposition 7

TBA

B.6 Proof of Proposition 3

- Start from Euler equation. Argue that for $B \leq \tilde{B}$ value of staying put is worse.
- Start from Stationary case...At this solution, the prices are $R = \beta^{-1}, q = q\left(1 - (1-\beta) B\right).$
- Given prices $R = \beta^{-1},\ q = q\left(1 - (1-\beta) B\right),$ the Euler equation is a necessary condition for individually optimal sequences. I show that staying put violates the

Euler equation for $B \in \left(B^{\star}, \tilde{B}\right)$. The equation is

$$\frac{C_{t+1}\left(B'\right)}{C_t\left(B'\right)} = \beta R \frac{q^E\left(B'\right)}{q^B\left(B'\right)}$$

and I assume that $B'' = B$, the following to next period I also stay put. The elements are

$$\begin{aligned}
C_t\left(B'\right) &= \tilde{B} - B + \frac{1}{q}\left(1 + \beta B' - \tilde{B}\right) \\
C_{t+1}\left(B'\right) &= \max\left\{\tilde{B} - B', 0\right\} + \frac{1}{q}\left[1 + \beta B - B' - \max\left\{\tilde{B} - B', 0\right\}\right] \\
q^E\left(B'\right) &= \mathbb{I}\left[B_{t+1} < B^{\star}\right] \cdot 1 + \mathbb{I}\left[B_{t+1} \geq B^{\star}\right] \cdot q \\
q^B\left(B'\right) &= \mathbb{I}\left[B_{t+1} < \tilde{B}\right] \cdot 1 + \mathbb{I}\left[B_{t+1} \geq \tilde{B}\right] \cdot q
\end{aligned}$$

and at $B' = B$ we have

$$\begin{aligned}
C_t\left(B'\right) &= \tilde{B} - B + \frac{1}{q}\left(1 + \beta B - \tilde{B}\right) \\
C_{t+1}\left(B'\right) &= \tilde{B} - B + \frac{1}{q}\left(1 + \beta B - \tilde{B}\right) \\
q^E\left(B'\right) &= q \\
q^B\left(B'\right) &= 1
\end{aligned}$$

so the Euler equation does not hold since

$$1 \neq q.$$

- I want to find the sequence that arrives to $B^{\star}$. I will use the final condition $B' = B^{\star}$ and the Euler equation backwards in time. As a first step I show that the set of values B that have optimal policy $B' = B^{\star}$ is the interval

$$\mathcal{B}_0 \equiv \left[B^{\star}, B^{\star} + \left(1 - \frac{1}{q}\right)\left(\tilde{B} - B^{\star}\right)\right]$$

where $B^{\star} = \beta^{-1}\left(\tilde{B} - 1\right)$. To show this I evaluate the sub-differential counterpart of the Euler equation at point $B' = B^\star$. The derivative from the left must be nonnegative

$$\lim_{B'\uparrow B^\star} \frac{C'(B')}{C(B')} \geq \frac{1}{1}$$

and the marginal prices are 1 since $B' < B^\star < \tilde{B}$. We know that the optimal policy once in $B^\star$ is $B^\star$ so $B'' = B^\star$ and consumptions at the limit satisfy

$$\frac{\tilde{B} - B^\star}{\tilde{B} - B} \geq 1$$

so $B \geq B^\star$ and $B^\star$ is the lower bound of the interval. The derivative from the right must be nonpositive

$$\lim_{B'\downarrow B^\star} \frac{C'(B')}{C(B')} \leq \frac{q}{1}$$

and the marginal expenditure and borrowing price are q (since $B' > B^\star$) and 1 (since $B' < \tilde{B}$), respectively. So consumptions at the limit satisfy

$$\frac{\tilde{B} - B^\star}{\tilde{B} - B} \leq q \iff B \leq B^\star + \left(1 - \frac{1}{q}\right)\left(\tilde{B} - B^\star\right).$$

- Recall that I want to characterize the optimal sequence arriving to $B^\star$. Let me define $B_{-1} \in \mathcal{B}_0$, as shown above, at this point consumption C_{-1} satisfies

$$C_{-1} = \tilde{B} - B_{-1}$$

In general, in the region with $B' \in \left(B^\star, \tilde{B}\right)$ the ratio of marginal prices is q then S

$$\frac{C_{t+1}}{C_t} = q \iff C_t = \frac{1}{q} C_{t+1}$$

so

$$C_{-t} = \frac{1}{q^{t-1}} C_{-1}.$$

and for each $B_{-1} \in \mathcal{B}_0$

$$\frac{1}{q^{t-1}}C_{-1} = C_{-t} = \underbrace{\tilde{B} - B_{-t}}_{S} + \underbrace{\frac{1}{q}\left(1 + \beta B_{-t+1} - \tilde{B}\right)}_{X}.$$

$$s.t.\; C_{-1} = \; \tilde{B} - B_{-1}, \quad B_{-1} \in \mathcal{B}_0$$

This defines a difference equation for B_{-t} with initial condition B_{-1} and non-homogeneous term $q^{-t+1}C_{-1}$, by using the Euler equation backwards. So it defines a whole sequence $\{B_{-t}\}$. Writing the equation in terms of B_{-t} and using $\mathcal{A} = q^{-1}$

$$B_{-t} = \tilde{B} - \mathcal{A}^{t-1}C_{-1} + \mathcal{A}\left(1 + \beta B_{-t+1} - \tilde{B}\right),$$

The coefficient of B_{-t+1} drives the dynamic and if $\mathcal{A}\beta < 1$, then as $-t \to -\infty$ the sequence converges

$$B_{-ss} \equiv \frac{1}{1 - \mathcal{A}\beta}\left(\tilde{B}\left(1 - \mathcal{A}\right) + 1\right) > \tilde{B} \iff \frac{q}{1-\beta} > \tilde{B}$$

which is true since $\tilde{B} < \overline{B}$. This equation also implies that $B_{-t} > B_{-t+1}$ e.g. $B_0 < B_{-1} < B_{-2} < \cdots < B_{-t}$. To show this, I take the case $t = 1$ and $t \geq 2$. First, explicit B_{-t} from the consumption expression and substitute $t = 2$,

$$\begin{aligned} B_{-t} &= \tilde{B} - \mathcal{A}^{t-1}C_{-1} + \mathcal{A}\left(1 + \beta B_{-t+1} - \tilde{B}\right), \\ B_{-2} - B_{-1} &= \tilde{B} - \mathcal{A}C_{-1} + \mathcal{A}\left(1 + \beta B_{-1} - \tilde{B}\right) - B_{-1} \\ B_{-2} - B_{-1} &= \left(1 - \mathcal{A}\right)\left(\tilde{B} - B_{-1}\right) + \mathcal{A}\beta\left(B_{-1} - B^{\star}\right) > 0 \end{aligned}$$

where the third equality is found rearranging the second and using the expression for C_{-1} and the relative position of B_{-1} with respect to $B^{\star}$.which is positive since

$C_0 < 1$ and $\tilde{B} > 1$. For the case $t \geq 3$ first assume that $B_{-t+1} < \tilde{B}$

$$\begin{aligned}
B_{-t} - B_{-t+1} &= \tilde{B} - \mathcal{A}^{t-1}C_{-1} + \mathcal{A}\left(1-\tilde{B}\right) - \left(1-\mathcal{A}\beta\right)B_{-t+1} \\
&> \tilde{B} - \mathcal{A}^{t-1}C_{-1} + \mathcal{A} - \mathcal{A}\tilde{B} - \left(1-\mathcal{A}\beta\right)\tilde{B} \\
&= -\mathcal{A}^{t-1}C_{-1} + \mathcal{A} + \left(1-\mathcal{A}-\left(1-\mathcal{A}\beta\right)\right)\tilde{B} \\
&= -\mathcal{A}^{t-1}C_{-1} + \mathcal{A} - \mathcal{A}\left(1-\beta\right)\tilde{B} \\
&= -\mathcal{A}^{t-1}C_{-1} + \mathcal{A} - \mathcal{A}\left(1-\beta\right)\tilde{B} \\
&> -\mathcal{A}^{t-1}C_{-1}^{\max} + \mathcal{A} - \mathcal{A}\left(1-\beta\right)\tilde{B} \\
&= -\mathcal{A}^{t-1}\left(\tilde{B} - B^{\star}\right) + \mathcal{A} - \mathcal{A}\left(1-\beta\right)\tilde{B} \\
&= -\mathcal{A}^{t-1}\beta^{-1}\left(1-\left(1-\beta\right)\tilde{B}\right) + \mathcal{A}\left(1-\left(1-\beta\right)\tilde{B}\right) \\
&= \left(-\mathcal{A}^{t-1}\beta^{-1} + \mathcal{A}\right)\left(1-\left(1-\beta\right)\tilde{B}\right) \\
&= \mathcal{A}\left(-\mathcal{A}^{t-2}\beta^{-1} + 1\right)\left(1-\left(1-\beta\right)\tilde{B}\right)
\end{aligned}$$

and this is positive if for $t \geq 3$

$$\begin{aligned}
-\mathcal{A}^{t-2}\beta^{-1} + 1 &> 0 \\
\mathcal{A}^{t-2} &< \beta
\end{aligned}$$

a sufficient condition is that $\delta < \beta$. So the sequence is strictly increasing and converging to a value larger than $\tilde{B}$, this implies that it reaches $\tilde{B}$ in finite time.

- Now, start from the debt level $B > \tilde{B}$ such that its debt policy is the debt level $B' < \tilde{B}$ that is the last point lower than $\tilde{B}$ in the sequence above. At this point the marginal expenditure price is q (since consumption is all chained) and the marginal borrowing price is 1 (since $B' < \tilde{B}$) This means that I don't need a new Euler equation so the debt level that we should target for deleverage is the first value of the sequence above that is larger than $\tilde{B}$, let's call it B^{target}

- Starting from that point the optimal deleverage to $B^{\star}$ is the one characterized exactly by the sequence presented. To specify the value call τ the known number of periods to arrive from B^{target} to $B^{\star}$ then for this individual problem (at prices

$R = \beta^{-1}$ and $q = q(1-(1-\beta)B)$ for some $B \in \left(\tilde{B}, B^h\right)$)

$$
\begin{aligned}
V\left(B^{target}\right) &= \log\left(C_{-\tau}\right) + \beta\log\left(C_{-\tau+1}\right)\cdots + \beta^{\tau-1}\log\left(C_{-1}\right) + \beta^{\tau}V\left(B^{\star}\right),\\
V\left(B^{target}\right) &= \sum_{t=1}^{\tau}\beta^{\tau-t}\log\left(C_{-t}\right) + \beta^{\tau}V\left(B^{\star}\right),\\
V\left(B^{target}\right) &= \sum_{t=1}^{\tau}\beta^{\tau-t}\log\left(\frac{C_0}{q^t}\right) + \beta^{\tau}V\left(B^{\star}\right),\\
V\left(B^{target}\right) &= \sum_{t=1}^{\tau}\beta^{\tau-t}\left[\log\left(\tilde{B}-B^{\star}\right) - t\log q\right] + \beta^{\tau}V\left(B^{\star}\right),\\
V\left(B^{target}\right) &= \sum_{t=1}^{\tau}\beta^{\tau-t}\left[\log\left(\tilde{B}-B^{\star}\right) - t\log q\right] + \beta^{\tau}\frac{\log\left(\tilde{B}-B^{\star}\right)}{1-\beta}.
\end{aligned}
$$

where $\tilde{B} - B^{\star} = 1-(1-\beta)B^{\star}$ by definition of $B^{\star}$.

- you can define a recursion in sets.
- Think more about the condition for B^h. B^h is such that $\underline{V}\left(B^h\right) \geq \max V\left(B^{target}\right)$ for some $B^{target} \in \left(\tilde{B}, \overline{B}\right)$and some finite τ. With monotonicity it will be characterize B^h nicely.
- With this I can identify exactly B^h.
- Index the sets by the upper and lower bound.
- Adapt the value equation for Targets that are n periods above $\tilde{B}$.
- Computationally we will get B^h. Analytically will we?
- For each $\tau \in \mathbb{N}$, $\mathcal{B}_0 = \left[B^{\star}, B^{\star} + \left(1-\frac{1}{q}\right)\left(\tilde{B}-B^{\star}\right)\right]$.

Start of hysteresis proof

In this proposition we want to show there is an interval $\left(\tilde{B}, B^h\right)$ where the solution to the Euler equation $B' = B$ (staying put) does not yield an individually optimal plan. We propose a deleveraging strategy to $B^{\star}$ that is optimal.

Because of consumption smoothing the worker will save and consume a constant amount each period to arrive to debt level $B^{\star}$. Then the value of the deleveraging with this strategy is

$$V^d(B) = \max_{E\in(0,1-(1-\beta)B)} \left\{ \log(E/q)\left(\frac{1-\beta^{\tau(E)}}{1-\beta}\right) + \beta^{\tau(E)} V(B^\star) \right\}. \tag{24}$$

where $\tau(E)$ is the number of periods to arrive at $B^\star$ and E is constrained in such set because if $E \geq 1-(1-\beta)B$ then the worker is not saving to reduce its debt.

The first term is the present value of $\tau(E)$ periods spending E and the second term is the continuation value after arriving at $B^\star$.

The problem is cast in a continuous set for expenditure, $E \in (0, 1-(1-\beta)B)$. However, the worker can only choose natural numbers for the periods τ, this corresponds to a discrete choice set for E. Nonetheless, the first order condition (FOC) gives us a criteria to identify local maxima in the original discrete problem. The objective is defined in an open interval and is differentiable; so we can use critical points (points of derivative zero) and compare the closest integer value at the right and the left to find the maximum of the discrete problem. I now focus in the continuous choice set problem.

For each B, the FOC with respect to E is

$$\frac{1}{E(B)}\frac{1-\beta^{\tau(E(B))}}{1-\beta} = -\log\beta\cdot\beta^{\tau(E(B))}\tau'(E(B))\left[V(B^\star) - \frac{\log(E(B)/q)}{1-\beta}\right]. \tag{25}$$

where $\tau(E(B))$ is obtained by a first-order difference equation resulting in[38]

$$\tau(E(B)) = -\frac{1}{\log\beta}\log\left(\frac{\frac{1-E(B)}{1-\beta}-B^\star}{\frac{1-E(B)}{1-\beta}-B}\right), \text{ and}$$
$$\tau'(E(B)) = \frac{1}{\log\beta}\frac{1}{1-\beta}\left(\frac{1}{\frac{1-E}{1-\beta}-B^\star} - \frac{1}{\frac{1-E}{1-\beta}-B}\right) > 0.$$

The roadmap of the proof *is as follows: For an interval* $\left(\tilde{B}, B^h\right)$ *I will show the existence and uniqueness of the solution,* $E(B)$*, to the above problem and that this is a*

[38] The difference equation is the classical solution to exponential growth

$$y_t - y_{ss} = a^t(x_t - x_{ss}) \iff \frac{1-E(B)}{1-\beta} - \tilde{B} = \beta^{-\tau(E(B))}\left(\frac{1-E(B)}{1-\beta} - B\right).$$

profitable deviation from the all chained consumption equilibrium. For existence of the critical point, I use the intermediate value theorem.

For uniqueness, I show that the objective is locally concave at any critical point and that the critical value must be strictly above the chained value $\underline{V}(B)$ (the value as $E \uparrow 1-(1-\beta)B$).

The existence condition for $E(B)$ will characterize the interval $\left(\tilde{B}, B^h\right)$ for which there is no hysteresis. The existence of the debt level where hysteresis starts, B^h, is given by the intermediate value theorem. The uniqueness of this debt level will be obtained by the strict concavity of the all chained value function $\underline{V}$. Finally, the local strict concavity of the problem shows that all values to the right of B^h are in hysteresis.

Let's start with existence of $E(B)$. For $E \downarrow 0$, $\tau(E)$ and $\tau'(E)$ are finite then the limit of the derivative in equation 25 takes the form of [39]

$$\lim_{x \downarrow 0} a\frac{1}{x} - b\log(x) + c$$

where $a, b \in \mathbb{R}_{++}$ and $c \in \mathbb{R}$. By a result of algebra of limits we have that if

$$\lim_{x \downarrow 0} \left(a\frac{1}{x} + b\log(x) - c\right) x = a > 0 \tag{26}$$

then the limit exists and is $+\infty$. This is the case since the limit $\lim_{x\downarrow 0} x\log(x) = 0$ implies the desired result. Thus for values of E close to zero the derivative is positive (the LHS is larger).

Then as $E \uparrow 1-(1-\beta)B$ by substitution of the expressions above

$$\lim_{E \uparrow 1-(1-\beta)B} -\log\beta \cdot \beta^{\tau(E(B))}\tau'(E(B)) = \frac{1}{1-\beta}\frac{1}{B-B^{\star}}.$$

Substituting in equation (25) then the condition for the existence of a critical point is

[39] It is

$$\frac{1}{E/q}\underbrace{\frac{1}{q}\cdot\frac{1-\beta^{\tau(0)}}{1-\beta}}_{=a} + \underbrace{-\frac{\log\beta}{1-\beta}\cdot\beta^{\tau(0)}\tau'(0)}_{=b}\log(E/q) + \underbrace{\log\beta\cdot\beta^{\tau(0)}\tau'(0)\,V\left(\tilde{B}\right)}_{=c}.$$

that $\forall B \in \left(\tilde{B}, B^h\right)$,[40]

$$\frac{1}{1-(1-\beta)B} < \frac{V(B^\star) - \underline{V}(B)}{B - B^\star}. \tag{28}$$

To establish uniqueness of the critical point I require a result about the local concavity of the problem. Let's denote the objective in equation 24 by U such that $V^d = \max_E U$. The second derivative of U has 5 terms

$$\begin{aligned}
&= -\frac{1}{E(B)}\frac{1}{E(B)}\frac{1-\beta^{\tau(E(B))}}{1-\beta} \\
&\quad + \frac{1}{1-\beta}\frac{1}{E(B)}(-1)\log\beta \cdot \beta^{\tau(E(B))}\tau'(E(B)) \\
&\quad + \log\beta \cdot \tau'(E(B)) \cdot \log\beta \cdot \beta^{\tau(E(B))}\tau'(E(B))\left[V(B^\star) - \frac{\log(E(B)/q)}{1-\beta}\right] \\
&\quad + \log\beta \cdot \beta^{\tau(E(B))}\tau''(E(B))\left[V(B^\star) - \frac{\log(E(B)/q)}{1-\beta}\right] \\
&\quad + \log\beta \cdot \beta^{\tau(E(B))}\tau'(E(B))(-1)\frac{1}{1-\beta}\frac{1}{E(B)}
\end{aligned}$$

and since

$$\begin{aligned}
\tau''(E) &= \frac{1}{\log\beta}\frac{1}{(1-\beta)^2}\left(\frac{1}{\left(\frac{1-E}{1-\beta} - B^\star\right)^2} - \frac{1}{\left(\frac{1-E}{1-\beta} - B\right)^2}\right) > 0. \\
&= \tau'(E)\frac{1}{(1-\beta)}\left(\frac{1}{\left(\frac{1-E}{1-\beta} - B^\star\right)} + \frac{1}{\left(\frac{1-E}{1-\beta} - B\right)}\right) > 0
\end{aligned}$$

[40] Note that it is enough to look for sets that are open connected intervals i.e. $\left(\tilde{B}, B^h\right)$. This is true because for strictly increasing continuous functions such as $g(B)$ (below) the pre-image (the set of B's) of an open interval $(0, V(B^\star))$ is an open interval $B \in \left(\tilde{B}, B^h\right)$

$$0 < g(B) \equiv \frac{1}{1-(1-\beta)B}(B - B^\star) + \underline{V}(B) < V(B^\star). \tag{27}$$

g is strictly increasing (has positive derivative) for all B because $\underline{V}_B(B) = -\frac{1}{1-(1-\beta)B}$

we have that the second derivative of U is equal to

$$
\begin{aligned}
&-\frac{1}{E(B)}\cdot\frac{1}{E(B)}\left(\frac{1-\beta^{\tau(E(B))}}{1-\beta}\right)\\
&+2\frac{1}{E(B)}\frac{1-\beta^{\tau(E(B))}}{1-\beta}(-1)\log\beta\cdot\frac{\beta^{\tau(E(B))}}{1-\beta^{\tau(E(B))}}\tau'(E(B))\\
&+(-1)\log\beta\cdot\tau'(E(B))\cdot(-1)\log\beta\cdot\beta^{\tau(E(B))}\tau'(E(B))\left[V(B^\star)-\frac{\log(E(B)/q)}{1-\beta}\right]\\
&+(-1)\frac{1}{(1-\beta)}\left(\frac{1}{\left(\frac{1-E}{1-\beta}-\tilde{B}\right)}+\frac{1}{\left(\frac{1-E}{1-\beta}-B\right)}\right)\cdot(-1)\log\beta\cdot\beta^{\tau(E(B))}\tau'(E)\left[V(B^\star)-\ldots\right]
\end{aligned}
$$

factorizing out $\frac{1}{E(B)}\left(\frac{1-\beta^{\tau(E(B))}}{1-\beta}\right)$ or, equivalently, the RHS of the FOC I obtain an expression with the same sign

$$
\begin{aligned}
&-\frac{1}{E(B)}\\
&-2\log\beta\cdot\frac{\beta^{\tau(E(B))}}{1-\beta^{\tau(E(B))}}\tau'(E(B))\\
&-\log\beta\cdot\tau'(E(B))\\
&-\frac{1}{(1-\beta)}\left(\frac{1}{\left(\frac{1-E}{1-\beta}-B^\star\right)}+\frac{1}{\left(\frac{1-E}{1-\beta}-B\right)}\right).
\end{aligned}
$$

then let me call

$$\frac{1}{f(X)}=\frac{1-E}{1-\beta}-X,$$

so the second derivative of U is has the same sign of

$$
\begin{aligned}
&=-\frac{1}{E(B)}-\log\beta\cdot\tau'(E(B))\frac{1+\beta^{\tau(E(B))}}{1-\beta^{\tau(E(B))}}-\frac{1}{(1-\beta)}\left(f(B^\star)+f(B)\right),\\
&=-\frac{1}{E(B)}-\frac{1}{1-\beta}\left(f(B^\star)-f(B)\right)\frac{f(B)+f(B^\star)}{f(B)-f(B^\star)}-\frac{1}{(1-\beta)}\left(f(B^\star)+f(B)\right),\\
&=-\frac{1}{E(B)}<0
\end{aligned}
$$

where the second equality is obtained using the expressions for τ' and τ and the function $f(\cdot)$. The last inequality comes from $E \in (0, 1-(1-\beta)B)$. Now, I will show uniqueness and that the value at this point must be above the disrupted value.

Let's denote the critical point of U as E_0 and suppose that $U(E_0)$ is below $\underline{V}(B)$. Since E_0 is a strict local maximum then the function is decreasing to the right of E_0 (negative derivative) but it must start increasing again (positive derivative) at some point because as $E \uparrow 1-(1-\beta)B$ then $U(E) \uparrow \underline{V}(B)$. This change of sign in the derivative requires that there is another critical point that is a local minimum to the right of E_0. This is a contradiction since we just proved that all critical points are strict local maximums. Thus I conclude that the value at E_0 is larger than $\underline{V}(B)$ and uniqueness follows from the same argument. Then $E(B)$ is a global maximum.

Let's prove that in fact such B^h exists. Again I use the intermediate value theorem, so I prove that there are B satisfying (28) and B satisfying it with the reversed sign. We will show that $\tilde{B}$ satisfies the condition with strict inequality ($<$), so the condition with $\leq$ is true for values above but close to $\tilde{B}$. The condition on $\tilde{B}$ is

$$\frac{1}{1-(1-\beta)\tilde{B}} < \frac{V(B^\star) - \underline{V}(\tilde{B})}{\tilde{B} - B^\star}.$$

Rewriting this inequality we will show

$$\frac{1}{1-(1-\beta)\tilde{B}}\left(\tilde{B} - B^\star\right) < V(B^\star) - \underline{V}(\tilde{B}).$$

Using the expression for $B^\star = \beta^{-1}\left(\tilde{B} - 1\right)$, the LHS is equal to β^{-1}. The RHS is

$$\begin{aligned} & V(B^\star) - \underline{V}(B^\star) + \underline{V}(B^\star) - \underline{V}(\tilde{B}), \\ &= \frac{\log q}{1-\beta} + \frac{1}{1-\beta}\log\left(\frac{1-(1-\beta)B^\star}{1-(1-\beta)\tilde{B}}\right), \\ &= \frac{\log q}{1-\beta} + \frac{1}{1-\beta}\log\left(\beta^{-1}\right) = \frac{\log\left(\beta^{-1}q\right)}{1-\beta}, \end{aligned}$$

where the second equality uses that $V(B^\star)$ is undisrupted. Since β is close to 1 and since $q > \delta^{-1} > 1$ we have[41]

$$\begin{aligned}
\beta \exp\left(\beta^{-1} - 1\right) &\approx 1 < q, \\
\exp\left(\beta^{-1} - 1\right) &< \beta^{-1} q, \\
\left(\beta^{-1} - 1\right) &< \log\left(\beta^{-1} q\right), \\
\beta^{-1} &< \frac{\log\left(\beta^{-1} q\right)}{1 - \beta},
\end{aligned}$$

which we wanted to show. Now take $B \uparrow \overline{B} = 1/(1-\beta)$ then since consumption goes to zero we apply the same argument as in equation (26)[42] and we have that the condition holds with the reversed sign, the LHS goes to plus infinity. This establishes existence and B^h is characterized by

$$\frac{1}{1 - (1-\beta) B^h} = \frac{V\left(\tilde{B}\right) - V^x\left(B^h\right)}{B^h - \tilde{B}}. \tag{29}$$

Now, uniqueness follows because equation (29) can be rewritten as

$$g\left(B^h\right) \equiv \underline{V}_B\left(B^h\right)\left(B^\star - B^h\right) + \underline{V}\left(B^h\right) = V\left(B^\star\right)$$

$$g'\left(B^h\right) = \underbrace{\underline{V}_{BB}\left(B^h\right)}_{<0}\underbrace{\left(B^\star - B^h\right)}_{<0} - \underline{V}_B\left(B^h\right) + \underline{V}_B\left(B^h\right)$$

and by strict concavity of $\underline{V}$, the LHS is a one-to-one mapping for B^h.

Finally, I prove that B's to the right of B^h are in hysteresis. Suppose that there is a $B > B^h$ such that $V^d(B) > \underline{V}(B)$ then I can draw two implications. First, that there is a local maximum E' (a point where the derivative changes from positive to negative) and that equation (28) holds with the reversed sign since $B > B^h$ (i.e. the derivative as $E \uparrow 1 - (1-\beta) B$ is positive). These two facts together imply that there is another (critical) point, say, $E'' > E'$ which is a local minimum since the derivative must change from negative to positive. This is a contradiction.

[41]Alternatively this could be written as a condition for the delay parameter $\beta \exp\left(\beta^{-1} - 1\right) < \delta^{-1}$.
[42]Notice that the log of the limit argument appears because $\underline{V}(B) = \log\left[\left(1 - (1-\beta) B\right)/q\right]$

C. Proofs of Section 4

C.1 Proof of Proposition 8

The strategy of the proof is to show that any solution to the Ramsey Problem satisfies the constraints in the Primal Problem (step 1), that solutions respect the optimal expenditure rules, and, finally, that any solution to the Primal Problem can be induced by a proper tax sequence $\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}_{t\geq 0}$ (step 2).

Step 1. The constraint set in Primal Problem contains constraints in Ramsey Problem. Take the household budget constraints and the government budget in the original Ramsey Problem:

$$\left(1+\tau_{t+1}^k\right)\frac{B_{t+1}}{R_{t+1}}+\left(1+\tau_t^c\right)C_t^s=B_t,\ \forall t\geq 0$$

$$B_t+\left(1+\tau_t^c\right)\left(S_t^w+q_tX_t^w\right)=\frac{B_{t+1}}{R_{t+1}}+1-\tau_{t+1}^\ell,\ \forall t\geq 0$$

and

$$\tau_{t+1}^k\frac{B_{t+1}}{R_{t+1}}+\tau_t^c\left(C_t^s+C_t^w\right)+\tau_{t+1}^\ell=0,\ \forall t\geq 0.$$

If we add the first two constraints and cancel common terms, we obtain:

$$\left(1+\tau_t^c\right)\left(S_t^w+q_tX_t^w\right)+\tau_{t+1}^k\frac{B_{t+1}}{R_{t+1}}+\left(1+\tau_t^c\right)C_t^s=1-\tau_{t+1}^\ell.$$

If we then subtract the government budget constraint from this last equation, we obtain:

$$-\left(\tau_{t+1}^k\frac{B_{t+1}}{R_{t+1}}+\tau_t^c\left(C_t^s+S_t^w+q_tX_t^w\right)+\tau_t^\ell\right)+\left(1+\tau_t^c\right)\left(S_t^w+q_tX_t^w\right)+\tau_{t+1}^k\frac{B_{t+1}}{R_{t+1}}+\left(1+\tau_t^c\right)C_t^s=1-\tau_{t+1}^\ell.$$

Cancelling terms, this condition further becomes:

$$S_t^w+q_tX_t^w+C_t^s=1. \tag{30}$$

Finally, using $q_t = \mathcal{A}(\mu_t)^{-1}$, we obtain:

$$S_t^w + \mathcal{A}(\mu_t)^{-1} X_t^w + C_t^s = 1 \text{ and } \mu_t = \mathcal{A}(\mu_t)^{-1} X_t^w.$$

This shows that any solution to the Ramsey Problem satisfies the constraints of the Primal Problem.

Step 2. The Ramsey Planner can implement the Primal Problem Solution. Next, observe that for any choice of C_t^w in the Primal Problem, the Primal Planner is better off maximizing S_t^w since $\mathcal{A}(\mu_t)^{-1} \geq 1$. Hence, it must be the case that:

$$S_t^w = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - C_t^s\right\}.$$

Then, by definition:

$$X_t^w = E_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - C_t^s\right\}/q_t.$$

In this expression, I used that $E_t^w = 1 - C_t^s$ exploiting the expenditure-income—equation (30). Since both the Ramsey and the Primal problems induce the same level of consumption for workers given a level of saver consumption, the value of both problems coincides if they can induce the same set of saver consumption paths.

In the primal problem, I use that

$$C_t^s = (1-\beta) B_t.$$

Thus, since the planner in the Primal Problem can choose the path of debt directly, it can chose saver expenditures as well. Since the constraint set in the Primal is a subset of the constraint in the Ramsey problem, the primal problem is more relaxed than the original Ramsey problem. Hence, If the Ramsey planner can achieve the same level of saver expenditures as the Primal planner, then, it must achieve the same value. The next Lemma is used to verify that claim.

Lemma 1. *Let $\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}_{t \geq 0}$ be a sequence of taxes in the Ramsey Problem. The solu-*

tion to the saver's problem is given by:

$$C_t^s = (1-\beta) B_t^o$$

where

$$B_0^o = \frac{B_0}{(1+\tau_0^c)}$$

$$B_{t+1}^o = \hat{R}_{t+k} \beta B_t^o$$

and

$$\hat{R}_{t+k} \equiv \frac{R_{t+1}}{\left(1+\tau_{t+1}^k\right)} \frac{1}{\left(1+\tau_{t+1}^c\right)/\left(1+\tau_t^c\right)}.$$

This Lemma is the solution to the saver's problem. The Lemma implies that any sequence of solutions to the Primal Problem can be reproduced by the the Ramsey Planner.

Indeed, let $\{B_t^o\}_{t\geq 0}$ be a solution to the primal problem. Then, the Ramsey planner can set

$$(1+\tau_0^c) = B_0/B_0^o$$

and set the sequence of taxes $\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}$ to satisfy,

$$\frac{B_{t+1}^o}{B_t^o} = \beta \frac{R_{t+1}}{\left(1+\tau_{t+1}^k\right)} \frac{1}{\left(1+\tau_{t+1}^c\right)/\left(1+\tau_t^c\right)},$$

given the equilibrium rate R_{t+1} induced by his solution. This equilibrium has to be found to provide an actual implementation. Recall from the previous step that once we determine the saver's consumption path, we have the worker's expenditures. Hence, we are free to treat C^w, S^w, μ. and C^w, as functions of B in the Primal and Ramsey problems (assuming that both problems produce the same path of B_t^o) as we do in the problem without taxes in the main text:

$$S^w\left(B,\tilde{B}\right) \equiv \min\left\{\max\left\{\tilde{B}-B,0\right\}, 1-(1-\beta)B\right\},$$

$$\mu\left(B,\tilde{B}\right) \equiv 1-(1-\beta)B - S^w\left(B,\tilde{B}\right),$$

$$X^w\left(B,\tilde{B}\right) \equiv \mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\left(1-(1-\beta)B - S^w\left(B,\tilde{B}\right)\right),$$

and

$$C^w\left(B,\tilde{B}\right)=X^w\left(B,\tilde{B}\right)+S^w\left(B,\tilde{B}\right).$$

Recall that the Ramsey planner must satisfies the two household Euler equations:

$$\frac{C^s_{t+1}}{C^s_t}\frac{1}{\beta}\left[\frac{1+\tau^c_{t+1}}{1+\tau^c_t}\right]\left(1+\tau^k_t\right)=R_{t+1} \tag{31}$$

and for the worker, at continuity points,

$$\frac{C^w_{t+1}}{C^w_t}\frac{1}{\beta}\left[\frac{1+\tau^c_{t+1}}{1+\tau^c_t}\right]\left[\frac{1+\left(\mathcal{A}\left(\mu_{t+1}\right)^{-1}-1\right)\mathbb{I}_{[S^w_{t+1}=0]}}{1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t>0]}}\right]=R_{t+1}, \tag{32}$$

where I express the indicators as a function of consumption since these conditions are equivalent to the ones in the main text and, likewise, I work directly with consumption. I verify below, in the solution to the Primal Planner's Problem, that the planner never chooses $B^o_t=\tilde{B}_t$ but may chose $B^o_t=B^\star_t$. Thus, the worker's Euler equation must satisfy:

$$\lim_{B^o_t\uparrow B^\star_t}\frac{C^w_{t+1}}{C^w_t}\frac{1}{\beta}\left[\frac{1+\tau^c_{t+1}}{1+\tau^c_t}\right]\left[\frac{1+\left(\mathcal{A}\left(\mu_{t+1}\right)^{-1}-1\right)\mathbb{I}_{[S^w_{t+1}=0]}}{1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t>0]}}\right]<R_{t+1},$$

and

$$\lim_{B^o_t\downarrow B^\star_t}\frac{C^w_{t+1}}{C^w_t}\frac{1}{\beta}\left[\frac{1+\tau^c_{t+1}}{1+\tau^c_t}\right]\left[\frac{1+\left(\mathcal{A}\left(\mu_{t+1}\right)^{-1}-1\right)\mathbb{I}_{[S^w_{t+1}=0]}}{1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t>0]}}\right]\geq R_{t+1}.$$

The numerator is the same in both cases, it equals 1, in the neighborhood of $B^\star_t$. If the worker's Euler equation holds with equality in the limit from above $B^\star_t$, we immediately verify that the inequality holds in the limit from below:

$$\lim_{B^o_t\downarrow B^\star_t}1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t>0]}=\delta^{-1}<0=\lim_{B^o_t\uparrow B^\star_t}1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t>0]}.$$

Hence, we are free to substitute the the strict inequality in the denominator for an equality:

$$\frac{C^w_{t+1}}{C^w_t}\frac{1}{\beta}\left[\frac{1+\tau^c_{t+1}}{1+\tau^c_t}\right]\left[\frac{1+\left(\mathcal{A}\left(\mu_{t+1}\right)^{-1}-1\right)\mathbb{I}_{[S^w_{t+1}=0]}}{1+\left(\mathcal{A}\left(\mu_t\right)^{-1}-1\right)\mathbb{I}_{[X_t\geq 0]}}\right]=R_{t+1}.$$

I use this modified Euler equation in the rest of the proof.

Substituting out $\left[\frac{1+\tau^c_t}{1+\tau^c_{t+1}}\right]\beta R_{t+1}$ from both (31) and (32), and replacing the saver's optimal expenditures, we obtain:

$$\left(1+\tau_t^k\right) = \frac{C_t^s}{C_{t+1}^s}\frac{C_{t+1}^w\left(B_{t+1},\tilde{B}_{t+1}\right)}{C_t^w\left(B_t,\tilde{B}_t\right)}\left[\frac{1+\left(\mathcal{A}\left(\mu\left(B_{t+1},\tilde{B}_{t+1}\right)\right)^{-1}-1\right)\mathbb{I}_{\left[S^w\left(B_{t+1},\tilde{B}_{t+1}\right)=0\right]}}{1+\left(\mathcal{A}\left(\mu\left(B_t,\tilde{B}_t\right)\right)^{-1}-1\right)\mathbb{I}_{\left[X_t\geq 0\right]}}\right].$$

We can treat the solution of this equation as a function mapping the sequence of solutions in the Primal Planner to the Ramsey planner:

$$\tau^k\left(B,B',\tilde{B},\tilde{B}'\right) = \frac{B}{B'}\frac{1-(1-\beta)B}{1-(1-\beta)B'}\frac{Q\left(B',\tilde{B}'\right)}{Q\left(B,\tilde{B}\right)}\left[\frac{1+\left(\mathcal{A}\left(\mu\left(B',\tilde{B}'\right)\right)^{-1}-1\right)\mathbb{I}_{\left[S^w\left(B',\tilde{B}'\right)=0\right]}}{1+\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)^{-1}-1\right)\mathbb{I}_{\left[X_t\geq 0\right]}}\right]-1. \tag{33}$$

As long as we have the sequence of debt obtained from the Primal Problem, we obtain a mapping from this solution to the sequence of capital taxes.

The equilibrium rate is deduced from (31),

$$R_{t+1} = \left[\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\right]\frac{B_{t+1}^o}{B_t^o}\frac{1}{\beta}\left(1+\tau^k\left(B_t^o,B_{t+1}^o,\tilde{B}_t,\tilde{B}_{t+1}\right)\right).$$

Hence, other than for time zero, expenditure taxes are indeterminate. Indeed, any sequence of expenditure taxes satisfies the saver's budget equation. If we substitute (31) into the saver budget constraint to obtain:

$$\frac{C_t^s}{C_{t+1}^s}\beta\left[\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\right]B_{t+1}+\left(1+\tau_t^c\right)C_t^s = B_t.$$

Using that $C_t^s = (1-\beta)B_t^o$ and that the debt induced by the Ramsey solution must satisfy $B_t = B_t^o/\left(1+\tau_t^c\right)$, the budget constraint is verified.

Since there are multiple paths for expenditure taxes and labor income taxes, the only condition need is that they jointly satisfy the government budget constraint. Replacing the results above, we obtain that any sequence $\left\{\tau_t^c, \tau_{t+1}^\ell\right\}$ that satisfies:

$$\frac{\tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}{1+\tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}\left[\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\right]\beta B_t^o+\tau_t^c\left((1-\beta)\,B_t^o+\frac{1-(1-\beta)\,B_t^o}{Q\left(B_t^o, \tilde{B}_t\right)}\right)+\tau_{t+1}^\ell=0,$$

implements the Primal Planner allocation. For an implementation where $\tau_t^c = 0, \forall t \geq 1$, Hence, we have:

$$\tau_t^\ell = -\frac{\tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}{1+\tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}\beta B_t^o,$$

also implements the solution.

C.1.1 Auxiliary Proofs

Proof of Lemma 1. To proof the result, I solve the saver's problem for an arbitrary sequence of taxes:

Problem 7. *The saver's problem with taxes is:*

$$V_t = \sum_{t\geq 0}\beta^t \log\left(C_t\right)$$

subject to: $\left(1+\tau_{t+1}^k\right)R_{t+1}^{-1}B_{t+1}+\left(1+\tau_t^c\right)C_t = B_t$, *with* B_0 *given.*

Dividing both sides of the budget constraint by $1+\tau_t^c$ and multiplying and dividing by $\left(1+\tau_{t+1}^c\right)$ in the first term, we obtain:

$$\left(1+\tau_{t+1}^k\right)\frac{B_{t+1}}{R_{t+1}\left(1+\tau_t^c\right)}\frac{\left(1+\tau_{t+1}^c\right)}{\left(1+\tau_{t+1}^c\right)}+C_t = \frac{B_t}{\left(1+\tau_t^c\right)}.$$

I introduce the following change of variable:

$$B_t^o \equiv \frac{B_t}{1+\tau_t^c}.$$

Using this change of variables, the budget constraint is modified to:

$$B_{t+1}^o = \hat{R}_{t+k}\left(B_t^o - C_t\right),$$

where

$$\hat{R}_{t+k} \equiv \frac{R_{t+1}}{\left(1+\tau_{t+1}^{k}\right)} \frac{1}{\left(1+\tau_{t+1}^{c}\right) / \left(1+\tau_{t}^{c}\right)}.$$

This change of variables implies that the original problem can be reformulated as follows.

Problem 8. *Equivalent problem*

$$V_t = \sum_{t \geq 0} \beta^t \log\left(C_t\right)$$

subject to: $\hat{R}_{t+k}^{-1} B_{t+1}^{o} + C_t = B_t^o$, ***where*** $B_0^o \equiv \left(1+\tau_0^c\right)^{-1} B_0$.

The solution to this problem is typical of log. Conjecture that:

$$V_t = V\left(B^o, t\right) = \frac{1}{1-\beta} \log\left(D\right) + v\left(t\right).$$

We thus have that:

$$\begin{aligned} V\left(B^o, t\right) &= \max_C \quad \log\left(C\right) + \beta \log\left(\hat{R}_{t+1}\left(B^o - C\right)\right) + \beta v\left(t+1\right). \\ &= \max_C \quad \log\left(C\right) + \frac{\beta}{1-\beta} \log\left(B^o - C\right) + \beta\left(\frac{1}{1-\beta} \log\left(\hat{R}_{t+1}\right) + v\left(t+1\right)\right). \end{aligned}$$

Taking first-order conditions with respect to C, we obtain:

$$\frac{1}{C} = \frac{1}{B^o - C} \frac{\beta}{1-\beta} \rightarrow C = \left(1-\beta\right) B^o.$$

We verify the conjecture by replacing the expenditure rule:

$$V\left(B^o, t\right) = \frac{\log\left(B^o\right)}{\left(1-\beta\right)} + \frac{\log\left(1-\beta\right)}{1-\beta} + \beta\left(\frac{1}{1-\beta} \log\left(\hat{R}_{t+1}\right) + v\left(t+1\right)\right),$$

where

$$v\left(t\right) = \frac{\log\left(1-\beta\right) + \beta \log\left(\hat{R}_{t+1}\right)}{1-\beta} + \beta v\left(t+1\right).$$

C.2 Proof of Proposition 9

I first state Proposition 9 in greater generality than as shown in the main body of the paper.

Proposition 12. (Solution of the Primal Problem): *The solution to the Primal Planner Problem is given by the solution to the following static problem:*

Problem 9.

$$\mathcal{P}^{\theta}\left(\tilde{B}\right) = \max_{B\in\left[0,\tilde{B}\right]} \mathcal{P}\left(B,\tilde{B}\right)$$

where

$$\mathcal{P}\left(B,\tilde{B}\right) \equiv \left\{(1-\theta)\log\left((1-\beta)B\right) + \theta\log\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\mu\left(B,\tilde{B}\right) + S^{w}\left(B,\tilde{B}\right)\right)\right\}$$

and $\mu\left(B,\tilde{B}\right)$ *and* $S^{w}\left(B,\tilde{B}\right)$*:*

$$\mu\left(B,\tilde{B}\right) \equiv 1-(1-\beta)B - \min\left\{\max\left\{\tilde{B}-B,0\right\},1-(1-\beta)B\right\}$$

$$S^{w}\left(B,\tilde{B}\right) \equiv \min\left\{\max\left\{\tilde{B}-B,0\right\},1-(1-\beta)B\right\}.$$

Let the solution to this problem be $\mathcal{B}^{\theta}$*. Then, the solution to the primal planner's problem is* $B_t = \mathcal{B}^{\theta}\left(\tilde{B}_t\right)$*. The function* $\mathcal{B}^{\theta}$ *satisfies:*

I. Efficiency Threshold. *For* $\tilde{B} \geq \frac{1+\theta\beta}{1-\beta}$*,* $\mathcal{B}^{\theta} = B_{ss}$*. Moreover, for this debt level* $X^{w} = 0$*.*

II. Inefficiency Threshold. *For* $\tilde{B} < \frac{1+\theta\beta}{1-\beta}$ *the planner's solution induces TFP losses. The solution to the Primal Planner's problem in this region depends on the threshold SBL,* $\tilde{B}^{e}$*.*

II.a Social Insurance complements Productive Efficiency. *For* $\tilde{B} \in \left[\tilde{B}^{e}, \frac{1+\theta\beta}{1-\beta}\right]$*, we have that* $\mathcal{B}^{\theta} = B^{\star}\left(\tilde{B}\right)$ *if*

$$\frac{1-(1-\beta)B^{\star}\left(\tilde{B}\right)}{(1-\beta)B^{\star}\left(\tilde{B}\right)} \leq \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}$$

Or is the unique solution $B^{\theta} < B_{ss}$ to:

$$\frac{1-(1-\beta)B^{\theta}}{(1-\beta)B^{\theta}} = \frac{\theta}{1-\theta}\frac{Q\left(B^{\theta},\tilde{B}\right)}{q\left(B^{\theta},\tilde{B}\right)}\left(\frac{q\left(B^{\theta},\tilde{B}\right)-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu\left(B^{\theta},\tilde{B}\right)\right)\right)}{1-\beta}\right).$$

Moreover, for this debt level $X^{w}, S^{w} > 0$.

II.b Social Insurance conflicts Productive Efficiency. *For $\tilde{B} \in \left[0, \tilde{B}^{e}\right]$, we have that $\mathcal{B}^{\theta}$ is the unique constant solution $B^{\theta} > B_{ss}$ to the equation:*

$$\frac{1-(1-\beta)B^{\theta}}{(1-\beta)B^{\theta}} = \frac{\theta}{1-\theta}\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-(1-\beta)B^{\theta}\right)\right).$$

Moreover, for this debt level $S^{w} = 0$.

Threshold value $\tilde{B}^{e}$. *Let $\underline{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}(0)$ and $\bar{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}\left(\bar{B}\right)$. The threshold $\tilde{B}^{e}$ solves:*

$$\bar{\mathcal{P}}^{\theta} = \underline{\mathcal{P}}^{\theta} + \theta\int_{\tilde{B}^{e}}^{\frac{1+\theta\beta}{1-\beta}}\left(\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_{\tilde{B}}\left(B^{p}(\tilde{B})\right)B_{\tilde{B}}^{\star}\left(\tilde{B}\right)\right)d\tilde{B}.$$

where for $\tilde{B} \in \left[\tilde{B}^{e}, \frac{1+\theta\beta}{1-\beta}\right]$ we have:

$$\mathcal{P}_{\tilde{B}}^{\theta}\left(\tilde{B}\right) = \frac{\theta}{1-(1-\beta)B^{p}(\tilde{B})}\frac{Q\left(B^{p}(\tilde{B}),\tilde{B}\right)}{q\left(B^{p}(\tilde{B}),\tilde{B}\right)}\log\left(\mathcal{A}\left(\mu\left(B^{p}(\tilde{B}),\tilde{B}\right)\right)-\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu\left(B^{p}(\tilde{B}),\tilde{B}\right)\right)\right)\right).$$

To begin the proof, let me start with the Primal Problem in the statement of Proposition 8. Taking the sequence of borrowing limits $\left\{\tilde{B}_{t}\right\}_{t\geq 0}$, the Primal Planner maximizes:

$$\max_{\{B_t\}_{t\geq 0}}\sum_{t\geq 0}\beta^{t}\left[(1-\theta)\log\left(C_{t}^{s}\right)+\theta\log\left(X_{t}^{w}+S_{t}^{w}\right)\right],$$

subject to the saver's budget constraint,

$$C_{t}^{s} = (1-\beta)B_{t},\ \forall t \geq \mathbf{0},$$

the income expenditure identity,

$$1 = \mu_t + S_t^w + (1-\beta) B_t,$$

the spot expenditure constraint,

$$S_t \leq \max\left\{\tilde{B}_t - B_t, 0\right\}, \;\; \forall t \geq \mathbf{0},$$

and the cost of chained goods,

$$X_t^w = \mathcal{A}(\mu_t)\mu_t,$$

and subject to $\mu_t \in [0,1]$.

We have that for any B_t, the consumption delivered to the savers is fixed. To maximize the worker's utility, we must

$$\max \log\left(X_t^w + S_t^w\right)$$

subject to:

$$1 - (1-\beta) B_t = \mu_t + S_t^w$$
$$S_t \leq \max\left\{\tilde{B}_t - B_t, 0\right\}$$
$$X_t^w = \mathcal{A}(\mu_t)\mu_t.$$

and subject to $\mu_t \in [0,1]$. The last constraint implies that:

$$1 - (1-\beta) B_t - S_t^w \geq 0,$$

or equivalently,

$$S_t^w \leq 1 - (1-\beta) B_t.$$

Hence, the Primal Planner respects the same constraint as the worker:

$$S_t^w = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - (1-\beta) B_t\right\}.$$

Therefore, the expenditures on chained goods are:

$$\mu_t = 1 - (1-\beta) B_t - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - (1-\beta) B_t\right\}.$$

Since the the maximization is static, we can solve it state by state as in the statement of the proposition. Thus, the objective of the Primal Planner is the same as solving the following problem at each date:

Problem 10. *The Primal Planner's problem is given by:*

$$\mathcal{P}^{\theta}\left(\tilde{B}\right) = \max_{B \in \left[0, \tilde{B}\right]} \mathcal{P}\left(B, \tilde{B}\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) = \left\{(1-\theta) \log\left((1-\beta) B\right) + \theta \log\left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right) \mu\left(B, \tilde{B}\right) + S^{w}\left(B, \tilde{B}\right)\right)\right\}$$

subject to:

$$\mu\left(B, \tilde{B}\right) \equiv 1 - (1-\beta) B - \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta) B\right\}$$

$$S^{w}\left(B, \tilde{B}\right) = \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta) B\right\}.$$

This is a problem where the planner can distribute wealth at will, but respects the constraints. Next, I proceed to solve this problem. Naturally, welfare depends on whether and how the planner may want to distort TFP to provide insurance. There are multiple policy regimes that depend on the SBL, $\tilde{B}$. I make use of two problems, the best and worst value problems:

$$\underline{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}(0) \text{ and } \bar{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}\left(\bar{B}\right).$$

Case 1. Values of $\tilde{B}$ such that all consumption is spot. Ideally, the planner wants to maximize spot consumption and set $\mu = 0$. The unconstrained solution to the Primal Planner's problem is given by the ratio of of Pareto weights:

$$\frac{1 - (1-\beta) B^{o}}{(1-\beta) B^{o}} = \frac{\theta}{1-\theta} \rightarrow B^{o} = \frac{1-\theta}{1-\beta} = B_{ss}.$$

This yields the same value as $\bar{\mathcal{P}}^{\theta}$.

This level of debt must satisfy the condition that all spot consumption must be feasible:

$$\max\left\{\tilde{B}-B^{o},0\right\}\geq 1-(1-\beta)\,B^{o}>0.$$

Thus, we need

$$\tilde{B}\geq B^{o}$$

and that:

$$\tilde{B}\geq 1+\beta B^{o}.$$

Combining both constraints we have:

$$\tilde{B}\geq B^{o}+\max\left\{1-(1-\beta)\,B^{o},0\right\}.$$

We know that the optimal debt B^{o} must be less than the natural borrowing limit. Hence, the inequality is just:

$$\tilde{B}\geq 1+\beta B^{o}=1+\frac{\beta}{1-\beta}\,(1-\theta)\,. \tag{34}$$

Thus, for these levels of the SBL, the planner can achieve the unconstrained solution. This corresponds to the debt in the efficient steady state level of the competitive equilibrium that produces the planner's Pareto weights.

Case 2. Values of $\tilde{B}$ such that some consumption is chained. Now consider the case where the constraint binds, $\tilde{B}<\frac{1+\theta\beta}{1-\beta}$. In this case, the planner cannot achieve the unconstrained solution. The amount of chained expenditures are therefore positive:

$$\mu\left(B,\tilde{B}\right)=1-(1-\beta)\,B-\min\left\{\max\left\{\tilde{B}-B,0\right\},1-(1-\beta)\,B\right\}>0.$$

We have critical values.

- If the planner chooses $B^{p}\geq B_{h}\left(\tilde{B}\right)=\tilde{B}$, there is no spot consumption.
- If the planner chooses $B^{p}<B_{l}\left(\tilde{B}\right)<\tilde{B}$, that there is no chained consumption.

This threshold $B_l\left(\tilde{B}\right)$ solves,

$$\max\left\{\tilde{B}-B_l,0\right\}=1-(1-\beta)\,B_l$$

when since $B_l<\tilde{B}$, we obtain:

$$\tilde{B}=1+\beta B_l\rightarrow B_l\left(\tilde{B}\right)=\max\left\{0,\beta^{-1}\left(\tilde{B}-1\right)\right\}=B^{\star}\left(\tilde{B}\right).$$

Obviously, $B^{\star}\left(\tilde{B}\right)<B^{o}$ since we are in the constrained region.

Consider now the planner problem that restricts choices to at least some of both goods are consumed by the worker:

Problem 11. *The Primal Planner's problem restricted to both types of consumption is:*

$$\tilde{\mathcal{P}}\left(\tilde{B}\right)=\max_{B\in\left[B_l\left(\tilde{B}\right),B_h\left(\tilde{B}\right)\right]}\left\{(1-\theta)\log\left((1-\beta)\,B\right)+\theta\log\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\cdot\mu\left(B,\tilde{B}\right)+S^{w}\left(B,\tilde{B}\right)\right)\right\}$$

subject to:

$$\mu\left(B,\tilde{B}\right)\equiv 1+\beta B-\tilde{B}$$

and

$$S^{w}\left(B,\tilde{B}\right)=\tilde{B}-B.$$

We have the following Lemma.

Lemma 2. *For any $\tilde{B}<\frac{1+\theta\beta}{1-\beta}$, the original planner problem satisfies:*

$$\mathcal{P}^{\theta}\left(\tilde{B}\right)=\max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right),\underline{\mathcal{P}}^{\theta}\right\}.$$

Proof. Indeed, in the region $B^{p}\in\left[0,B_l\left(\tilde{B}\right)\right]$ the objective of the planner is equivalent to the objective when the SBL is most relaxed, $\mathcal{P}\left(B,\bar{B}\right)$. Thus, since $\tilde{B}<\frac{1+\theta\beta}{1-\beta}$, the planner's objective is increasing in the region $\left[0,B_l\left(\tilde{B}\right)\right]$. Thus, the planner's solution must fall in between $B^{p}\in\left[B_l\left(\tilde{B}\right),\bar{B}\right]$. For any $B^{p}\geq B_h\left(\tilde{B}\right)=\tilde{B}$, the objective function in $\mathcal{P}\left(B,\tilde{B}\right)$ is independent of $\tilde{B}$ and hence, must coincide with the value of $\underline{\mathcal{P}}^{\theta}$. Hence, we can partition $\mathcal{P}^{\theta}\left(\tilde{B}\right)$ according to the Lemma. □

To prove the main result, I solve the problems $\underline{\mathcal{P}}^{\theta}$ and $\tilde{\mathcal{P}}\left(\tilde{B}\right)$

Auxiliary Problem $\underline{\mathcal{P}}^{\theta}$: no spot consumption. The planner's problem with the tightest SBL $\underline{\mathcal{P}}^{\theta} = \mathcal{P}^{\theta}(0)$ is given by:

$$\underline{\mathcal{P}}^{\theta} = \max_{B \in \left[0, \bar{B}\right]} \underline{\mathcal{P}}(B, 0)$$

where

$$\underline{\mathcal{P}}^{\theta} = \max_{B \in \left[0, \bar{B}\right]} \left\{(1-\theta)\log\left((1-\beta)B\right) + \theta \log\left(\mathcal{A}\left(\mu(B,0)\right)\mu(B,0)\right)\right\}$$

subject to:

$$\mu(B,0) \equiv 1 - (1-\beta)B.$$

To solve this problem, I perform some calculations. First, note that:

$$\frac{\partial\left[\mathcal{A}(\mu)\mu\right]}{\partial\mu} = \mathcal{A}(\mu)\left(1 + \epsilon^{\mathcal{A}}\right)$$

where,

$$\epsilon_{\mu}^{\mathcal{A}} \equiv \frac{\partial\mathcal{A}(\mu)}{\partial\mu}\frac{\mu}{\mathcal{A}(\mu)}.$$

The derivative $\underline{\mathcal{P}}_B$ is therefore given by:

$$\underline{\mathcal{P}}_B = (1-\theta)\frac{(1-\beta)}{(1-\beta)B} + \theta\frac{\mathcal{A}(\mu)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\mu_B(B,0)}{\mathcal{A}(\mu)\mu} = \frac{(1-\theta)}{B} - \theta\frac{\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)(1-\beta)}{1-(1-\beta)B}.$$

The second equation uses that $\mu_B(B,0) \equiv -(1-\beta)$.

The first term in $\underline{\mathcal{P}}_B$, $(1-\theta)/B$, is decreasing in B. The second term,

$$\theta\frac{\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)(1-\beta)}{1-(1-\beta)B}, \tag{35}$$

is increasing. We know this because denominator is decreasing in $B \in \left[0, \bar{B}\right]$ and the elasticity of TFP is itself decreasing in μ,

$$\epsilon_{\mu\mu}^{\mathcal{A}} = \frac{\partial}{\partial\mu}\left[\frac{\mathcal{A}'(\mu)\mu}{\mathcal{A}(\mu)}\right] = \frac{\mathcal{A}''(\mu)\mu}{\mathcal{A}(\mu)} + \frac{\mathcal{A}'(\mu)}{\mathcal{A}(\mu)} - \frac{\left[\mathcal{A}'(\mu)\right]^2}{\left[\mathcal{A}(\mu)\right]^2} < 0.$$

Hence, $\epsilon^{\mathcal{A}}_{\mu\mu}\mu_B(B,0) > 0$, since the product of two numbers thus, the numerator of the second term (35). Thus, $\underline{\mathcal{P}}$ is concave and therefore $\underline{\mathcal{P}}^{\theta}$ has a unique solution:

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{1-\theta}\left(1+\epsilon^{\mathcal{A}}_{\mu}(\mu(B,0))\right)$$

and recall that $\mu(B,0) = 1-(1-\beta)B$. I call this solution $\underline{B}^{p}$: the planner debt level under the most tight SBL. We have the following Lemma:

Lemma 3. *The solution $\underline{B}^{p} > B^{o}$.*

Proof. The proof is immediate from $1+\epsilon^{\mathcal{A}}_{\mu} < 1$ and the fact that

$$\frac{1-(1-\beta)B^{o}}{(1-\beta)B^{o}} = \frac{\theta}{(1-\theta)}.$$

□

Next, I solve $\tilde{\mathcal{P}}\left(\tilde{B}\right)$.

Auxiliary Problem $\tilde{\mathcal{P}}\left(\tilde{B}\right)$: spot and chained consumption. Consider now the planner problem where at least some of both goods are consumed by the worker:

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) = \max_{B\in\left[B^{\star}\left(\tilde{B}\right),\tilde{B}\right]} \tilde{\mathcal{P}}\left(B,\tilde{B}\right)$$

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) \equiv \left\{(1-\theta)\log((1-\beta)B) + \theta\log\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\cdot\mu\left(B,\tilde{B}\right) + S^{w}\left(B,\tilde{B}\right)\right)\right\}$$

subject to:

$$\mu\left(B,\tilde{B}\right) \equiv 1+\beta B-\tilde{B}$$

and

$$S^{w}\left(B,\tilde{B}\right) = \tilde{B}-B.$$

The derivative of the objective in $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is:

$$\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)=(1-\theta)\frac{1}{B}+\theta\frac{\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B,\tilde{B}\right)\right)\right)\beta-1}{\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\mu\left(B,\tilde{B}\right)+\tilde{B}-B}.$$

Recall that,

$$C^w\left(B,\tilde{B}\right)=\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\mu\left(B,\tilde{B}\right)+\tilde{B}-B.$$

Hence, using the definition of Q and q we rewrite:

$$\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)=(1-\theta)\frac{1}{B}-\theta\frac{Q}{q}\frac{q-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B,\tilde{B}\right)\right)\right)}{1-(1-\beta)B}$$

We can multiply both sides by the ratio of $1-(1-\beta)B$ and divide by $(1-\beta)$ and obtain:

$$\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)\frac{1-(1-\beta)B}{(1-\beta)B}=\frac{1-(1-\beta)B}{(1-\beta)B}-\frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right).$$

This function must have the same sign as $\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)$, since at was obtained by multiplication of positive numbers. The first term is decreasing in B. In turn, $Q_\mu\mu_B$ is increasing in B. Hence, as long as

$$\mathcal{A}(\mu)\left(1+\epsilon_\mu^{\mathcal{A}}\right)=\mathcal{A}(\mu)+\mathcal{A}'(\mu)\mu$$

is decreasing in B, the second term is increasing. The second term is indeed decreasing in μ since its derivative is:

$$2\mathcal{A}'(\mu)+\mathcal{A}''(\mu)\mu<0,$$

where the sign follows immediately from the concavity and monotone decreasing properties of $\mathcal{A}$. Hence, the objective function $\tilde{\mathcal{P}}\left(B,\tilde{B}\right)$ is concave in B. Furthermore, since

$$Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right)>1,$$

and we have an interior maximum in the region $B \in \left[B^{\star}\left(\tilde{B}\right), \tilde{B}\right]$, B must solve:

$$\frac{1-(1-\beta)\,B}{(1-\beta)\,B} = \frac{\theta}{1-\theta} Q\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right),$$

and it is such that $B < B^{o}$.

Next, we establish properties regarding the limits of this function at the boundaries of the set $\left[B^{\star}\left(\tilde{B}\right), \tilde{B}\right]$. First, the limit at the left boundary:

$$\lim_{B\downarrow B^{\star}\left(\tilde{B}\right)} \frac{1-(1-\beta)\,B}{(1-\beta)\,B} - \frac{\theta}{1-\theta}\frac{Q}{q}\left(\frac{q}{1-\beta} - \frac{\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)}{1-\beta}\right) = \lim_{B\downarrow B_{l}\left(\tilde{B}\right)} \frac{1-(1-\beta)\,B}{(1-\beta)\,B} - \frac{\theta}{1-\theta}\left(\frac{1}{1-\beta} - \frac{\beta\left(1+1\right.}{}\right.$$

In Appendix A, I show that $\lim_{\mu\downarrow 0} 1+\epsilon_{\mu}^{\mathcal{A}} = 0$. However, I also show that

$$\lim_{\mu\downarrow 0} \mathcal{A}\left(\mu\right) = \delta.$$

Thus, the limit of the objective function at the left boundary has the sign of:

$$\frac{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)} - \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta} = \frac{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)} - \frac{1-(1-\beta)\,B^{o}}{(1-\beta)\,B^{o}}\frac{1-\beta\delta}{1-\beta}.$$

The function $B^{\star}\left(\tilde{B}\right) < B^{o}$ but $\frac{1-\beta\delta}{1-\beta} > 1$, hence the sign is ambiguous. The solution is at this corner if this derivative is negative.

Next, we consider the limit of the derivative of the objective to the right of the constraint set $\left[B^{\star}\left(\tilde{B}\right), \tilde{B}\right]$. If

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}} - \frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-(1-\beta)\,\tilde{B}\right)\right)}{1-\beta}\right) \geq 0$$

the solution to this problem is $B = \tilde{B}$. Otherwise, the solution must fall at some value $B < \tilde{B}$:

$$\frac{1-(1-\beta)\,B}{(1-\beta)\,B} = \frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-(1-\beta)\,B\right)\right)}{1-\beta}\right).$$

Moreover, we know that since

$$Q\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)>1,$$

The corner solution $\tilde{B}$ is chosen only if

$$\frac{1-\left(1-\beta\right)\tilde{B}}{\left(1-\beta\right)\tilde{B}}>\frac{1-\left(1-\beta\right)B^{o}}{\left(1-\beta\right)B^{o}}$$

which implies that this occurs only for some $\tilde{B}<B^{o}$. Collecting the results, up to this point, we have the following Lemma.

Lemma 4. *The solution B_{ℓ}^{p} to $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is as follows.*

I. $B_{\ell}^{p}=\tilde{B}$ *if*

$$\frac{1-\left(1-\beta\right)\tilde{B}}{\left(1-\beta\right)\tilde{B}}\geq\frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-\left(1-\beta\right)\tilde{B}\right)\right)}{1-\beta}\right),$$

II. $B_{\ell}^{p}=B^{\star}\left(\tilde{B}\right)$

$$\frac{1-\left(1-\beta\right)B^{\star}\left(\tilde{B}\right)}{\left(1-\beta\right)B^{\star}\left(\tilde{B}\right)}\leq\frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}.$$

III. *Otherwise,* B_{ℓ}^{p} solves:

$$\frac{1-\left(1-\beta\right)B_{\ell}^{p}}{\left(1-\beta\right)B_{\ell}^{p}}=\frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(1-\left(1-\beta\right)B_{\ell}^{p}\right)\right)}{1-\beta}\right).$$

Overall Solution. Recall that we showed above that:

$$\mathcal{P}^{\theta}\left(\tilde{B}\right)=\max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right),\underline{\mathcal{P}}^{\theta}\right\}.$$

The following Lemma shows that the solution B^{p} to the Planner's problem is never a corner solution.

Lemma 5. *The planner never chooses a solution at $B^{p}=\tilde{B}$.*

Proof. $\mathcal{P}^{\theta}\left(\tilde{B}\right) = \max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right), \underline{\mathcal{P}}^{\theta}\right\}$. To proof this Lemma observe that the left limit as $B \uparrow \tilde{B}$ satisfies

$$\frac{q-1-\epsilon^{\mathcal{A}}}{1-\beta} + 1 + \epsilon^{\mathcal{A}} > 1 + \epsilon^{\mathcal{A}},$$

where the inequality follows from $q > 1$ and $\epsilon^{\mathcal{A}} < 0$. As a consequence,

$$\frac{1-(1-\beta)\tilde{B}}{(1-\beta)\tilde{B}} - \frac{\theta}{1-\theta}\lim_{B\uparrow\tilde{B}} \Lambda\left(B, \tilde{B}\right) \geq 0$$

implies

$$\frac{1-(1-\beta)\tilde{B}_t}{(1-\beta)\tilde{B}_t} - \frac{\theta}{1-\theta}\lim_{B\downarrow\tilde{B}_t} \Lambda\left(B, \tilde{B}\right) > 0.$$

Hence, although the derivative of the objective is discontinuous at $\tilde{B}$, we know that if the derivative is weakly increasing at the left, it is increasing from the right and this implies that $B = \tilde{B}$ is never an optimal choice for the planner. □

Next, observe that the problem $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ has a compact-valued and continuous constraint correspondence with a continuous objective. It satisfies the conditions for the Theorem of the Maximum. In addition it is immediate to verify that:

$$\lim_{\tilde{B}\uparrow\frac{1+\theta\beta}{1-\beta}} \tilde{\mathcal{P}}\left(\tilde{B}\right) = \bar{\mathcal{P}}^{\theta} \text{ and } \lim_{\tilde{B}\uparrow\frac{1+\theta\beta}{1-\beta}} B^{p}(\tilde{B}) = B^{o}.$$

We can employ the Envelope Theorem on $\tilde{\mathcal{P}}\left(\tilde{B}\right)$. In the region where the solution to $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is not at a corner solution, the Envelope Theorem yields:

$$\tilde{\mathcal{P}}_{\tilde{B}} = \frac{\theta}{1-(1-\beta)B^{p}(\tilde{B})} Q\left(B^{p}(\tilde{B}), \tilde{B}\right)\left(1 - \left(1 + \epsilon^{\mathcal{A}}_{\mu}\left(\mu\left(B^{p}(\tilde{B}), \tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B^{p}(\tilde{B}), \tilde{B}\right)\right)\right),$$

since $\tilde{B}$ appears directly through μ and S in the objective. The function is strictly increasing in $\tilde{B}$ since $\epsilon^{\mathcal{A}}_{\mu} < 0$ and $\mathcal{A} < 1$. In the region where the function is at the lower corner of the constraint:

$$B = B^{\star}\left(\tilde{B}\right),$$

the value of this term is:

$$\tilde{\mathcal{P}}_{\tilde{B}} = \frac{\theta}{1-(1-\beta)\, B^{\star}\left(\tilde{B}\right)}\,(1-\delta)$$

and

$$\tilde{\mathcal{P}}_{B} = \frac{1-\theta}{B^{\star}\left(\tilde{B}\right)} - \frac{\theta\,(1-\beta\delta)}{1-(1-\beta)\, B^{\star}\left(\tilde{B}\right)}.$$

Thus, we have that the marginal objective value is:

$$\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_{B}\left(B^{p}(\tilde{B})\right) B^{\star}_{\tilde{B}}\left(\tilde{B}\right)$$

Hence, there exists a threshold value $\tilde{B}^{e}$ such that:

$$\tilde{\mathcal{P}}\left(\tilde{B}^{e}\right) = \underline{\mathcal{P}}^{\theta}$$

because notice that $\underline{\mathcal{P}}^{\theta}$ is finite and lower than the value of the problem without constraints, $\tilde{\mathcal{P}}\,(0)$ tends to $-\infty$ and $\tilde{\mathcal{P}}\,(\overline{B})$ tends to the value of problem without constraints. This implies that $\tilde{\mathcal{P}}\,(0) < \underline{\mathcal{P}}^{\theta} < \tilde{\mathcal{P}}\,(\overline{B})$ and from continuity of the value function (Theorem of the Maximum) and the Intermediate Value Theorem, the existence of $\tilde{B}^{e}$ is guaranteed. By the fundamental theorem of calculus:

$$\bar{\mathcal{P}}^{\theta} = \underline{\mathcal{P}}^{\theta} + \theta \int_{\tilde{B}^{e}}^{\frac{1+\theta\beta}{1-\beta}} \left(\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_{B}\left(B^{p}(\tilde{B})\right) B^{\star}_{\tilde{B}}\left(\tilde{B}\right)\right) d\tilde{B}.$$

This concludes the proof of the general version of Proposition 9.

C.3 Proof of Proposition 10

Let's define the indirect social utility function of the Planner Problem with expenditures $\mathcal{P}^g$:

$$\mathcal{P}^g\left(B,\tilde{B},G^s,G^x\right) = (1-\theta)\log\left(C^s\left(B\right)\right) + \theta\log\left(S^w\left(B,\tilde{B},G^s,G^x\right) + X^w\left(B,\tilde{B},G^s,G^x\right)\right).$$

The following functions determine the allocation:

$$\begin{aligned}
C^s\left(B\right) &\equiv (1-\beta)\,B \\
E^w\left(B,G^s,G^x\right) &\equiv 1 - E^s\left(B\right) - G^x - G^s \\
S^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\} \\
X^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \frac{E^w - \min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\}}{q} \\
q\left(B,\tilde{B},G^s,G^x\right) &\equiv \mathcal{A}^{-1}\left(\mu\right) \\
\mu\left(B,\tilde{B},G^s,G^x\right) &\equiv G^x + X^w q.
\end{aligned}$$

The worker's total consumption is:

$$C^w = S^w\left(B,\tilde{B},G^s,G^x\right) + X^w\left(B,\tilde{B},G^s,G^x\right).$$

Because

$$\mathcal{Y} = C^w + C^S + G^s + \frac{G^x}{q\left(\mu\right)},$$

but $C^s\left(B\right)$ is independent of the government's expenditure, we obtain: $d\mathcal{Y} = dC^w + dG^s + d\left(G^x/q\left(\mu\right)\right)$..

Thus, we have that the government's expenditure multiplier for expenditure of type $i = x, s$ relates to the worker's consumption as follow:

$$\mathcal{M}^s\left(B,\tilde{B}\right) \equiv \frac{d\mathcal{Y}}{dG^s} = \frac{dC^w}{dG^s} + 1$$

and

$$\mathcal{M}^x\left(B,\tilde{B}\right) \equiv \frac{d\mathcal{Y}}{dG^x} = \frac{dC^w}{dG^x} + \frac{1}{q} - \frac{G^x}{q^2}\frac{dq}{d\mu}\frac{d\mu}{dG^x}$$
$$= \frac{dC^w}{dG^x} + \mathcal{A}\left(\mu\right).$$

where the last equation is because we are interested in obtaining the government's infinitesimal multiplier, evaluated at $G^x = G^s = 0$. So we have defined them as

$$\mathcal{M}^i\left(B,\tilde{B}\right) \equiv \left.\frac{d\mathcal{Y}}{dG^i}\right|_{G^x=G^s=0},$$

and relating to the indirect social utility function,the change in the objective is:

$$\frac{\theta}{C^w}\frac{dC^w}{dG^i},$$

for expenditure i.

Case 1. All spot consumption $B < B^{\star}$. If there is only spot consumption:

$$C^s\left(B\right) \equiv \left(1-\beta\right)B$$
$$C^w\left(B,\tilde{B},G^s,G^x\right) \equiv 1-\left(1-\beta\right)B - G^x - G^s$$
$$X^w\left(B,\tilde{B},G^s,G^x\right) \equiv 0$$
$$q\left(B,\tilde{B},G^s,G^x\right) \equiv \mathcal{A}^{-1}\left(\mu\right)$$
$$\mu\left(B,\tilde{B},G^s,G^x\right) \equiv G^x.$$

Since worker consumption is independent of q, we have that:

$$\frac{dC^w}{dG^i} = -1$$

for both $i \in \{x,s\}$. In turn, we have that

$$\mathcal{M}^s\left(B,\tilde{B}\right) = \frac{dC^w}{dG^s} + 1 = 0.$$

Likewise, for chained expenditures we have:

$$\mathcal{M}^x\left(B,\tilde{B}\right) = \frac{dC^w}{dG^x} + \mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right) = -\left(1-\mathcal{A}\left(\mu\right)\right)+\epsilon_\mu^{\mathcal{A}} < 0.$$

Case 2. Some chained consumption. If there are some chained expenditures:

$$\begin{aligned}
C^s\left(B\right) &\equiv \left(1-\beta\right)B \\
E^w\left(B,G^s,G^x\right) &\equiv 1-\left(1-\beta\right)B-G^x-G^s \\
C^w\left(B,\tilde{B},G^s,G^x\right) &\equiv S^w+X^w \\
S^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \max\left\{\tilde{B}-B,0\right\} \\
X^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \frac{E^w-\max\left\{\tilde{B}-B,0\right\}}{q} \\
q\left(B,\tilde{B},G^s,G^x\right) &\equiv \mathcal{A}^{-1}\left(\mu\right) \\
\mu\left(B,\tilde{B},G^s,G^x\right) &\equiv G^x+X^w q.
\end{aligned}$$

Rewriting the last three identities using $\mathcal{A}\left(\mu\right)$ instead of q we have

$$\begin{aligned}
X^w &\equiv \mathcal{A}\left(\mu\right)\left(E^w-\max\left\{\tilde{B}-B,0\right\}\right) \\
\mu &\equiv G^x+E^w-\max\left\{\tilde{B}-B,0\right\}.
\end{aligned}$$

Substituting E^w we have, naturally,

$$\mu = 1-\underbrace{\left(\left(1-\beta\right)B+G^s+\max\left\{\tilde{B}-B,0\right\}\right)}_{\text{spot exp.}} \tag{36}$$

From here we obtain that:

$$\frac{dX^w}{dG^x} = -\mathcal{A}\left(\mu\right)+\mathcal{A}'\left(\mu\right)\frac{d\mu}{dG^x}\left(E^w-\max\left\{\tilde{B}-B,0\right\}\right).$$

Since $\frac{d\mu}{dG^x} = \frac{dS^w}{dG^x} = 0$, we have that:

$$\frac{dC^w}{dG^x} = -\mathcal{A}(\mu).$$

Hence, the government multiplier for chained expenditures is:

$$\mathcal{M}^x\left(B, \tilde{B}\right) = \epsilon_\mu^{\mathcal{A}} < 1.$$

Next, observe that:

$$X^w \equiv \mathcal{A}(\mu)\left(1 - (1-\beta)B - G^x - G^s - \max\left\{\tilde{B} - B, 0\right\}\right).$$

Hence,

$$\begin{aligned} dX^w &= -\mathcal{A}(\mu)\, dG^s + \mathcal{A}'(\mu)\left(\frac{X^w}{\mathcal{A}(\mu)}\right) d\mu \\ &= -\mathcal{A}(\mu)\, dG^s + \mathcal{A}(\mu)\,\epsilon_\mu^{\mathcal{A}}\left(\frac{X^w}{\mathcal{A}(\mu)}/\mu\right) d\mu. \end{aligned}$$

The second line follows from:

$$\mathcal{A}(\mu)\,\mu \equiv G^x + X^w.$$

Also following this condition, we have that:

$$\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) d\mu \equiv dX^w.$$

Combining the differentials evaluated at $G^x = 0$, we obtain:

$$d\mu = -dG^s.$$

Hence,

$$\frac{dC^w}{dG^s} = \frac{dX^w}{dG^s} = -\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right).$$

Following the relationship with the fiscal multiplier, we obtain:

$$\mathcal{M}^s\left(B,\tilde{B}\right) = 1-\mathcal{A}\left(\mu\right)-\mathcal{A}\left(\mu\right)\epsilon_{\mu}^{\mathcal{A}}.$$

Summary. We summarize the results:

$$\frac{dC^w}{dG^x}=\begin{cases}-1 & B<B^{\star}\left(\tilde{B}\right)\\ \\ -\mathcal{A}\left(\mu\right) & B>B^{\star}\left(\tilde{B}\right)\end{cases}$$

and

$$\frac{dC^w}{dG^s}=\begin{cases}-1 & B<B^{\star}\left(\tilde{B}\right)\\ \\ -\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right) & B>B^{\star}\left(\tilde{B}\right).\end{cases}$$

Finally, recall that the government multipliers relate to the change in consumption as follows:

$$\mathcal{M}^s\left(B,\tilde{B}\right)\equiv\frac{d\mathcal{Y}}{dG^s}=\frac{dC^w}{dG^s}+1$$

and

$$\mathcal{M}^s\left(B,\tilde{B}\right)\equiv\frac{dC^w}{dG^x}+\mathcal{A}\left(\mu\right)$$

with:

$$\frac{d\mu}{dG^x}=\begin{cases}1 & B<B^{\star}\left(\tilde{B}\right)\\ \\ 0 & B>B^{\star}\left(\tilde{B}\right).\end{cases}$$

Therefore, adding terms:

$$\mathcal{M}^x\left(B,\tilde{B}\right)=\begin{cases}-\left(1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\right) & B<B^{\star}\left(\tilde{B}\right)\\ \\ 0 & B>B^{\star}\left(\tilde{B}\right)\end{cases}$$

Hence,

$$\mathcal{M}^s\left(B,\tilde{B}\right)=\begin{cases}0 & B<B^\star\left(\tilde{B}\right)\\ \\ 1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right) & B>B^\star\left(\tilde{B}\right).\end{cases}$$