

Infinite Debt Rollover in Stochastic Economies

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Abstract

This paper shows that there is *more* scope for a borrower to engage in a *sustainable infinite debt rollover* (a “Ponzi scheme”) when interest/growth rates are stochastic. In this context, I prove that the relevant “r vs. g” comparison uses the yield r_{long} to an infinite-maturity zero-coupon bond. I show that r_{long} is lower than the (risk-neutral) expectation of the short-term yield when it is variable, and that r_{long} is close to the minimal realization of the short-term yield when it is highly persistent. The paper applies these results to illustrative heterogeneous agent dynamic stochastic general equilibrium models to obtain weak sufficient conditions for the existence of public debt bubbles.

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1 Introduction

Suppose that a borrower faces a constant interest rate \bar{r} and a supply of available loanable funds that is growing deterministically at rate \bar{g} . If $\bar{g} \geq \bar{r}$, it is possible for the borrower to engage in what I will term a *sustainable infinite debt rollover*. In such a (Ponzi) scheme, the borrower issues debt in the current period, repays the principal and interest by issuing new debt next period, and then so on ad infinitum (literally). Of course, this infinite rollover plan is not sustainable if $\bar{g} < \bar{r}$ because the requisite repayments necessarily must eventually exceed the funds accessible to the borrower’.

But what are the analogs of these conditions on interest rates and/or growth rates if they are stochastic? This paper tackles this question in a general Markovian setting. Its main finding is that there is *more* scope for a sustainable infinite debt rollover when interest rates and growth rates exhibit persistent fluctuations.

The specifics are as follows. As described in the next Section 2, interest rates and growth rates are governed by a discrete-time time-homogeneous Markov process (with respect to risk-neutral probabilities, which are treated as exogenous until the last section). The key variable is the yield r_{long} on a zero-coupon bond with arbitrarily long maturity (the far right end of the yield curve). Consistent with much earlier work (notably, Hansen and Scheinkman (2009) and Alvarez and Jermann (2005)), the paper assumes that r_{long} is really a parameter, in the sense that it is constant across dates and states. Section 2 shows that this assumption of the constancy of r_{long} is satisfied whenever the driving Markov process is in fact a finite-state Markov chain with a positive transition matrix.

The paper establishes its two main sets of results in Section 3.

1. Suppose the growth rate g_t (of available loanable funds) is a deterministic constant \bar{g} . There is a sustainable infinite debt rollover if and only if the long-term yield r_{long} is less than or equal to \bar{g} . **Thus, when interest rates are stochastic, the relevant “r vs. g” comparison is “ r_{long} vs. g”.**

2. The long-term yield r_{long} is less than the (risk-neutral) expectation of one-period bond yields if they are stochastic. If they are highly persistent, then r_{long} is well-approximated by the *lowest* possible realization of the one-period yield. **It is in this sense that allowing for volatility and persistence of short-term interest rates makes sustainable infinite debt rollover *more* possible.**

The following three findings, developed in Sections 4-6, extend the scope of the two main results described above.

- If the growth rate g_t is stochastic, construct a *detrended* economy with deterministically zero growth by subtracting the realized growth rate in each date and state from the one-period riskfree bond yield in that date and state. Then, there is a sustainable infinite debt rollover in the original (undetrended) economy if and only if the arbitrage-free long-term yield in the detrended economy (\hat{r}_{long}) is non-positive.
- It is typically not possible to accomplish a sustainable infinite debt rollover using short-term riskfree debt. But the rollover can always be implemented using short-term risky debt or using money (a zero-coupon infinite maturity asset). The sustainability of the rollover requires that the value of the outstanding debt varies (possibly considerably) with current and expected short-term yields.
- Section 6 considers two leading example heterogeneous agent dynamic stochastic general equilibrium models (overlapping generations and incomplete financial markets). I show that there is a public debt bubble in general equilibrium if the bond prices in a bubbleless autarkic equilibrium admit a sustainable infinite debt rollover.

Why is r_{long} the appropriate benchmark interest rate, as is shown in Result 1? Pick any horizon T . A borrower who rolls over debt for T years, and then stops doing so, is receiving resources today but has to give up resources in T years. A T -year debt rollover is hence equivalent in a cash flow sense to issuing a T -year discount (zero coupon) bond. Taking limits, an infinite debt rollover is equivalent to issuing an infinite horizon zero coupon bond.

Accordingly, the arbitrage-free yield on an infinite debt rollover is the interest rate at the far right end of the yield curve - that is, r_{long} .

The intuition behind the first part of Result 2 (r_{long} is low when short term yields are volatile) arises in the economics of fixed income securities¹ and also plays a key role in environmental economics.² Consider a situation in which the riskfree yield from the current period to the next is 0%, but thereafter could, with probability 50%, jump upward permanently to 10%. In this setting, the yield curve is flat after period 2 (either at 0% or at 10%). A risk-neutral investor in period 1 wants to use the available bonds to generate an expected payoff of \$1000 in period 32. How much does the investor need to have available in period 1?

It may seem like the answer to this question should at least be well-approximated by $1000/1.05^{30}$ - that is, the present value of \$1000 dollars calculated using the average yield of 5%. But this averaging ignores the fundamental *convexity* of the compounding of interest, which makes it very costly to generate \$1000 if the yields don't jump up in period 2. That convexity means that the investor actually needs:

$$1000\left(\frac{0.5}{1.1^{30}} + \frac{0.5}{1}\right) = 528 > 231 = 1000/1.05^{30}.$$

This kind of consideration implies that the yield on a long-term investment can be much lower than the average of the expected one-year yields over the investment's life. Thus, in this example, the annual yield on the thirty-year bond is only 2.1%.

The second part of Result 2 (that r_{long} is near the minimal short-term yield when the latter is highly persistent) echoes the logic of Weitzman (1998). Intuitively, suppose that, instead of 30 years, we use 3000 years in the above calculation. Then, the yield on the long-term bond falls to essentially 0. Note this conclusion is true even if the probability of getting heads is very close to 1 (say, 0.9999). This calculation illustrates the second part of Result 2: if short-term interest rates are highly persistent, then the long-term yield is close to the

¹See, among others, Litterman, Scheinkman and Weiss (1991) and Gilles (1996) for discussions of what is often termed the *convexity* factor in the determination of longer-term yields.

²See Newell and Pizer (2003).

minimal short-term yield.

This paper is related to two main strands of literature. The first is the pricing theory for long-term assets, as developed by Alvarez and Jermann (2005), Hansen and Scheinkman (2009), and Martin (2012). This research is particularly salient because, in much of the paper, I follow Jiang, et al. (2022) and treat asset prices as exogenous.

The second relevant line of research dates back to the seminal work of Samuelson (1958). It studies what Brunnermeier, Merkel, and Sannikov (2022) term public debt bubbles, in which government liabilities have positive value even though primary surpluses are non-positive. There has been a recent revival of interest in this phenomenon, spurred in no little part by Blanchard (2019)'s provocative paper.

However, the research on public debt bubbles in the presence of stochastic interest/growth rates is (surprisingly) limited. Peled (1982) and Manuelli (1990) provide sufficient conditions for the existence of a monetary equilibrium (that is, a public debt bubble) in classes of stochastic overlapping generations (OG) economies without growth. Their conditions are both strictly stronger than the " $r_{long} \leq 0 (= g)$ " restriction developed in this paper.

Chattopadhyay and Gottardi (1999) (CG) also study a class of non-growing stochastic OG models. Their Theorem 4 provides a simple condition on autarkic contingent claims prices under which there exists an allocation that is (conditionally) Pareto superior to autarky. Their condition is mathematically equivalent to requiring r_{long} to be non-positive (as is derived in this paper). However, CG do not make any connection between their characterization and the yield curve (or to public debt bubbles).³

2 Model

This section presents the baseline model, which treats asset prices as exogenous.

Time is discrete. Let (Ω, F, Pr) be a probability space. For now, I assume that financial

³See also Aiyagari and Peled (1991). Of course, the connections to the yield curve are now much clearer thanks to research done in the decade after CG was published.

markets are complete with respect to this probability space. Later, I describe how this assumption can be relaxed considerably.

The risk-neutral probability measure implied by asset prices is denoted by Pr^* . As is standard, Pr^* and Pr are assumed to be equivalent, meaning that the characterization “almost everywhere” has the same content for the two measures. Economically, this means that a claim to consumption in some event has a positive price if and only if that event has a positive true probability of occurring.

Let the stochastic process $\{x_t\}_{t=1}^\infty$ be a time-homogeneous Markov process⁴, with respect to Pr^* , that has state space X . Consider a one-period bond that pays off one unit without risk. (I am deliberately agnostic about what “units” mean here, so that the bonds could be nominal or real.) I assume that the price of this one-period bond at any date is a time-invariant positive function q^1 of the Markov state, so that:

$$q_t^1 = q^1(x_t) \equiv \exp(-y^1(x_t))$$

Here, $y^1(x_t) = -\ln(q^1(x_t))$ is the yield on the one-period bond in state x_t .

Consider an N -period zero-coupon bond trading at date t that pays off one unit without risk in period $(t + N)$. Denote its price by q_t^N and its continuously compounded yield by:

$$y_t^N = -\ln(q_t^N)/N$$

⁴Formally, let X be a Borel subset of a Euclidean space and $B(X)$ represent the Borel subsets of X . Suppose there is a Markov kernel p^* so that:

$$p^* : X \times B(X) \rightarrow [0, 1]$$

For any B in $B(X)$, $p^*(\cdot, B)$ is Borel-measurable

For any x , $p^*(x, \cdot)$ is a probability measure over $B(X)$.

Let μ_1 be a probability measure over $B(X)$. Then, the joint probability Pr^* of any event $(A_1 \times A_2 \times A_3 \times \dots \times A_t)$, where A_t is in $B(X)$, can be computed as:

$$\int_{A_1} \int_{A_2} \dots \int_{A_{t-1}} \int_{A_t} p^*(x_{t-1}, dx_t) p^*(x_{t-2}, dx_{t-1}) \dots p^*(x_1, dx_2) \mu_1(dx_1).$$

A N -period bond is a one-period promise to receive an $(N - 1)$ -period bond. Hence, yields satisfy the following recursion:

$$\exp(-Ny_t^N) = \exp(-y^1(x_t))E^*(\exp(-(N - 1)y_{t+1}^{N-1})|x_t).$$

It follows that the yield y_t^N at date t is a time-and-state-invariant function y^N of the Markov state x_t , which takes the form:

$$\begin{aligned} y^N(x_t) &= -\ln(E^*(\prod_{s=1}^N \exp(-y^1(x_{t+s-1})|x_t))/N) \\ &= -\ln(E^*(\exp(-\sum_{s=1}^N y^1(x_{t+s-1}))|x_t))/N \end{aligned}$$

With these representations in hand, I make the following assumption about long-term bond yields.

Assumption 1: There exists a constant r_{long} and a bounded function $\phi : X \rightarrow \mathbb{R}$ such that

$$\lim_{N \rightarrow \infty} N(y^N(x) - r_{long}) = \phi(x) \tag{1}$$

almost everywhere.

Recall that Pr and Pr^* agree on what they imply about “almost everywhere”. Hence, Assumption 1 can be viewed as being stated in terms of either the risk-neutral or true probabilities.

Assumption 1 immediately implies that the (very) long-term yield does not vary with the state x :

$$\lim_{N \rightarrow \infty} y^N(x) = r_{long}.$$

However, it also has the stronger requirement that the rate of convergence of $y^N(x)$ to r_{long} , with respect to N , is sufficiently fast so that the sequence $N(y^N(x) - r_{long})$ does not

explode to infinity (in absolute value) as N grows to infinity.

The following proposition applies the Perron-Frobenius Theorem to show that Assumption 1 is satisfied whenever x_t follows a finite-state Markov chain with a positive⁵ transition matrix P^* . The statement of the proposition uses the notation $\max(\text{eig}(M))$ to refer to the maximal eigenvalue of a matrix M .

Proposition 1. *Suppose that, under Pr^* , $\{x_t\}_{t=1}^\infty$ is governed by a Markov chain with a state space $\{1, 2, \dots, J\}$ and a positive transition matrix P^* . Let q_i^1 be the price of a one-period bond in state i . Define a (positive) matrix Q^* via $Q_{ij}^* = P_{ij}^* q_i^1$, $i, j = 1, \dots, J$. Then Assumption 1 is satisfied, with:*

$$r_{long} = -\ln(\max(\text{eig}(Q^*))).$$

Proof. In Appendix. □

Note that in the statement of Proposition 1, the matrix element Q_{ij}^* is equal to the stochastic discount factor from state i to state j .

More generally, Assumption 1 is a consequence of the analysis of Hansen and Scheinkman (2009, p. 214). It is also similar to Assumption 1 in Alvarez and Jermann (2005, p. 1982). It is implied too by the long-run risk asset pricing model of Bansal and Yaron (2004), which is itself a generalization of the standard representative agent power utility model.⁶

Like Newell and Pizer (2003), Gollier (2015) argues that short-term interest rates should be viewed as following a random walk. This specification violates Assumption 1. However,

⁵The proposition also applies to any primitive transition matrix (so that there exists a natural number τ such that $P^{*\tau}$ is a positive matrix). In this way, it can be extended to cover second-order Markov chains.

⁶See Bansal and Shaliastovich (2013) for technical details. Their model (equations (26)-(27) on page 17) implies that for both real and nominal yields,

$$\lim_{N \rightarrow \infty} (Ny_t^N - Nr_{long})$$

is a time-invariant linear function (which is ϕ in Assumption 1) of the four state variables (which constitute x_t in Assumption 1). The long-run yield r_{long} can be computed as $\lim_{N \rightarrow \infty} (B_{0,N} - B_{0,N-1})$ in their equation (A13) on page 31.

Assumption 1 does accommodate processes that are very close to being random walks. I discuss the potentially important implications of examples of such processes in Section 3.6.

It will sometimes be useful to strengthen Assumption 1 by adding the following uniform boundedness restriction.

Assumption 1*: Assumption 1 is satisfied. In addition, there exists a constant $k > 0$ such that if $N \geq N^*$, then $|Ny^N(x) - Nr_{long} - \phi(x)| \leq k$ almost everywhere, where ϕ is defined as in (1).

Readers who are willing to proceed under the assumption that the state space X is finite can ignore Assumption 1*, as it is implied by Assumption 1 in that case. If X is infinite, the stronger Assumption 1* provides a justification for applying the Bounded Convergence Theorem to the sequence:

$$\{exp(-Ny^N(x) + Nr_{long})\}_{n=1}^{\infty},$$

which allows key limits to be passed from outside to inside (conditional) expectations.

3 Infinite Debt Rollover

This section contains the main results. It considers a setting in which growth is deterministic but bond yields are stochastic, and asks under what conditions an infinite debt rollover is sustainable. It shows that an infinite debt rollover is sustainable if and only if the growth rate is no smaller than r_{long} , the interest rate at the far right of the yield curve. It shows too that r_{long} is, in at least a couple of senses, low relative to one-period yields.

3.1 Definition

This subsection defines what is meant by a sustainable debt rollover. As noted in the introduction, the relevant growth is not that of the borrower's income: the borrower's position is self-financing, and so they need never use their income. What matters instead is that the

growth of available loanable funds is sufficiently large to allow the borrower to keep rolling over their debt.

Suppose that the available loanable funds $\{L_t\}_{t=1}^{\infty}$ grow at a constant rate g , so that:

$$L_t = \exp(tg)L_0$$

for some positive L_0 . There is a *sustainable infinite debt rollover* if the borrower can construct a self-financing chain of debt issues while keeping the debt to available loanable funds ratio bounded. Mathematically, an infinite debt rollover is sustainable if there exists a real number $\lambda \geq 1$ and a bounded function $v : X \rightarrow \mathbb{R}$ so that:

$$\exp(v(x)) = \frac{\exp(-y^1(x))\exp(g)}{\lambda} E^*(\exp(v(x'))|x). \quad (2)$$

Here, $\exp(v(x_t))L_t$ represents the amount of debt issued in period t in state x_t . Notice that v is only determined up to an arbitrary constant⁷, so that the upper bound on $\exp(v(x))$ can be made as small as is deemed plausible.⁸

Why is (2) the appropriate formulation to think about infinite debt rollover? Consider a borrower who owes $\frac{\exp(v(x_t))L_t}{\lambda}$ units in state x_t . The borrower sells a bond that promises to pay:

$$(\exp(v(x_{t+1}))L_{t+1}/\lambda)$$

in period $(t + 1)$. That sale will raise:

$$L_t \exp(g) \frac{\exp(-y^1(x_t)) E^*(\exp(v(x_{t+1}))|x_t)}{\lambda} = \exp(v(x_t)) L_t$$

⁷Consider the linear functional operator $W(f)(x) = \exp(-y^1(x))\exp(g)E^*(f(x')|x)$. Then, $\exp(v)$ is an eigenfunction of this operator W with eigenvalue λ .

⁸In particular, the upper bound can be chosen to be less than one.

units in state x_t . The borrower can use that to pay off its obligations in state x_t , because:

$$\exp(v(x_t))L_t \geq \exp(v(x_t))L_t/\lambda.$$

In this sense, debt can be rolled over forever.

3.2 Non-Random Yields

The following proposition can be seen as a proof of concept. It shows that a usual result works in this setting: if the one-period interest rate is non-random, infinite debt rollover is sustainable if and only if that interest rate is no larger than the growth rate g .

Proposition 2. *Suppose the one-period yield $y^1(x) = \bar{r}$ for almost all x in X . Then, an infinite debt rollover is sustainable if and only if $g \geq \bar{r}$.*

Proof. Suppose $g \geq \bar{r}$. Pick any constant \bar{v} and let $\lambda = \exp(g - \bar{r}) \geq 1$. Then:

$$\exp(\bar{v}) = \frac{\exp(g)\exp(-y^1(x))}{\lambda} \exp(\bar{v})$$

for all x , which implies that an infinite debt rollover is sustainable.

Conversely, suppose that an infinite debt rollover is sustainable. There exists $\lambda \geq 1$ and a bounded real-valued function v such that:

$$\begin{aligned} \exp(v(x_t)) &= \frac{\exp(g)\exp(-\bar{r})}{\lambda} E^*(\exp(v(x_{t+1})|x_t) \\ \exp(v(x_t)) &= \frac{\exp(Ng)\exp(-N\bar{r})}{\lambda^N} E^*(\exp(v(x_{t+N})|x_t). \end{aligned}$$

Let $\{v_{min}, v_{max}\}$ be lower and upper bounds for v . Taking logs:

$$v(x_t) \geq Ng - N\bar{r} - N\ln(\lambda) + v_{min}$$

$$v(x_t) \leq Ng - N\bar{r} - N\ln(\lambda) + v_{max}$$

Dividing by N and taking limits, we obtain:

$$(g - \bar{r} - \ln(\lambda)) = 0,$$

which establishes the result. □

3.3 Main Result

Proposition 2 restricts attention to situations in which the short-run interest rate is constant at \bar{r} . In these situations, the yield curve is also flat, so all longer-term yields also equal \bar{r} .

But what happens when interest rates are stochastic? The following (main) proposition demonstrates that the relevant comparison is between g and r_{long} .

Proposition 3. *Suppose that Assumption 1* is satisfied. An infinite debt rollover is sustainable if and only if the growth rate g is no smaller than the long-term yield r_{long} . In that case, the rollover factor λ (in (2)) is equal to $\exp(g - r_{long})$.*

Proof. Suppose an infinite debt rollover is sustainable. Then there exists $\lambda \geq 1$ and a bounded real-valued function v such that:

$$\begin{aligned} \exp(v(x_t)) &= \lambda^{-1} \exp(-y^1(x_t)) \exp(g) E^*(\exp(v(x_{t+1})) | x_t) \\ &= \lambda^{-N} \exp(Ng) E^*\left(\prod_{n=1}^N \exp(-y^1(x_{t+n-1})) \exp(v(x_{t+n}))\right) | x_t \end{aligned}$$

where the second equality is a consequence of recursing forwards. The function v is bounded from above by some v_{max} and from below by some v_{min} . Hence, taking logs:

$$\begin{aligned} v(x_t) &\leq -N \ln(\lambda) + Ng - Ny^N(x_t) + v_{max} \\ v(x_t) &\geq -N \ln(\lambda) + Ng - Ny^N(x_t) + v_{min} \end{aligned}$$

Divide by N on both sides and take limits. We get:

$$\begin{aligned} 0 &= -\ln(\lambda) + g - \lim_{N \rightarrow \infty} y^N(x_t) \\ 0 &= -\ln(\lambda) + g - r_{long}. \end{aligned}$$

Hence, $g - r_{long} = \ln(\lambda) \geq 0$.

Now suppose that $g \geq r_{long}$ and define $\hat{\lambda} = \exp(g - r_{long}) \geq 1$. Then:

$$\begin{aligned} \lim_{N \rightarrow \infty} \hat{\lambda}^{-N} \exp(Ng) E^* \left(\prod_{n=1}^N \exp(-y^1(x_{t+n-1})) \middle| x_t \right) \\ &= \lim_{N \rightarrow \infty} \hat{\lambda}^{-N} \exp(Ng) \exp(-Ny^N(x_t)) \\ &= \lim_{N \rightarrow \infty} \exp(Nr_{long}) \exp(-Ny^N(x_t)) \\ &= \lim_{N \rightarrow \infty} \exp(-N(y^N(x_t) - r_{long})) \\ &= \exp(-\phi(x_t)) \end{aligned}$$

where ϕ is defined as in (1). Hence, ϕ satisfies:

$$\begin{aligned} \exp(-\phi(x_t)) &= \hat{\lambda}^{-1} \exp(g) \lim_{N \rightarrow \infty} \hat{\lambda}^{-N+1} \exp((N-1)g) E^* \left(\prod_{n=1}^N \exp(-y^1(x_{t+n-1})) \middle| x_t \right) \\ &= \hat{\lambda}^{-1} \exp(g) \exp(-y^1(x_t)) \lim_{N \rightarrow \infty} \hat{\lambda}^{-N+1} \exp((N-1)g) E^* \left(\left(\prod_{n=2}^N \exp(-y^1(x_{t+n-1})) \right) \middle| x_{t+1} \right) \middle| x_t \\ &= \hat{\lambda}^{-1} \exp(g) \exp(-y^1(x_t)) \lim_{N \rightarrow \infty} E^* \left(\exp((N-1)r_{long} - (N-1)y^{N-1}(x_{t+1})) \middle| x_t \right) \\ &= \hat{\lambda}^{-1} \exp(g) \exp(-y^1(x_t)) E^* \left(\exp(-\phi(x_{t+1})) \middle| x_t \right) \end{aligned}$$

The last step is an application of the bounded convergence theorem (justified by Assumption 1*) to the sequence:

$$\left\{ \exp((N-1)r_{long} - (N-1)y^{N-1}(x_{t+1})) \right\}_{N=1}^{\infty}$$

It follows that $v = -\phi$ and $\lambda = \hat{\lambda}$ jointly satisfy the restriction (2). \square

Thus, it is possible to keep rolling over debt as long as the growth rate is no smaller than the long-term yield.

3.4 An Upper Bound For r_{long}

This subsection shows that r_{long} is weakly bounded below the risk-neutral expectation of shorter-term yields. The bound is strict for one-period yields if they are are volatile.

The first result derives a (weak) upper bound in terms of S -period bond yields, for $S \geq 1$.

Proposition 4. *Let $\{x_t\}_{t=1}^{\infty}$ be strictly stationary under Pr^* , and suppose Assumption 1* is satisfied. Then:*

$$E^*(y^S(x_t)) \geq r_{long}$$

for any $S \geq 1$.

Proof. In Appendix. \square

Using E^* rather than E eliminates the impact of risk on yields. The convexity effect then tilts the risk-neutral expectation of the yield curve downward.

Mathematically, Proposition 4 is a simple consequence of Jensen's inequality. With that motivation in mind, the following corollary shows that the upper bound for one-period yields becomes strict if they are volatile (and the Markov process satisfies a regularity restriction⁹, so that x_t does not always fully reveal x_{t+1}).

Corollary 1. *Suppose that the hypotheses of Proposition 4 are satisfied, and that the one-period yields y^1 satisfy $Var^*(y^1(x_t)) > 0$. Suppose in addition that the Markov process $\{x_t\}_{t=1}^{\infty}$ satisfies the restriction that for any $f : X \rightarrow \mathbb{R}$, $Var^*(f(x_t)) > 0 \Rightarrow E^*(Var^*(f(x_{t+1})|x_t)) > 0$. Then $E^*(y^1(x_t)) > r_{long}$.*

⁹Corollary 1 can be stated in terms of S -period yields, for $S > 1$, given an appropriate alteration in the regularity restriction.

Proposition 2 shows that the upper bound in Proposition 4 is attained in the case of non-random one-period interest rates. Corollary 1 shows that, given the “variance \rightarrow conditional variance” restriction on the Markov process, the upper bound is attained only in that case.

Proposition 4 is formulated in terms of $E^*(y_t^S)$. This parameter is not the same as $E(y_t^S)$ (the unconditional expectation with respect to the true probability measure Pr). Under (substantive) stationary and ergodicity assumptions, the latter can be estimated consistently using time-series averages. Estimating the former requires an additional model of risk. However, short-term US government bond returns are often treated as having negligible risk premia. Under this assumption:

$$E^*(y_t^1) \approx E(y_t^1).$$

and Corollary 1 can then be seen as implying that the estimable (true) expectation of (sufficiently) short-term interest rates should exceed r_{long} .

It is illustrative to see how Corollary 1 works when $\{x_t\}_{t=1}^{\infty}$ is i.i.d. over time (with respect to Pr^*). Then, it can be shown that:

$$\begin{aligned} r_{long} &= \lim_{N \rightarrow \infty} y^N(x_t) \\ &= \lim_{N \rightarrow \infty} -N^{-1} \ln(E^*(\prod_{n=0}^{N-1} \exp(-y^1(x_{t+n})) | x_t)) \\ &= \lim_{N \rightarrow \infty} -N^{-1} \ln(\prod_{n=0}^{N-1} E^*(\exp(-y^1(x_t)))) \\ &= -\ln(E^*(\exp(-y^1(x_t)))). \end{aligned}$$

From Jensen’s inequality, $r_{long} < E^*(y^1(x_t))$, which is just what Corollary 1 says.

But in this i.i.d. case, the difference between r_{long} and $E^*(y_t^1)$ is small for plausible specifications of short-term yields. (After all, we typically think nothing of approximating both \exp and \ln linearly in this context.) The next subsection will highlight the role of persistence in generating a large gap between r_{long} and (starred) expectations of one-period yields.

3.5 A Lower Bound for r_{long}

This subsection provides an extreme lower bound for r_{long} . More importantly, it shows in two important classes of models that this extreme lower bound is approximately attained when the process for one-period yields is highly persistent.

The following proposition shows that the long-term yield r_{long} can be no smaller than the lowest realization of one-period yields.

Proposition 5. *Suppose that $r_{min} = \inf_{x \in X} y^1(x)$. Then $r_{long} \geq r_{min}$.*

Proof. The supposition implies that for all x :

$$\exp(-y^1(x)) \leq \exp(-r_{min}).$$

Recall that:

$$\begin{aligned} -r_{long} &= \lim_{N \rightarrow \infty} y^N(x) \\ &= \lim_{N \rightarrow \infty} N^{-1} \ln(E_t^* \exp(\sum_{n=0}^{N-1} -y^1(x_{t+n}))) \\ &\leq -r_{min} \end{aligned}$$

which proves the theorem. □

The lower bound is intuitive.

What is more interesting is that the lower bound is approximately attained when the process for short-term yields is highly persistent. The intuition is similar to that discussed in the introduction and described in Weitzman (1998). Under a highly persistent process, a buy-and-hold-forever purchaser of a long-term bond compares that opportunity to (eventually) rolling over a sequence of short-term bonds given a randomly determined, but known to be nearly flat, yield curve. The price of the long term bond (that pays off one unit in N periods)

thus takes the form:

$$E^* \exp(-Ny^1)$$

where y^1 is a random one-period yield. When N is large, this expectation is dominated by (the high price outcome) $\exp(-Nr_{min})$, where r_{min} is the lowest possible realization of the one-period yield y^1 (and assuming that realization has a positive probability).

The next two propositions make this point more formally and are, as far as I know, new to this paper. The first deals with Markov chains, while the second handles the case in which the one-period yield follows an autoregression (akin to Vasicek (1977)'s Gaussian model, but with bounded support). To establish the Markov chain result, suppose (as in Proposition 1) that $\{x_t\}_{t=1}^\infty$ is governed by a Markov chain with state space $\{1, 2, \dots, J\}$. Consider a sequence of economies indexed by the natural numbers. In any economy m , the (state-contingent) one-period bond prices are given by $\{q_i^1\}_{i=1}^J$ (and so are independent of m), and the positive¹⁰ transition matrix (with respect to Pr^*) is P_m^* . Then, the following proposition describes what happens as the Markov chain becomes increasingly persistent.

Proposition 6. *Suppose that the sequence $\{P_m^*\}_{m=1}^\infty$ of Markov matrices converges (in the sup-norm) to the identity matrix. Let $r_{long,m}$ be the long-term yield in economy m . Then:*

$$\lim_{m \rightarrow \infty} r_{long,m} = r_{min} \equiv \min_j -\ln(q_j^1).$$

Proof. In each economy m , define the matrix Q_m^* by setting $Q_{m,ij}^* = P_{m,ij}^* q_i^1$. We know from Proposition 1 that:

$$r_{long,m} = -\ln(\max(\text{eig}(Q_m^*)))$$

¹⁰As with Proposition 1, this restriction can be relaxed to the requirement that P_m^* is primitive for all m .

The limit $Q_\infty^* = \lim_{m \rightarrow \infty} Q_m^*$ is a diagonal matrix, with $Q_{\infty,ii}^* = q_i^1$. Hence:

$$\begin{aligned} \lim_{m \rightarrow \infty} r_{long,m} &= -\ln(\max(\text{eig}(Q_\infty^*))) \\ &= -\ln(\max_i q_i^1) \\ &= \min_i -\ln(q_i^1) \\ &= r_{min}, \end{aligned}$$

which proves the proposition. □

The proposition is a straightforward application of Proposition 1 to near-diagonal transition matrices - that is, highly persistent shocks.¹¹

The next proposition covers the case in which one-period bond yields follow an autoregression. To be specific, suppose that the one-period bond yields are governed by the process:

$$y_{t+1}^1 = (1 - \rho)\mu_y + \rho y_t^1 + \varepsilon_{t+1}(1 - \rho^2)^{1/2}, t \geq 1, 0 < \rho < 1 \quad (3)$$

where, under Pr^* , $\{\varepsilon_t\}_{t=1}^\infty$ is an i.i.d sequence of random variables that have mean zero and have bounded support. The initial bond yield y_1^1 is drawn from the stationary distribution for bond yields (which is given by the distribution of $(1 - \rho^2)^{1/2} \sum_{n=0}^\infty \rho^n \varepsilon_n$, where $\{\varepsilon_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables drawn from the distribution for ε_t).

Here, again, we will be interested in the properties of the long-run yield $r_{long}(\rho)$ when ρ is near 1. Note that the scaling factors $(1 - \rho)$ and $(1 - \rho^2)^{1/2}$ ensure that the first two moments of the stationary distribution of one-period bond yields are independent of ρ . This scaling helps clarify that it is the persistence of the process that is driving the result, not its mean or variance. (However, the proposition remains valid even if the scaling factors are

¹¹Proposition 6 is using a strong notion of persistence, as *all* eigenvalues of P_m^* are near 1 for m large. Intuitively, when all eigenvalues are close to 1, it takes a long time for the process to return to its stationary distribution, given any initial state. It would be more standard to measure persistence of a Markov chain through the size of the second-highest eigenvalue of the transition matrix. Proposition 6 is not valid with this notion of persistence.

dropped.)

Proposition 7. *Consider a set of economies indexed by $\rho \in (0, 1)$. Suppose that in economy ρ , the one-period bond yields follow the process (3). Suppose that (under Pr^*), ε has an atom at its lower bound $\varepsilon_{min} < 0$ OR it has a continuous density that is positive at ε_{min} . Then:*

$$\lim_{\rho \rightarrow 1} \frac{r_{long}(\rho)}{r_{min}(\rho)} = 1,$$

where, in the economy indexed by ρ , $r_{min}(\rho)$ is the lowest possible realization of y_t^1 and $r_{long}(\rho)$ is the long-term yield.

Proof. In Appendix. □

Because of the scaling factors, the unconditional mean and variance of y_t^1 are both independent of ρ :

$$\begin{aligned} E^*(y_t^1) &= \mu_y \\ Var^*(y_t^1) &= Var^*(\varepsilon_t^1) \end{aligned}$$

Nonetheless, the proof of the proposition shows that: -

$$\begin{aligned} \lim_{\rho \rightarrow 1} (\mu_y - r_{min}(\rho)) &= \infty \\ \lim_{\rho \rightarrow 1} (\mu_y - r_{long}(\rho)) &= \infty. \end{aligned}$$

As ρ nears 1, the lowest possible realization of y_t^1 is becoming small (that is, highly negative) and so is the yield on a long-term bond. The point of the proposition is that their ratio nears 1 as ρ approaches 1.

The proposition (or at least its proof) requires that the (starred) distribution of ε_{t+1} has sufficient mass near ε_{min} (an atom or a positive continuous density). The needed assumption applies to a wide range of commonly used distributions with bounded support, including

any truncations of unbounded distributions. Note that the restriction is equivalent whether stated in terms of Pr^* or Pr .

3.6 A Quantitative Illustration

This subsection illustrates the possibility and properties of sustainable infinite debt rollovers in a simple quantitative model. The example makes four main points.

- As suggested by Proposition 6, r_{long} can be much lower than the (starred) expectation of short-term yields when they are persistent.
- The thirty-year yield may differ considerably from r_{long} .
- Relatedly, the thirty-year yield may have substantial variance, even though the long-term yield is a constant.
- A sustainable debt rollover with stochastic interest/growth rates may rely on debt levels being highly sensitive to the relevant shocks if they are very persistent.

In this example, the one-month interest rate follows a Markov chain. I set the one-month growth rate to be constant at $1.5\%/12$.

3.6.1 Case 1: “Low” Persistence

Suppose that the one-month bond yield y_t^1 follows a two-state Markov chain with state space $\{0.06/12, -0.02/12\}$ and transition matrix:

$$P^* = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix} \quad (4)$$

where all probabilities are with respect to the risk-neutral measure. The initial y^1 is drawn from the stationary distribution $(\frac{1}{2}, \frac{1}{2})$.

In this setting, the price $q_i^T, i = 1, 2$, of a T -period bond is a function of the short-term bond price and satisfies:

$$\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} = (\hat{Q}^*)^{T-1} \begin{bmatrix} q_1^1 \\ q_2^1 \end{bmatrix}$$

where $q_i^1 = \exp(-y_i^1), i = 1, 2$, and $Q_{ij}^* = P_{ij}^* q_i^1, i = 1, 2$. The annualized long-term yield r_{long} is then given by:

$$-12 * \ln(\max(\text{eig}(Q^*))) = 1.35\%$$

where $\text{eig}(Q^*)$ represents the eigenvalues of Q^* . Note that, as in Corollary 1, this is smaller than the unconditional expectation of annualized one-month yields (calculated using the (starred) stationary distribution):

$$6\%/2 - 2\%/2 = 2\%$$

Since $r_{long} = 1.35\% < 1.5\% = g$, it is possible to construct a sustainable infinite debt rollover. In that construction, the debt-loanable funds ratio depends on the Markov state. The dependence is governed by the eigenvector $(0.72, 1)'$ of \hat{Q}^* that corresponds to its largest eigenvalue. As a result, the debt-loanable funds ratio must fall by 28% from state 2 (when yields are low) to state 1 (when they are high).

Regardless of the realization of the Markov state, the long-term yield is constant at 1.35%. However, it is important to note that, because of the persistence of short-run yields, there is nontrivial variability in the 30-year (360 month) yield. In state 1, the 30-year yield is 2%. In state H , the 30-year yield is 0.9%. Hence, the 30-year yield may not be a good guide to r_{long} .

3.6.2 Case 2: High Persistence

It is instructive to perturb the above example to illustrate the effects of near-unit root behavior in short-term interest rates. Consider the following transition matrix:

$$P' = \begin{bmatrix} 0.99999 & 0.00001 \\ 0.0001 & 0.9999 \end{bmatrix}$$

instead of the specification in (4). The stationary distribution associated with P' is now $(10/11, 1/11)$ and so puts considerably more weight on the high-interest-rate state. However, there is a lot more persistence embedded in P' than in P . As shown in Proposition 6, this degree of persistence implies that the annualized long-term yield (calculated as above, except with the substitution of P' for P) is close to the minimal realization of the short-run yield:

$$\hat{r}_{long} = -1.88\%$$

This is much below the (starred) expectation of the short-term interest rate (which is now 5.3%).

The thirty-year yield displays considerable variance. If the annualized short-term interest rate is high (6%), the annualized thirty-year yield is 5.97%. If the annualized short-term yield is low (-2%), the annualized thirty-year yield is -1.93%. Finally, the debt-to-loanable funds ratio in the high-yield state is about 0.15% (that is, less than one-seven-hundredth) of that in the low-yield state.

4 Stochastic Growth

Until now, the growth of available loanable funds has been assumed to be deterministic. This section extends the above analysis to the case in which that growth is stochastic.

4.1 A Basic Equivalence

Suppose that the growth rate of loanable funds L_{t+1} is a fixed bounded function g of the state x_{t+1} :

$$L_{t+1}/L_t = \exp(g(x_{t+1})).$$

Given this definition, we can extend (2) to say that an infinite debt rollover is sustainable if there exists a bounded function v and a constant $\lambda \geq 1$ such that:

$$\exp(v(x)) = \frac{\exp(-y^1(x))}{\lambda} E^*(\exp(v(x'))\exp(g(x'))|x) \quad (5)$$

for almost all x . By multiplying through by $\exp(g(x))$, we can say that an infinite debt rollover is sustainable if there is a bounded function $\hat{v}(= v + g)$ and a constant $\lambda \geq 1$ such that:

$$\exp(\hat{v}(x)) = \frac{\exp(-y^1(x))\exp(g(x))}{\lambda} E^*(\exp(\hat{v}(x'))|x). \quad (6)$$

In (6), the (logged) ratio of period $(t + 1)$ debt to period t (lagged) loanable funds ratio is given by $\hat{v}(x)$ (which is only determined up to a constant).

The above expression suggests that adding stochastic growth is equivalent to subtracting growth rates from yields. Along those lines, consider a *detrended economy* in which growth $\hat{g} = 0$ (so that L_t does not have a stochastic or deterministic trend) and the one-period bond yield is given by:

$$\hat{y}^1(x) = y^1(x) - g(x)$$

in state x . The following proposition shows that the sustainability of infinite rollover is equivalent in the original economy and the detrended economy.

Proposition 8. *Consider a detrended economy in which the available loanable funds are constant over time ($g = 0$) and the one-period bond yield is given by:*

$$\hat{y}^1(x) = y^1(x) - g(x).$$

Then, an infinite debt rollover is sustainable in the detrended economy if and only if an infinite debt rollover is sustainable in the original economy (with stochastic growth).

Proof. It follows from a straightforward comparison of (2) and (6). \square

The absence of arbitrage implies that in the detrended economy, the yield to an N -period bond in state x_t is given by:

$$\begin{aligned}\hat{y}^N(x_t) &= N^{-1} \ln(E^*(\exp(\sum_{s=1}^N \hat{y}^1(x_{t+s-1}) | x_t)) \\ &= N^{-1} \ln(E^*(\exp(\sum_{s=1}^N (y^1(x_{t+s-1}) - g(x_{t+s-1})) | x_t))\end{aligned}$$

I assume that Assumption 1 (and, when necessary, Assumption 1*) apply to the detrended yields \hat{y}^N . Hence, there exists a constant \hat{r}_{long} such that:

$$\lim_{N \rightarrow \infty} \hat{y}^N(x) = \hat{r}_{long}.$$

Here, \hat{r}_{long} is the (very) long-term (zero-coupon bond) yield in the detrended economy. As before, Assumption 1 ensures that it is independent of the state x .

Note that we can readily extend Proposition 1 to show that Assumption 1 is satisfied in the detrended economy if x_t is governed by a Markov chain with a strictly positive transition matrix P^* under the risk-neutral measure Pr^* . In that case:

$$\hat{r}_{long} = -\ln(\max(\text{eig}(\hat{Q}^*)),$$

where \hat{Q}^* is defined via:

$$\begin{aligned}\hat{Q}_{ij}^* &= P_{ij}^* \exp(-\hat{y}_i^1) \text{ for all } i, j \\ &= P_{ij}^* \exp(-y_i^1) \exp(g_i) \text{ for all } i, j\end{aligned}$$

4.2 Ensuring A Sustainable Infinite Debt Rollover

The following proposition is a simple application of Proposition 3.

Proposition 9. *Suppose Assumption 1* is satisfied in the detrended economy. An infinite debt rollover is sustainable in the economy with stochastic growth if and only if the long-term yield \hat{r}_{long} in the detrended economy is non-positive. The factor λ (in (5)) is equal to $\exp(-\hat{r}_{long})$.*

Proof. Proposition 8 showed that an infinite debt rollover is sustainable in the economy with stochastic growth if and only if an infinite debt rollover is sustainable in the detrended economy without growth. But Proposition 3 implies that an infinite debt rollover is sustainable in the detrended economy if and only if the long-term yield \hat{r}_{long} is non-positive. \square

Note that if $g(x) = \bar{g}$ for almost all x , $\hat{r}_{long} = r_{long} - \bar{g}$, where r_{long} is the long-term yield in the original (undetrended) economy. Hence, Proposition 9 nests Proposition 3.

4.3 Bounds on \hat{r}_{long}

It is straightforward to extend Proposition 4 to the detrended economy to show that (under an appropriate stationarity assumption):

$$\hat{r}_{long} \leq E^*(y^1(x)) - E^*(g(x))$$

The long-term yield in the detrended economy is less than the (risk-neutral) expectation of the difference between one-period yields and growth rates.

Proposition 5 implies that:

$$\hat{r}_{long} \geq \hat{r}_{min} = \inf_{x \in X} (y^1(x) - g(x)).$$

More importantly, as was shown in Section 3.5,

$$\hat{r}_{long} \approx \hat{r}_{min}$$

if $(y_t^1 - g_t)$ is highly persistent.¹² In an economy in which y_t^1 is constant at \bar{r} (over dates and states), and growth is highly persistent, we can conclude that:

$$\hat{r}_{long} \approx \bar{r} - \sup_x g(x).$$

In this (very special) case, the maximal growth rate is what determines the sustainability of the infinite debt rollover.

5 Implementations

This section again considers the economy with stochastic growth described in Section 4. It starts by assuming that \hat{r}_{long} - the long-term bond yield in the detrended economy - is non-positive. In this case, Proposition 9 shows that there exists a constant $\lambda = \exp(-\hat{r}_{long}) \geq 1$ and a function \hat{v} such that:

$$\exp(\hat{v}(x)) = \frac{\exp(-y^1(x) + g(x))}{\lambda} E^*(\exp(\hat{v}(x'))|x) \quad (7)$$

for almost all x . I consider what kinds of borrowing schemes give rise to the borrower's achieving a ratio $(\exp(\hat{v}(x)))$ of period $(t + 1)$ debt to period t loanable funds across Markov states x .

Proposition 4 provides an important starting point for the analysis, as it creates the following possibility. Suppose that $\hat{y}_t^1 = (y_t^1 - g_t)$ is random (so it has positive variance).

¹²Note that $\hat{r}_{min} \geq \inf_{x \in X} y^1(x) - \sup_{x \in X} g(x)$, so that it is important to take account of the covariance between interest rates and growth rates.

Suppose too that its (starred) expectation exceeds 0, which in turn is larger than \hat{r}_{long} .

$$E^*(\hat{y}_t^1) > 0 > \hat{r}_{long}.$$

Then, given any T , there is some positive probability (under either Pr or Pr^*) event in which $y_t^1 > g_t$ for T consecutive periods. Hence, any attempt to simply roll over one-period risk-free debt will exhaust the available funds.

However, we shall see that the borrower can use:

- risky short-term debt
- OR zero-coupon consols (that is, money)

In the former implementation, the payoff of the risky debt exhibits a potentially rich dependence on the realization of the state next period. The latter implementation requires only a single infinitely-lived asset that makes no mandated payments of any kind.

5.1 Risky Short-Term Debt

This subsection shows how a borrower can use risky short-term debt to sustain a debt to lagged loanable funds ratio of $exp(\hat{v}(x))$, as a function of the Markov state.

Suppose that in state x_t the borrower issues one-period risky debt that promises:

$$\frac{exp(\hat{v}(x_{t+1}))L_t}{\lambda}$$

next period (as a function of the Markov state x_{t+1} and period t loanable funds L_t). The value of this debt in state x_t is:

$$\begin{aligned} & \frac{L_t exp(-y^1(x_t)) E^*(exp(\hat{v}(x_{t+1})|x_t))}{\lambda} \\ & = L_t exp(\hat{v}(x_t) - g(x_t)) \\ & = L_{t-1} exp(\hat{v}(x_t)). \end{aligned}$$

where the first equality is an implication of (7). Since $\lambda \geq 1$, the borrower can use these funds to pay off existing obligations $\frac{L_{t-1} \exp(\hat{v}(x_t))}{\lambda}$. In this way, the borrower can roll over this risky debt forever.

The basic idea here is that the state x_{t+1} captures the current and future evolution of bond prices. The borrower's repayment in period $(t + 1)$ responds to that risk. Thus, if x_{t+1} indicates that current and future bond prices will be high (low), then it is possible to borrow a lot (very little). The borrower's promised repayment in that state is correspondingly high (low).

Note that if \hat{r}_{long} is strictly negative, then this infinite debt rollover generates an income stream for the borrower in each Markov state that is proportional to loanable funds L_t . In particular, the borrower raises funds $\exp(\hat{v}(x_t))L_{t-1}$ and has to repay $\exp(\hat{v}(x_t))L_{t-1}\exp(\hat{r}_{long})$. Hence, the borrower generates income:

$$\exp(\hat{v}(x_t))L_{t-1}(1 - \exp(\hat{r}_{long}))$$

in each Markov state x_t .

5.2 Money

In this subsection, the borrower issues a single infinitely durable and divisible asset, which makes no coupon payments. I refer to this asset as being money.

The borrower enters period $(t + 1)$ with M_t units of money outstanding. The borrower creates and spends:

$$(\lambda - 1)M_t$$

units of money, so that:

$$M_{t+1} = \lambda M_t.$$

Here, $\lambda \geq 1$ is defined as in (7). (Note that λ may equal 1.) The desired debt-(lagged)

loanable funds ratio implies that the price of each unit of money in period $(t + 1)$ is given by:

$$\frac{\exp(\hat{v}(x_{t+1}))L_t}{M_{t+1}}.$$

How much is money worth in the prior period t ? We can discount its period $(t + 1)$ price to conclude that the price of each unit of money in period t is:

$$\frac{q^1(x_t)L_tE^*(\exp(\hat{v}(x_{t+1})|x_t)}{M_{t+1}}$$

The equation (7) then implies that this price is equal to:

$$\frac{L_t\exp(\hat{v}(x_t) - g(x_t))\lambda}{M_{t+1}} = \frac{L_{t-1}\exp(\hat{v}(x_t))}{M_t}.$$

Multiplying through by M_t/L_{t-1} , we find that the borrower's debt to (lagged) loanable funds ratio in period t is given by $\exp(\hat{v}(x_t))$, as was desired.

As in the prior risky debt implementation, the state x_{t+1} captures the current and future evolution of bond prices. But the payoff of the liability is independent of that risk. Instead, the price of money fluctuates in response to the Markov state. Thus, if x_{t+1} indicates that current and future bond prices will be high (low), then the price of money is high (low).

6 Public Debt Bubbles in Dynamic Stochastic General Equilibrium

The above results are all for a partial equilibrium setting, in which the borrower treats asset prices as given. This section uses these prior findings to understand the conditions that give rise to public debt bubbles in dynamic stochastic general equilibrium settings. The main theme of the analysis is that if a sustainable debt rollover is possible given autarkic asset prices, then there is an equilibrium with a stochastic public debt bubble. The first subsection

deals with overlapping generations (OG) economies¹³, while the second treats a (simple) class of Aiyagari (1994)-Bewley (1977)-Huggett (1993) (ABH) model economies.

6.1 An Overlapping Generations Model With Growth Shocks

This subsection analyzes overlapping generations models with shocks to lifetime income growth and to cohort-to-cohort income growth.

6.1.1 Model Description

Consider an overlapping generations economy in which a unit measure of agents are born at each date, and live for two periods (“young” and “old”). Their utility functions are given by:

$$\ln(c_y) + E(\ln(c_o))$$

where (c_y, c_o) are consumptions when young and old, respectively. In period t , young agents have an endowment e_{yt} and old agents have an endowment e_{ot} . There is an initial old cohort who prefers to consume more to less. The endowments evolve stochastically according to the laws of motion:

$$\ln(e_{y,t}) = g_t + \ln(e_{y,t-1}), t \geq 1$$

$$\ln(e_{o,t+1}) = r_t + \ln(e_{y,t}), t \geq 1$$

given $e_{y1} = 1$. I assume that (g_t, r_t) follow a time-homogeneous Markov chain with J states $(\bar{g}_i, \bar{r}_i)_{i=1}^J$ and a positive $J \times J$ transition matrix P .

¹³My analysis of OG economies in this paper abstracts from capital and considerations of dynamic efficiency. Abel and Panageas (2022) show how, in the presence of aggregate risk, an infinite debt rollover can be Pareto improving even though the level of capital is dynamically efficient. See also Kocherlakota (2022).

6.1.2 Benchmark Autarkic Equilibrium

Consider an autarkic equilibrium in this overlapping generations economy. In this equilibrium, the (shadow) one-period bond yield in period t is given by:

$$y_t^1 = \ln(e_{o,t+1}/e_{y,t}) = r_t$$

Hence, it evolves stochastically according to the Markov chain. Since the young agents face no future endowment risk, there is no difference between E^* and E . As well, the young agents are the ones that buy government debt, and so the growth rate g_t of their endowments from cohort to cohort determines the growth rate of available loanable funds.

With these considerations in mind, we can (as in Section 4) define:

$$\hat{Q}_{ij} = P_{ij} \exp(-\bar{r}_i + \bar{g}_i)$$

As in Proposition 1, the autarkic long-term yield in the detrended version of this economy is then:

$$\hat{r}_{long}^{aut} = -\ln(\max(\text{eig}(\hat{Q}))). \quad (8)$$

6.1.3 Monetary (Public Debt Bubble) Equilibrium

Now suppose that each initial old agent is endowed with one unit of a divisible durable good called money. Money is intrinsically useless, as it provides no direct utility to any agent.

The following proposition shows that if $\hat{r}_{long}^{aut} < 0$, there exists an equilibrium in the *monetary OG economy* in which money has positive value. Since there are no tax collections in the economy, the positive value of money implies that there is a public debt bubble.

Proposition 10. *Suppose \hat{r}_{long}^{aut} (defined as in (8)) is negative. Then there is an equilibrium in the monetary OG economy in which money has a strictly positive price $\Gamma_i e_{y,t-1}$ in period t as a function of the Markov state i and the young agents' endowment in the prior period.*

Proof. In Appendix. □

In the equilibrium described in Proposition 10, if the difference $(\bar{r}_i - \bar{g}_i)$ is persistent, the price of money moves inversely with $(\bar{r}_i - \bar{g}_i)$. Thus, inflation is high when the economy transits to a high $(r - g)$ state.

As noted above, this example is one in which E^* and E are the same in the autarkic equilibrium. But this restriction is easily relaxed. Suppose instead that there is a $J \times J$ matrix Ψ such that:

$$\ln(e_{o,t+1}) = \ln(e_{yt}) + \Psi_{i_t j_{t+1}}$$

where i_t is the Markov state in period t and j_{t+1} is the Markov state in period $(t + 1)$. Now young agents face endowment risk when they are old, so that the starred transition matrix is:

$$P_{ij}^* = \frac{P_{ij} \exp(-\Psi_{ij})}{\sum_{j=1}^J P_{ij} \exp(-\Psi_{ij})}.$$

Define \hat{Q}^* as:

$$\begin{aligned} \hat{Q}_{ij}^* &= P_{ij}^* \left(\sum_{j=1}^J P_{ij} \exp(-\Psi_{ij} + \bar{g}_i) \right) \\ &= P_{ij} \exp(-\Psi_{ij} + \bar{g}_i), i, j = 1, \dots, J \end{aligned}$$

In this economy, the relevant long-term yield for Proposition 10 is:

$$\hat{r}_{long}^{aut} = -\ln(\max(\text{eig}(\hat{Q}^*)). \tag{9}$$

6.1.4 Inflation and Deficits

Proposition 10 demonstrates the existence of an equilibrium with a public debt bubble in which the supply of money is constant. In that equilibrium, the price of money has a common stochastic trend with the endowments of the young agents (that is, the available loanable funds). But the primary deficit is zero.

The following proposition show that it is possible to construct an equilibrium public debt bubble that co-exists with a positive primary deficit. In particular, let π be an element of the interval $[0, -\hat{r}^{aut})$. The initial old each have M_0 units of money in period 1, where $M_0 = 1$. But now there is a government that runs a primary deficit: in particular, it makes a lump-sum transfer of $(exp(\pi) - 1)M_{t-1}$ units of money to each old agent at each date t . Then, there is an equilibrium in this economy in which the price of money has two trends: a stochastic trend from the endowment of the young agents and a deterministic (negative) trend from the growth of money.

Proposition 11. *Suppose \hat{r}_{long}^{aut} (defined as in (8)) is negative, and $\pi \in [0, -\hat{r}_{long}^{aut})$. The money supply grows at rate π through lump-sum transfers to the old agents. Then there is an equilibrium in which the price of money in period t is $\Gamma_i e_{y,t-1} exp(-\pi t) > 0$ in period t as a function of the Markov state i and the young agents' endowment.*

Proof. It is similar to the proof of Proposition 10, and is in the Appendix. □

With the growth in the supply of money, the government is able to run a primary deficit. The size of the deficit is restricted by how negative \hat{r}_{long}^{aut} is. The above model assumes that the government is using that deficit to make transfers to the old agents. But the proposition can be extended to an economy in which the government uses the newly created money to buy consumption from the young.

6.2 Tail Risk Economy

In this subsection, I add aggregate shocks to an ABH model similar to that analyzed in Kocherlakota (2021). There is no growth in this setup. I build on Proposition 3 to show that there is a monetary equilibrium in this economy if the *autarkic* long-term yield is negative (that is, less than the growth rate of 0). As above, positively valued money represents a public debt bubble.

6.2.1 Model Description

Suppose there is a unit measure of infinitely-lived agents. The agents' individual states evolve according to stochastically independent Markov chains with state space $\{H, L\}$ and positive transition matrix Π , where $\Pi_{HL} = \Pi_{LH} = p$. There is always an equal fraction of agents in the two states.

In state H , an agent is endowed with e_H units of consumption and has momentary utility function over consumption:

$$\ln(c_H).$$

An agent in state L in period t is endowed with $e_L > 0$ units of consumption and has momentary utility function:

$$\bar{\nu}_{t-1} c_L.$$

Here, ν_{t-1} is a *common* shock revealed in period $(t - 1)$; assume $\nu_0 = 1$. It then follows a Markov chain with state space $\{\bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_J\}$ and transition matrix P . Agents in state L are hand-to-mouth, so that they consume all available resources at each date.¹⁴ The common shocks affect the precautionary demand for savings at the different dates.

6.2.2 Autarkic Benchmark Equilibrium

Kocherlakota (2021) shows that if $J = 1$, and there exists $\Delta > 0$ such that:

$$\frac{1}{e_H - \Delta} = \frac{\beta(1 - p)}{e_H - \Delta} + \beta \bar{\nu}_1 p$$

then there is a bubbly stationary equilibrium in which the real interest rate is constant at zero, or equivalently where the price of money is constant at some $\Gamma > 0$. In this equilibrium, any agent in state H chooses to buy Δ/Γ additional units of money. They then divest their

¹⁴In contrast, in Kocherlakota (2021), agents in state L are allowed to choose their asset positions freely subject to a borrowing limit. The hand-to-mouth restriction simplifies the construction of a monetary equilibrium in the presence of aggregate shocks.

moneyholdings immediately upon transiting to state L .

Under what conditions does this kind of equilibrium exist if $J > 1$ (so that there are aggregate shocks to the precautionary demand for savings)? To address this question, consider an autarkic equilibrium in this economy, under the presumption that the agents in state H are the marginal asset buyers. In this equilibrium, the risk-neutral and true expectations are aligned (because the agents in state H know their marginal utility next period). The one-period yield evolves according to:

$$\exp(-r_i^{ABH}) = \beta(1 - p) + \beta p \bar{v}_i e_H, i = 1, \dots, J.$$

The growth of available loanable funds (from agents in state H) is deterministically zero.

Given these elements of the equilibrium, define (as in Section 2) a $(J \times J)$ matrix Q^{ABH} where:

$$Q_{ij}^{ABH} = P_{ij} \exp(-r_i^{ABH}), i, j = 1, \dots, J. \quad (10)$$

The long-term yield in the autarkic equilibrium is then:

$$r_{long}^{ABH} = -\ln(\max(\text{eig}(Q^{ABH}))). \quad (11)$$

6.2.3 Existence of a Monetary (Public Debt Bubble) Equilibrium

As suggested by Proposition 3, the following proposition shows that there is a stationary monetary equilibrium of the kind discussed above if $r_{long}^{ABH} < 0$.

Proposition 12. *Suppose $r_{long}^{ABH} < 0$ and $\lambda^{ABH} = \max(\text{eig}(Q^{ABH})) > 1$, where r_{long}^{ABH} and Q^{aut} are as defined in (10)-(11). Then there is a monetary equilibrium in which the price of money across aggregate states is a positive vector $(\Gamma_j)_{j=1}^J$ that satisfies:*

$$\frac{\Gamma_i}{e_H - \Gamma_i} = \beta \sum_{j=1}^J P_{ij} \left((1 - p) \left(\frac{\Gamma_j}{e_H - \Gamma_j} \right) + p \bar{v}_i \Gamma_j \right), i = 1, \dots, J$$

Proof. In Appendix. □

In the equilibrium characterized in Proposition 12, each agent in state H uses their resources to buy one unit of money and then divests their moneyholdings upon transit to (the hand-to-mouth) state L . The resulting distribution of money is geometric: a measure $p(1-p)^{n-1}/2$ agents have n units of money. (Note that this distribution is unaffected by the aggregate shocks.)

The per-capita quantity of money in this equilibrium is $0.5/p$. But money is neutral. Hence, if the quantity of money is fixed at $M > 0$, there is an equilibrium with a state-contingent vector of money prices given by:

$$\frac{0.5/p}{M} \Gamma$$

In this case, each agent in state H in date t buys $\frac{M}{0.5/p}$ additional units of money (regardless of the price of money).

7 Conclusions

I close with a caveat and a path forward.

This paper argues that persistent fluctuations in interest rates and growth rates should be seen as broadening the scope for sustainable infinite debt rollovers (and public debt bubbles). The caveat to this argument is that in a risky world, a sustainable infinite debt rollover is necessarily stochastic. Intuitively, when short-term interest rates are expected to remain high relative to growth rates, the value of debt being rolled over is small. When short-term interest rates are expected to remain low relative to growth rates, the value of debt being rolled over is large. (Section 3.6 illustrates that the implied fluctuations in outstanding debt are potentially large.) Interestingly, if the relevant debt instrument is money, then this intuition provides a linkage between (real) interest/growth rates and inflation.

In terms of a path forward: This paper provides a one-parameter check for whether asset prices and growth rates are such that an infinite debt rollover is sustainable. A natural next step for future research is to use information in asset pricing data to measure this single parameter.

Some work along these lines has been done.¹⁵ The results are mixed. In the context of a nonparametric recursive utility model of risk, Christensen (2017) estimates r_{long} - the real yield on a zero-coupon bond with infinite maturity - to be between 1.5% and 2% per year in the US (the parameter y in Table III on p. 1525 of his paper). His estimate makes no use of yield data for bonds that have maturities longer than 90 days.¹⁶

In contrast, Balter, Pelsser, and Schotman (2021) use only data on euro swap rates with five-year and twenty-year maturities to estimate r_{long} (nominal) for Europe. They find that the implied bond yields are highly persistent. As a result, the point estimate of r_{long} (what they call θ in the tables on p. 208 of their paper) is notably negative, but the associated standard errors are enormous.

Both of these papers represent attempts to estimate r_{long} in the context of arbitrage-free asset pricing models. But the current paper emphasizes the need to estimate \hat{r}_{long} - that is, the long-term bond yield in a detrended economy in which realized growth rates (of available loanable funds) are netted out of short-term yields. A first step in estimating \hat{r}_{long} (or at least an upper bound for \hat{r}_{long}) is the specification and estimation of a model of the co-movements of short-term bond yields and growth rates. The requisite model would need to be especially reliable in its descriptions of the *low-frequency* joint behavior of these two variables.

¹⁵There is of course an enormous literature on yield curve estimation. I have chosen two recent papers that treat r_{long} as a fixed parameter and estimate that parameter in the context of arbitrage-free models.

¹⁶Perhaps not unrelatedly, Christensen's estimates seem high to me. Corollary 1 uses an arbitrage relationship between short-term and long-term bonds to show that $E^*(y_t^1) > r_{long}$. Christensen's estimation cannot make use of this relationship. In US data, a typical sample average of real yields on 90 day Treasury bills is less than 1%. It seems unlikely that these short-lived assets have sufficient risk to nearly double the starred expectations of their yields over their unstarred expectations.

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Appendix

This appendix collects the remaining proofs (of Corollary 1, and of Propositions 1, 4, 7, 10, 11, and 12).

Proof of Proposition 1

The Perron-Frobenius Theorem implies that the maximal eigenvalue of Q^* is positive, and so $\ln(\max(\text{eig}(Q^*)))$ is well-defined. The rest of the proof shows that $r_{long} = -\ln(\max(\text{eig}(Q^*)))$ satisfies Assumption 1.

The vector q^N of state-contingent prices of N -period zero-coupon bonds can be recursively calculated as:

$$\begin{aligned} q^N &= Q^* q^{N-1}. \\ &= (Q^*)^{N-1} q^1 \end{aligned}$$

Define the matrix $Q^{**} = Q^* \exp(r_{long})$. Since $\exp(-r_{long})$ is the maximal eigenvalue of Q^* , repeated exponentiation of this matrix results in a well-defined limit:

$$\lim_{N \rightarrow \infty} (Q^{**})^N = Q^\infty$$

where Q^∞ is a positive matrix. Then:

$$\begin{aligned} &\lim_{N \rightarrow \infty} q^N \exp(Nr_{long}) \\ &= \lim_{N \rightarrow \infty} \exp(r_{long})(Q^{**})^{N-1} q^1 \\ &= \exp(r_{long})Q^\infty q^1. \end{aligned}$$

Taking logs (on a component by component basis) proves the proposition:

$$\begin{aligned}
& \lim_{N \rightarrow \infty} N(-y^N + r_{long}) \\
&= \lim_{N \rightarrow \infty} (\ln(q^N) + Nr_{long}) \\
&= \ln(Q^\infty q^1).
\end{aligned}$$

Proof of Proposition 4

A N -period zero-coupon bond is a promise to receive a $(N - S)$ -period zero coupon bond in S periods. Hence, the yields satisfy the recursion:

$$-Ny^N(x_t) + Nr_{long} = -Sy^S(x_t) + Sr_{long} + \ln(E^*(\exp(-(N-S)y^{N-S}(x_{t+S}) + (N-S)r_{long})|x_t))$$

Let N converge to infinity, and (using assumption 1*) substitute in terms of the bounded function ϕ defined in (1):

$$-\phi(x_t) = -Sy^S(x_t) + Sr_{long} + \ln(E^*(\exp(-\phi(x_{t+S}))|x_t))$$

Jensen's inequality implies that:

$$-\phi(x_t) \geq -Sy^S(x_t) + Sr_{long} - E^*(\phi(x_{t+S})|x_t).$$

Take unconditional expectations on both sides:

$$-E^*(\phi(x_t)) \geq -E^*(Sy^S(x_t)) + Sr_{long} - E^*(\phi(x_{t+S})).$$

The stationarity of $\{x_t\}_{t=1}^\infty$ implies that the two unconditional expectations are equal. Thus, we arrive at:

$$E^*(y^S(x_t)) \geq r_{long}.$$

Proof of Corollary 1

As in the proof of Proposition 4, we can use Assumption 1* to derive the relationship:

$$-\phi(x_t) = (-y^1(x_t) + r_{long}) + \ln(E^*(\exp(-\phi(x_{t+1}))|x_t)). \quad (12)$$

Suppose $\phi(x)$ is constant for almost all x . Then, (12) implies that $y^1(x)$ is equal to r_{long} almost everywhere, which contradicts the hypothesis in the proposition. So, we can conclude that $Var^*(\exp(\phi(x))) > 0$. From the restriction in the proposition on the Markov process, there is some positive (starred) probability such that the conditional (starred) variance:

$$Var^*(\exp(-\phi(x_{t+1}))|x_t) > 0.$$

Applying Jensen's inequality to (12), we obtain:

$$-\phi(x_t) \geq -y^1(x_t) + r_{long} - E^*(\phi(x_{t+1})|x_t) \quad (13)$$

where the inequality is strict with some positive starred probability.

Taking unconditional expectations, we get:

$$-E^*(\phi(x_t)) > -E^*(y^1(x_t)) + r_{long} - E^*(\phi(x_{t+1}))$$

Given the stationarity of $\{x_t\}_{t=1}^{\infty}$, we can cancel to arrive at the desired conclusion:

$$E^*(y^1(x_t)) > r_{long}$$

Proof of Proposition 7

In the economy indexed by ρ , the minimal one-period bond yield is given by:

$$r_{min}(\rho) = \mu_y + \frac{\varepsilon_{min}(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}}.$$

To find $r_{long}(\rho)$, let y_t^N be the yield on a N -period bond. It satisfies the recursive relationship:

$$Ny_t^N = y_t^1 - \ln(E_t^* \exp(-(N - 1)y_{t+1}^{N-1})).$$

Given the Markovian structure, Ny_t^N is a time-homogeneous function of y_t^1 . We can guess and verify that this function is affine, so that:

$$Ny_t^N = A_0^N + A_1^N y_t^1.$$

where $A_0^1 = 0$ and $A_1^1 = 1$. Plugging this representation into the recursive relationship, we get:

$$\begin{aligned} & A_0^N + A_1^N y_t^1 \\ &= y_t^1 - \ln(E_t^* \exp(-A_0^{N-1} - A_1^{N-1}(1 - \rho)\mu_y - A_1^{N-1}\rho y_t^1 - A_1^{N-1}\varepsilon_{t+1}(1 - \rho^2)^{1/2})) \\ &= y_t^1 + A_1^{N-1}\rho y_t^1 + A_0^{N-1} + A_1^{N-1}(1 - \rho)\mu_y - \ln(E_t^* \exp(-A_1^{N-1}\varepsilon_{t+1}(1 - \rho^2)^{1/2})) \end{aligned}$$

Hence, the constants $\{A_0^N, A_1^N\}_{N=1}^\infty$ satisfy the recursive restrictions:

$$\begin{aligned} A_1^N &= 1 + \rho A_1^{N-1} \\ A_0^N &= A_0^{N-1} + A_1^{N-1}(1 - \rho)\mu_y - \ln(E_t^* \exp(-A_1^{N-1}\varepsilon_{t+1}(1 - \rho^2)^{1/2})). \end{aligned}$$

where $A_1^N = 1$ and $A_0^N = 0$. It follows that:

$$\begin{aligned} \lim_{N \rightarrow \infty} A_1^N &= \frac{1}{(1 - \rho)} \\ \lim_{N \rightarrow \infty} (A_0^N - A_0^{N-1}) &= \mu_y - \ln(E_t^* \exp(-\varepsilon_{t+1} \frac{(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}})). \end{aligned}$$

Hence, we can find the long-term yield as:

$$\begin{aligned} r_{long}(\rho) &= \lim_{N \rightarrow \infty} y_t^N \\ &= \lim_{N \rightarrow \infty} N^{-1} (A_0^N + A_1^N y_t^1) \\ &= \lim_{N \rightarrow \infty} N^{-1} A_0^N \\ &= \mu_y - \ln(E_t^* \exp(-\varepsilon_{t+1} \frac{(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}})). \end{aligned}$$

Now, rewrite by adding and subtracting $\frac{\varepsilon_{min}(1+\rho)^{1/2}}{(1-\rho)^{1/2}} = (r_{min}(\rho) - \mu_y)$:

$$r_{long}(\rho) - \mu_y = (r_{min}(\rho) - \mu_y) - \ln(E_t^* \exp(-\varepsilon_{t+1} \frac{(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}} + \varepsilon_{min} \frac{(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}})). \quad (14)$$

We next derive a lower bound on the last term of (14). The nature of the lower bound depends on the two distinct assumptions being made about the distribution of ε_{t+1} .

Case 1 (atom): Since ε_{t+1} has an atom at ε_{min} , $Pr^*(\varepsilon_{t+1} = \varepsilon_{min}) = \pi^* > 0$. This allows us to rewrite the last term of (14) as:

$$-\ln(\pi^* + (1 - \pi^*) E_t^* (\exp((\varepsilon_{min} - \varepsilon_{t+1}) \frac{(1 + \rho)^{1/2}}{(1 - \rho)^{1/2}}) | \varepsilon_{t+1} > \varepsilon_{min})))$$

This term is bounded from above by:

$$-\ln(\pi^*).$$

It follows that:

$$0 \leq (r_{long}(\rho) - \mu_y) - (r_{min}(\rho) - \mu_y) \leq -\ln(\pi^*).$$

Dividing through by $(r_{min}(\rho) - \mu_y) < 0$ and taking limits, we get:

$$0 \geq \lim_{\rho \rightarrow 1} \left(\frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} - 1 \right) \geq \lim_{\rho \rightarrow 1} \frac{-\ln(\pi^*)}{\frac{\varepsilon_{min}(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} = 0$$

so that:

$$\begin{aligned} 1 &= \lim_{\rho \rightarrow 1} \frac{r_{long}(\rho)/r_{min}(\rho) - \mu_y/r_{min}(\rho)}{1 - \mu_y/r_{min}(\rho)} \\ &= \lim_{\rho \rightarrow 1} \frac{r_{long}(\rho)/r_{min}(\rho)}{1} \end{aligned}$$

which proves the proposition in the atom case.

Case 2 (positive continuous density): Suppose ε_{t+1} has a positive continuous density at ε_{min} . There then exists $h > 0$ and $\delta^* > 0$ such that the density of ε_{t+1} is above h over the interval $[\varepsilon_{min}, \varepsilon_{min} + \delta^*]$. It follows that:

$$\begin{aligned} &E_t^* \exp(-(\varepsilon_{t+1} - \varepsilon_{min}) \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}) \\ &\geq \int_0^{\delta^*} \exp(-x \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}) h dx \\ &= -\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}} \exp(-x \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}) h \Big|_0^{\delta^*} \\ &= \frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}} h [1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})]. \end{aligned}$$

We can use this inequality to bound the last term of (14) from above:

$$\begin{aligned} &-\ln(E_t^* \exp(-\varepsilon_{t+1} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}} + \varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})) \\ &\leq -\ln\left(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}\right) - \ln(h) - \ln\left(1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})\right) \end{aligned}$$

Rearranging (14), we get:

$$\begin{aligned}
r_{long}(\rho) - \mu_y &\leq r_{min}(\rho) - \mu_y - \ln\left(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}\right) - \ln(h) - \ln(1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})) \\
\frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} &\geq 1 + \frac{-\ln\left(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}\right) - \ln(h) - \ln(1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \\
1 - \frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} &\leq \frac{-\ln\left(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}\right) - \ln(h) - \ln(1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{-\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \\
1 - \frac{r_{long}(\rho)/r_{min}(\rho) - \mu_y/r_{min}(\rho)}{1 - \mu_y/r_{min}(\rho)} &\leq \frac{-\ln\left(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}\right) - \ln(h) - \ln(1 - \exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{-\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \quad (15)
\end{aligned}$$

Note that $r_{long}(\rho)/r_{min}(\rho) \leq 1$ for ρ near 1 (because $r_{min}(\rho)$ is more negative than $r_{long}(\rho)$).

Hence, taking limits of both sides of (15), we find that:

$$0 \geq \lim_{\rho \rightarrow 1} 1 - \frac{r_{long}(\rho)/r_{min}(\rho)}{1} \geq 0$$

which proves the proposition for the positive continuous density case.

Proof of Proposition 10

A monetary equilibrium Γ satisfies the Euler equations of the young agents in the various states:

$$\frac{\Gamma_i \exp(-\bar{g}_i)}{1 - \Gamma_i \exp(-\bar{g}_i)} = \sum_{j=1}^J P_{ij} \frac{\Gamma_j}{\exp(\bar{r}_i) + \Gamma_j}$$

or equivalently:

$$\frac{\Gamma_i}{\exp(\bar{g}_i) - \Gamma_i} = \sum_{j=1}^J P_{ij} \frac{\Gamma_j}{\exp(\bar{r}_i) + \Gamma_j}.$$

Define a nonlinear operator:

$$T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

$$T_i((\Gamma_j)_{j=1}^J) = \exp(\bar{g}_i) \left(\sum_{j=1}^J P_{ij} \frac{\Gamma_j}{\exp(\bar{r}_i) + \Gamma_j} \right)^{-1} + 1)^{-1}.$$

A monetary equilibrium Γ is a fixed point of T .

The Jacobian of T at $\Gamma = 0$ is \hat{Q} . Hence, for Γ sufficiently close to zero:

$$T(\Gamma) \approx \hat{Q}\Gamma.$$

Because $\hat{r}_{long}^{aut} < 0$, \hat{Q} has a maximal eigenvalue larger than 1. By the Perron-Frobenius Theorem, there is a strictly positive eigenvector w associated with this maximal eigenvalue. Since T is well-approximated by \hat{Q} for Γ near zero, there exists a small but positive scalar ξ such that

$$T_i(\xi w) > \xi w_i$$

for all i .

T is a strictly monotone operator and its range is bounded above by $(\exp(\bar{g}_i))_{i=1}^J$. Hence the sequence $\{T^N(\xi w)\}_{N=1}^\infty$ converges:

$$\lim_{N \rightarrow \infty} T^N(\xi w) = w^*.$$

The limit w^* is a positive fixed point of T and so is the desired monetary equilibrium price vector.

Proof of Proposition 11

A monetary equilibrium price vector Γ satisfies the Euler equations of the young agents in the various states:

$$\frac{\Gamma_i \exp(-\bar{g}_i)}{1 - \Gamma_i \exp(-\bar{g}_i)} = \sum_{j=1}^J P_{ij} \frac{\Gamma_j \exp(-\pi)}{\exp(\bar{r}_i) + \Gamma_j}.$$

This expression makes use of the observation that the money transfer to old agents exactly offsets the fall in the value of the money induced by π . Equivalently, we can write:

$$\frac{\Gamma_i}{\exp(\bar{g}_i) - \Gamma_i} = \exp(-\pi) \sum_{j=1}^J P_{ij} \frac{\Gamma_j}{\exp(\bar{r}_i) + \Gamma_j}.$$

As in the proof of Proposition 10, we can define a nonlinear operator:

$$T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

$$T_i((\Gamma_j)_{j=1}^J) = \exp(\bar{g}_i) \left(\exp(-\pi) \sum_{j=1}^J P_{ij} \frac{\Gamma_j}{\exp(\bar{r}_i) + \Gamma_j} \right)^{-1} + 1)^{-1}.$$

A monetary equilibrium price vector Γ is a fixed point of T .

The Jacobian of T at $\Gamma = 0$ is $\hat{Q}_{pos} = \exp(-\pi) \hat{Q}$. Hence, for Γ sufficiently close to zero:

$$T(\Gamma) \approx \hat{Q}_{pos} \Gamma.$$

But the maximal eigenvalue of \hat{Q}_{pos} is $\exp(-\hat{r}_{long}^{aut} - \pi)$, which is larger than 1 from the restriction that $\pi < -\hat{r}_{long}^{aut}$. The remainder of the proof follows that of Proposition 10.

Proof of Proposition 12

The restrictions in the proposition are the Euler equations for agents in state H that ensure that those agents' optimum involves buying one additional unit of money (at price Γ_i in state i).

Proof. Define a nonlinear operator $T : \mathbb{R}_+^J \rightarrow \mathbb{R}_+^J$ by:

$$T_i((\Delta_j)_{j=1}^J) = e_H \left(\left(\beta \sum_{j=1}^J P_{ij} \left((1-p) \left(\frac{\Delta_j}{e_H - \Delta_j} \right) + p \bar{v}_i \Delta_j \right) \right)^{-1} + 1 \right)^{-1}.$$

The Jacobian of T at $\Gamma = 0$ is given by Q^{ABH} , and so the behavior of T near $\Gamma = 0$ is well-approximated by Q^{ABH} .

Let $\vec{\delta} = (\delta_1, \delta_2, \dots, \delta_J)$ be the positive eigenvector of Q^{ABH} , with unit Euclidean norm, associated with $\lambda^{ABH} > 1$. (The Perron-Frobenius Theorem implies that such a positive eigenvector exists.) Since T is well-approximated by Q^{ABH} for Γ small, there is a sufficiently small but positive scalar ξ such that.

$$T_i(\xi \vec{\delta}) > \xi \delta_i$$

for all i . The operator T is strictly monotone and is bounded from above by y_H . Hence the sequence $\{T^N(\xi \vec{\delta})\}_{N=1}^\infty$ converges:

$$\lim_{N \rightarrow \infty} T^N(\xi \vec{\delta}) = \Gamma.$$

The limit Γ is a positive fixed point of T and so satisfies the restriction in the proposition. \square