Optimal Information Disclosure and Quality Ratings[*]

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Abstract

This paper addresses two central questions in markets with adverse selection: How does information impact the welfare of market participants (sellers and buyers)? Also, relatedly, what is the optimal rating policy and how is it affected by the objective function of the planner? In addition, we study the optimal design and performance of simple mechanisms that consist of a small number of ratings. Particularly, we compare the revenue generated in the market under these two schemes and find that the value loss due to using simple ratings is small and drops sharply as the number of signals grows.

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1 Introduction

Reputation mechanisms and ratings are widely used in markets with adverse selection. While relevant for any market with asymmetric information (e.g., hygiene ratings for restaurants or doctors' performance ratings), information design is a key consideration for the overall performance of the ever more popular online trading platforms, where transactions are decentralized and rarely repeated. Despite the importance of these mechanisms, little is known about their optimal design and how it might depend on the characteristics of the market, such as supply, demand, and the distribution of sellers' quality. This paper sheds light on this question by considering the design of an optimal information disclosure mechanism and how it relates to market characteristics.

In particular, the paper addresses two central questions. First, how does information impact the welfare of market participants (i.e., sellers and buyers)? Also, relatedly, what is the optimal rating policy and how is it affected by the objective function of the planner? Secondly, and motivated by fact that rating systems tend to be very simple in practice, we consider the question of optimal design and performance of rating systems when limited to a small number of ratings.

Our baseline model considers a competitive market with a large set of buyers and sellers.[1] Firms are endowed with different levels of quality, which is the only source of product differentiation.[2] The model exhibits two features that are common to adverse selection settings. First, low-quality sellers benefit from being pooled with high-quality ones, while adversely affecting them. Second, high-quality sales are crowded out by low-quality ones. Information disclosure, and in particular a rating system, helps reallocate sales from lower- to higher-quality producers, thus mitigating the problem of adverse selection. Our analysis focuses on two main sources of market heterogeneity: the distribution of firm qualities and the responsiveness of sellers' supply to prices. Intuitively, the heterogeneity and skewness of seller quality affect the spread of prices across ratings, while the responsiveness of supply determines the resulting reallocation of output across these categories. To our best knowledge, this is the first paper that systematically considers the interaction of these

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[1]We also consider the case of Cournot competition with constant marginal costs and show the results are the same as those for perfect competition with linear supply.

[2]While moral hazard might be a critical consideration in some markets, in others adverse selection might play a more critical role, as suggested by an empirical study on eBay (see Hui et al. (2018)). Optimal rating design with moral hazard and adverse selection is considered in Saeedi and Shourideh (2019) in a simplified market environment.

two factors and their impact on optimal information disclosure and rating design.

As a result of improved information, prices become more strongly associated with true quality of sellers and thus more dispersed. Demand is reallocated from lower- to higher-quality firms, which has a positive effect on the average quality of goods consumed and total surplus. However, the effect of improved information on total market size and consumer surplus is ambiguous and depends on the properties of the supply function. When supply is concave, the higher spread in prices results in a decrease in total output and lowers consumer surplus. The opposite occurs when the supply is convex.

Regarding optimal information design, we find that better information has opposing welfare effects on consumers and producers that could lead to limited disclosure depending on the social objective. For example, in regions where the supply function is concave, pooling can mitigate the reduction in output from improved information and its negative impact on consumer surplus. Where the supply function is convex, pooling decreases total output and increases prices, which might have a positive impact on producers. For those cases where full information is not optimal, we find that the region of pooling increases with the strength of the bias in the planner's preference for one or the other group.

In the second part, we turn to the question of optimal rating design, when limited to a small number of ratings. In practice, rating systems usually provide coarse signals of quality to buyers. For example, in California, restaurants are given grades A, B, C, or none based on the score obtained after a hygiene inspection is conducted. Airbnb awards its top-quality hosts the Superhost badge, and eBay's high-quality sellers are classified as Top Rated Sellers. Many governmental and non-profit agencies certify firms that meet certain standards.[3] These examples raise two critical questions about coarse rating design: First, given a number of ratings, what are the criteria for setting the boundaries between them? In particular, when there are only two tiers, how stringent should the standards for certification be?[4] Second, what is the welfare loss of using a coarse rating

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[3]For example the website ecolabelindex.com currently lists 455 certifiers for food and consumer products across 199 countries and 25 industry sectors. To the best of our knowledge, they all use these simple mechanisms mostly with certifying only a subset of the firms in the sector that meet some minimum requirements. Accessed June 11, 2021.

[4]Hui et al. (2021) examine the effect of an increase in the requirements to become a badged seller on eBay. They find that this increase leads to a higher market share of high-quality sellers while decreasing the sales of sellers in the medium range of quality.

system instead of using the unconstrained optimal mechanism?

We first derive a necessary condition defining the thresholds that correspond to an intuitive criterion. Consider a marginal firm with quality at the threshold between two adjacent intervals. For this threshold to be optimal, the planner should be indifferent between pooling this firm with those in the intervals above or below. This decision ultimately affects the demand faced by the firm, and thus its total output. The benefit of the increased output is the extra value generated by the additional sales, which at the optimum should be equated to the extra cost of production. Therefore, one of the key determinants of this trade-off is the supply behavior of firms, in particular, the curvature of the supply function. We find a simple characterization for the optimal thresholds in the case of linear supply, which provides a useful benchmark.[5] These optimal thresholds are the solution to a standard clustering problem that involves only information regarding the distribution of qualities.[6]

Regarding the performance of ratings, we show that a one-threshold partition closes at least half of the surplus gap between no information and full information for quality distributions with log-concave density. In our numerical computations, we find that this partition closes from 46% to nearly 77% of the gap, depending on the underlying distribution of qualities. The loss due to coarse ratings diminishes rapidly as the number of thresholds increases, implying that a simple and cost-effective system with a few tiers can achieve a large part of the full-disclosure value.

The design of a rating system faces the following challenges, which we address. How strict should the standards for certification be? How selective should they be? And how does their choice depend on the distribution of quality and supply considerations? We find that an increase in right (resp. left) skewness reduces (resp. increases) the share of producers with high ratings and increases (resp. decreases) the share of those with lower ratings. Similar considerations apply to the degree of convexity of the supply function. An intuition for this result is that optimal ratings trade off pooling in different regions. Pooling is more costly where there is more quality dispersion or where supply is more responsive to prices.

The last part of the paper examines a series of extensions. We first consider a demand system

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[5]Equivalently, Cournot competition with constant marginal cost.

[6]This clustering problem can be solved by the $k-$means algorithm as introduced by MacQueen et al. (1967) and used extensively in machine learning and statistics.

where agents have heterogeneous preference for quality, and firms have inelastic supply. Second, we consider Cournot competition and show that all the results obtained under perfect competition for our benchmark case apply to this setting. In our final extension, we add entry to the baseline model.

Related Literature   Our paper is related to two strands of literature: first, the papers considering the impact of information disclosure on consumer and producer surplus; second, those concerning the determinants of coarse rating systems as well as their performance.[7]

Most papers belonging to the first strand of literature consider the case where there is a single seller, or auctioneer, and multiple buyers, as opposed to multiple agents on both sides. Similar to our results, Schlee (1996) shows that information can hurt consumers when the cost function is sufficiently convex. Bergemann et al. (2015) consider the impact of information in third-degree price discrimination. They show that any distribution of surplus that is between the ones achieved by optimal pricing with none and full information can be attained with some information structure. Bergemann and Pesendorfer (2007) show that in a private value setting, bidders can be worse off with better information even though total surplus increases. Board (2009) shows that this result depends on the number of bidders. Hoppe et al. (2011) consider a matching problem where for some distribution of types, consumers can be worse off with better information. In our paper we show that better information always increases total surplus, but it might decrease consumer or producer surplus depending on properties of the supply function. These considerations are absent in the matching framework, where supply is inelastic.

There is a large literature on certification and quality disclosure. Dranove and Jin (2010) provide an excellent survey of the earlier papers. Most of the literature has focused either on the incentives for firms to reveal their information or the incentives of certifiers to do so. The main question in this literature is how much information will be revealed in equilibrium and how this might depend on the nature of competition in the product or certification markets. As an example, Lizzeri (1999) finds that while a monopoly certifier chooses to provide coarse information with a single and

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[7]Our paper focuses on a setting where uncertainty is about seller quality, and information is provided to consumers. There is a growing literature that focuses on the reverse channel, where an intermediary transmits information about buyers to sellers. For a survey see Bergemann and Bonatti (2019).

low threshold, competition among certifiers can lead to full information. Ostrovsky and Schwarz (2010) consider equilibrium information structures where colleges strategically choose how much information to reveal about their students ability. DeMarzo, Kremer, and Skrzypacz (2019) consider a Bayesian game where agents choose the informativeness of testing but can withhold bad results.

Our paper differs from this certification literature in several dimensions. First, in our setting information is freely provided by a single informed certifier, and in particular it is exogenous to the firm, as occurs in the examples mentioned above. Secondly, information affects payoffs of firms through two channels. The first is a standard one, where certification provides a signal of expected quality to consumers, directly affecting the price faced by the firm. The second one is that certification affects total equilibrium output and thus the equilibrium prices received by all firms, thus impacting both, producer and consumer surplus. This effect is absent in most papers on certification in markets, that usually assume inelastic supply. Another implication of elastic supply, is that certification reallocates sales across firms, to a degree that is affected by supply elasticity. This plays an important role in the value and design of an optimal certification mechanism.

Coarse ratings have also been justified in the literature by their simplicity and overall performance. Wilson (1989) shows that losses relative to full information are of order $1/n^2$ for a partition with n classes. This finding is consistent with our computed bounds in Section 4.2. Our theoretical bound on the gains from a two-tier certification is also related to the bounds found by the coarse matching literature such as in McAfee (2002), Hoppe et al. (2011), and Shao (2016).

Information disclosure is the focus of the literature on Bayesian persuasion, where an informed sender chooses an information structure to influence the behavior of a receiver. Kamenica (2018) and Bergemann and Morris (2019) provide a great survey of this literature. Kolotilin (2018) and Dworczak and Martini, (2018) provide conditions on payoffs so that interval partitions are the optimal information structure. Onuchic and Ray (2021) study the problem of monotonic categorization when sender and receiver have different priors. In contrast to most of this literature, where a single receiver takes an action, in our setting the outcome is the result of the equilibrium choices of multiple agents, introducing a non-linearity across states[8]

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[8]While other papers have studied settings with multiple receivers, the analysis has often been suitable for games where a low-dimensional source of aggregate information is observed by a sender. For example, Bergemann and Morris (2013, 2016) characterize the outcome of all Bayesian persuasion games with multiple receivers. In principle,

The most relevant empirical papers related to our theory are Saeedi (2019), Elfenbein et al. (2015), Fan et al. (2013), and Jin and Leslie (2003). Saeedi (2019) studies the value of reputation mechanisms and establishes a positive signaling value for the certification done by eBay. Elfenbein et al. (2015) study the value of certification badges across different markets. They find that certification provides more value when the number of certified sellers is low and when markets are more competitive. Fan et al. (2013) analyze the effect of badges on Taobao.com. They find sellers offer price discounts to move up to the next reputation level. Jin and Leslie (2003) use data on restaurant hygiene ratings to examine the effect of an increase in product quality information to consumers on firms' choices of product quality. Our paper also relates to the literature that analyzes the effects of changes in marketplace feedback mechanisms on price and quality (e.g., Hui et al. (2016), Filippas et al. (2018), and Nosko and Tadelis (2015)).

Section 2 describes the model. Section 3 considers the optimal information disclosure problem. Section 4 finds the optimal coarse ratings when the number of signals is limited, and it also finds a bound on information loss due to this constraint. Section 5 studies the extensions to the baseline model, and Section 6 concludes. Proofs are relegated to the appendix unless otherwise specified.

2 The Model

There is a unit mass of firms with qualities z distributed according to a continuous cumulative distribution function (cdf) $F(z)$. Production technology is the same for all firms and is given by a continuous, strictly increasing, and strictly convex cost function $c(q)$, and, correspondingly, a strictly increasing supply function $s(p)$. On the demand side, there is mass M of consumers who face a discrete choice problem, with preferences

$$U(z, \theta, p) = z + \theta - p,$$

our problem could be potentially mapped into this framework, with an omniscient sender that observes the quality of a continuum of firms, but it would be impractical to solve it this way. Even for a simple two-player game, Bhaskar et al. (2016) show that computing the optimal public signal is NP-hard.

where z is the quality of the good purchased, θ is a taste parameter measuring the preference for goods offered in this market vis a vis an outside option, and p is the price of the good. The taste parameter θ is distributed according to a continuous and strictly increasing cdf $\Psi(\theta)$, while the outside good's utility (no purchase) is normalized to zero.[9] Goods are differentiated only by quality, which is equally valued by all consumers.[10] Given the linearity of the utility function in z, the same ordering is obtained for the consumption of a good of expected quality z. We assume all market participants have the same information about the expected qualities of firms, represented by the distribution function $G(z)$.[11] In particular, when considering a finite rating system as in Section 4, we assume that G is a discrete distribution with point masses at the conditional mean qualities associated to each rating. We will say that a firm has expected quality z if conditional on all signals received, that is the quality expected by consumers.

Given expected quality z, equilibrium prices take the form $p(z) = p(0) + z$, where $p(0)$ corresponds to the demand price of a hypothetical good of quality zero. This expression for prices guarantees that all consumers are indifferent between goods with different signal realizations, which is a necessary condition for an equilibrium. Given a baseline price $p(0)$, the marginal consumer's θ is found by setting $U(0, \theta, p(0)) = 0$, or simply $\theta(p(0)) = p(0)$. All consumers with $\theta \geq p(0)$ will consume some good, so aggregate demand is $Q = M(1 - \Psi(p(0)))$. Inverting this function, we can define an inverse baseline demand function

$$ P(Q) = \Psi^{-1}(1 - Q/M). \tag{1} $$

On the supply side, each firm with expected quality z chooses its output, $q = s(p(z))$, so aggregate supply $Q = \int s(p(z)) \, dG(z)$.

Definition. An (interior) *equilibrium,* given the distribution of expected qualities $G(z)$, is given by prices $p(z) = P(Q) + z$, where total quantity $Q = \int s(p(z)) \, dG(z)$.

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[9]Alternatively, one can consider $-\theta$ to be the value of the outside good to consumers. Also note that we do not need to make any particular assumption on the distribution of θ; additionally, linearity in z can be relaxed by modifying the distribution of qualities.

[10]In Section 5.3, we consider a case where consumers are heterogeneous with respect to their taste for quality.

[11]This representation of the information structure is consistent with the approach followed in Ganuza and Penalva (2010) and Gentzkow and Kamenica (2016). Given a common prior $F(z_0)$ over firm qualities and a signal structure π, we can let $G(z)$ be the distribution of the expected posterior of firm quality.
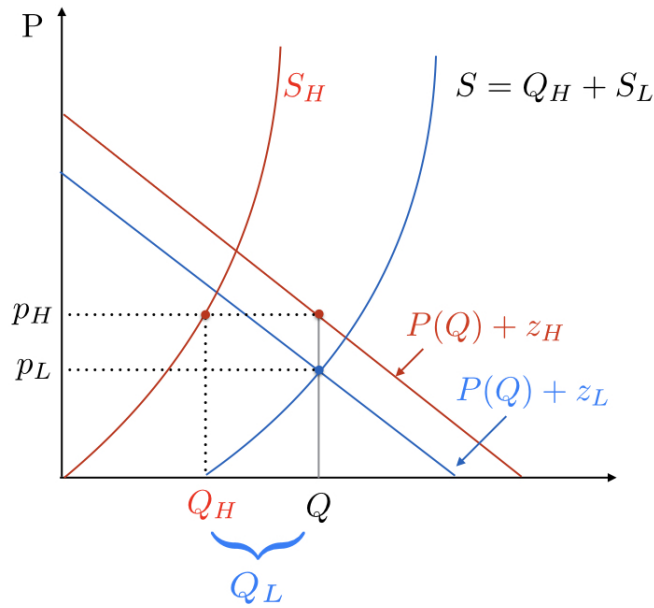
Figure 1: Equilibrium

Figure 1 shows graphically the derivation of an interior equilibrium for the case of a two-tier partition, where L represents the group of firms with quality below a threshold z^*, and H those above.[12] Denote by z_L (resp. z_H) the average quality of sellers in the L group (resp. H group). The two curves depict the demand curve for the goods in the L and H segment, respectively. Since all consumers value quality identically, the price difference between the goods in the two segments is the same as the difference between the two respective average qualities $p_H - p_L = z_H - z_L$. The first upward sloping curve is the supply function of the firms in the H group, $S_H = (1 - F(z^*)) s(p_H)$. The second one is the supply function of the firms in the L segment, $S_L = F(z^*) s(p_L)$, displaced to the right by the equilibrium quantity of the H group, Q_H. The marginal consumer Q is the one that is indifferent between consuming either of these goods or none at the equilibrium prices; Q is also the total market supply of both goods.

To prove the existence of an interior equilibrium, we make the following assumption.

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[12] Alternatively, this can be interpreted as a case of having two types of sellers with two levels of qualities.

Assumption 1. *There exists $\widetilde{\theta}$ in the support of Ψ such that*

$$M > \int s\left(\widetilde{\theta} + z\right) dG\left(z\right)$$

for all distributions G such that F is a mean-preserving spread of G.

This assumption rules out the possibility that all consumers make purchases in this market; in other words, we assume that the consumers are on the long side of the market.[13] While a corner equilibrium, if it exists, is also unique, we rule this out as a matter of convenience.

Lemma 1. *Under Assumption 1, there exists a unique interior equilibrium for all expected quality distributions G such that F is a mean-preserving spread of G.*

Proof. Given that the cdf Ψ is strictly increasing and continuous, the function $P\left(Q\right)$ is strictly decreasing and continuous. Define function $f\left(Q\right) = \int s\left(P\left(Q\right) + z\right) dG\left(z\right)$. This function is strictly decreasing and continuous. It follows immediately that $f\left(0\right) > 0$. By Assumption 1, $f\left(M\right) < M$ since $P\left(M\right) \leq \widetilde{\theta}$. Hence, there exists a unique fixed point Q^* for this mapping. $\qquad\square$

3 Information Disclosure

Our previous analysis takes the distribution of mean qualities, G, as a primitive. Given the linearity of payoffs, this is a sufficient representation of information, as two products with the same posterior mean qualities are equivalent to consumers. As in Ganuza and Penalva (2010) and Gentzkow and Kamenica (2016), improvements in information can be represented by mean-preserving spreads of the distribution of mean qualities. This section considers two related questions: (1) the impact of improved information on producer and consumer surplus and (2) optimal information disclosure by an informed principal.

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[13]As explained below, the assumption spans the set of all possible information structures.

3.1 Improved Information

This section examines the impact of improved information on producer and consumer surplus. Given total quantity Q, equilibrium prices are given by $p(Q) + z$, with mean $p(Q) + \bar{z}$. A mean-preserving spread of G increases the spread of prices around the mean while possibly changing the mean, too, as the equilibrium quantity Q changes.

The increased dispersion of prices has a direct positive effect on average profits, as a result of the convexity of the profit function. In turn, an increase (resp. decrease) in market size as measured by the change in total quantity Q has a negative (resp. positive) effect on profits. In contrast, as we now show, consumer surplus is affected only by changes in total quantity Q, and in the opposite direction of profits.

Consider a consumer of type θ who buys a good of quality z, with utility $\theta + z - p(z)$. Given the equilibrium price $p(z) = P(Q) + z$, the consumer's net utility is $\theta - P(Q)$. It follows that total consumer surplus is

$$\int_{P(Q)} (\theta - P(Q)) \, d\Psi(\theta) = \int_0^Q (P(x) - P(Q)) \, dx$$

where the equality follows from the change of variables $x = M(1 - \Psi(\theta))$, and our definition of $P(Q)$ given by equation (1). This implies that consumer surplus will move in the same direction as market size, as given by total quantity Q. It is worth noting that this general equilibrium effect has opposite impacts on consumer and producer surplus. This observation will become quite relevant in Section 3.2 when considering optimal information disclosure.

The impact of improved information on market size Q depends on the properties of the supply function. If it is linear, the increase in price dispersion has no effect on total output, so there is no change in Q. In contrast, when the supply function is convex (resp. concave), total output increases (resp. decreases) with price dispersion. More generally, the direction and magnitude of output change will depend on the shape of the supply function (convex or concave) and the magnitude of the changes in spread.

The following proposition summarizes these results.

Proposition 1. *An improvement in information quality, as given by a mean-preserving spread of G, has the following effects:*

1. *It increases (resp. decreases) total output if the supply function is convex (resp. concave).*

2. *Consumer surplus changes in the same direction as total output.*

3. *Producer surplus increases if total output does not increase.*

4. *Total surplus increases.*

In particular, in the case of concave supply functions, consumers are better off with no information. There are some related results in the literature, though in different settings. For example, Schlee (1996) considers a single product monopoly seller in a vertically differentiated market. The quality of the good offered is exogenous and privately observed by the monopolist, who must choose the informativeness of a signal to be provided to consumers before observing the quality realization. It is shown that if the cost function is sufficiently convex, consumers are worse off ex ante with a more informative signal. Hoppe et al. (2011) consider a matching problem and show that under some conditions on the distribution of types, one of the sides (e.g., consumers) can be worse off by having a more precise information structure regarding the type of the other side.

Regarding producer surplus, there is an additional direct contribution of price dispersion to profits. So, total profits can still increase when output increases. While examples can be constructed where producer surplus decreases, for this to occur, the degree of convexity of supply needs to be very strong relative to the convexity of the profit function.[14]

While improved information has ambiguous effects on consumer and producer surplus, it always increases total surplus. The intuition is as follows. Firstly, a social planner, subject to the same information structure, cannot improve on the competitive equilibrium allocation, which is thus optimal. Secondly, equipped with better information, a social planner can always increase

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[14]For example, consider the following setup: Marginal cost 0 for $q \leq 1$ and $1+\varepsilon$ for $q > 1$ up to a capacity constraint of 3. Mass of firms equal 1; equal weights of qualities 0 and 1. Baseline inverse demand function is $1/2$ up to $Q = 1$ and drops to 2ε after that. Initial equilibrium $p = 1$; total output equals 1, and total demand is also 1. Total profits are equal to 1. Equilibrium with full information: $p = 1 + 2\varepsilon$, and total profits equal $\frac{1}{2} \times (3 \times (2\varepsilon + 1) - 2(1 + \varepsilon))$, which is approximately equal to 1/2 for small ε.

total surplus. This result also implies that better information must benefit either producers or consumers, or both. In particular, average profits must rise when consumer surplus does not increase, as in the case of concave supply.

3.2 Optimal Information Disclosure

This section considers optimal information disclosure by a market designer, which we refer to as the planner. The information structure is as follows. To motivate the analysis, we start with two examples which capture in a stylized way some realistic features.

Example 1

All firms inelastically supply $\bar{q}$ units provided price is above marginal cost $c > 0$. Note that because output is inelastically supplied, the only role of information is to exclude some low-quality producers from the market. This scenario might represent a market where retailers can acquire the good at a wholesale price c, at a limited capacity. Let z^c, z^o, z^p denote the optimal thresholds for consumers, an equal weights planner, and producers, respectively. Consumers are interested in maximizing output, subject to the participation constraint for producers being above the threshold

$$P\left((1 - F\left(z^c\right))\bar{q}\right) + M\left(z^c\right) - c \geq 0,$$

where $M\left(z^*\right)$ denotes the average quality above z^*. An equal weights planner will exclude all producers that contribute negative value, and will thus choose

$$P\left((1 - F\left(z^o\right))\bar{q}\right) + z^o - c = 0.$$

Finally, producers would want the threshold to maximize total profits. It is easy to verify that the corresponding necessary condition is

$$P\left(\cdot\right) + z^p - c = -P'\left(\cdot\right)(1 - F\left(z^p\right))\bar{q}.$$

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It follows easily that $z^c < z^o < z^p$. In order to implement the threshold z^c, pooling above z^c is needed, while to implement producers' preferred threshold z^p, pooling below z^p is needed. Full separation can be used in the complementary regions.[15] Full separation for all producers is sufficient to implement the planner's preferred threshold z^o.

Example 2

Suppose the supply function is linear up to capacity constraint $\bar{q}$ but there is a fixed cost and, correspondingly, a breakeven price $p_b > 0$. This example captures in a stylized way realistic features for many activities (e.g., Uber rides), where there is a minimum strike price and an intensive margin beyond this price up to a capacity limit.

This supply function is convex in the lower end and concave above it. Following the intuition from the previous section, in order to maximize total output, consumers would want to separate firms in the lower end below some threshold z^c and pool those above it. Assuming the upper bound does not bind, total output is given by

$$Q = S\left(P\left((Q) + M\left(z^c\right)\right)\right)\left(1 - F\left(z^c\right)\right).$$

It follows after differentiation that dQ/dz^c has the same sign as $-P(Q) + z^c$. Consumers will thus set z^c to be the minimum such that $P(Q) + z^c \geq 0$ and $P(Q) + M(z^c) \geq p_b$. As shown before, the equal-weights social planner will want full information and an efficient threshold such that $P(Q) + z^o = p_b$. It follows easily that $z^o > z^c$. Moreover, because consumer surplus is decreasing at z^o, it must be that producer surplus is increasing, so the threshold for producers will be higher, with full separation above it.

General Theory

The planner's information is summarized by a distribution $F(z)$ of expected qualities across sellers with mean $\bar{z}$. This represents the maximal information that the planner could provide to buyers.

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[15] The latter is sufficient but might not be necessary in this example.

Any partial revelation of information can be represented by a distribution $G \epsilon \mathcal{G}$, where $\mathcal{G}$ is the set of garblings or mean-preserving contractions of F. Buyers have symmetric priors about seller quality.[16] We first consider the extreme case where buyers have no information about firms, sharing a degenerate prior with mass 1 at mean quality $\bar{z}$. Results are then extended to non-degenerate priors in Section 3.3.

For any $G \in \mathcal{G}$, let $Q(G)$ denote total equilibrium output. Letting $0 \leq \gamma \leq 1$ denote the weight given to consumers, the planner's problem is

$$\max_{G \in \mathcal{G}} (1 - \gamma) \int \pi(p + x) \, dG(x) + \gamma \int_0^{Q(G)} (P(q) - p) \, dq, \tag{2}$$

where $p = P(Q(G))$ and π is the profit function of the sellers. The first term corresponds to producer surplus and the second term to consumer surplus, as explained above.

Changes in the information structure, as represented by G, have two effects on the planner's objective: a direct effect on expected profits and a general equilibrium effect, operating through the change of Q and p. To provide some intuition, consider a small mean-preserving spread of G around quality x. The direct effect will be an increase in profits of a magnitude that depends on the curvature of the profit function around $p + x$, $(1 - \gamma) \pi''(p + x)$. The equilibrium effect will be given by a marginal change in prices dp with welfare effect

$$\left[(1 - \gamma) \int \frac{\partial \pi(p + x)}{\partial p} dG(x) - \gamma Q(G) \right] dp = (1 - 2\gamma) Q(G) \, dp,$$

which follows from the envelope condition $\frac{\partial \pi(p+x)}{\partial p} = s(p + x)$. Thus the general equilibrium effect will affect firms and consumers in opposite and equal directions. Moreover, since $dp = P'(Q) \, dQ$,

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[16]This notion of garbling of information has been used repeatedly in the literature, for example in Ganuza and Penalva (2010), who order the quality of information by the dispersion of beliefs. The distribution of expected qualities $\tilde{G}$ is more informative than distribution G if it is a mean-preserving spread of G. We will refer to this ordering as better information. As the maximal signal structure corresponds to perfect information, the class of all information structures can be represented by all garblings of F, i.e., all distributions G such that F is a mean-preserving spread of G. This corresponds to the ordering of integral precision of signal structures defined in Ganuza and Penalva (2010) and the ordering in Gentzkow and Kamenica (2016). Starting from a prior F_0, signal structure $\tilde{t}$ is more *integral precise* than signal t if the induced distribution of expected qualities $G(z)$ generated by $\tilde{t}$ is a mean-preserving spread of the one generated by t. In general, integral precision ordering is weaker than the likelihood ratio and other related orderings considered in the literature (see Ganuza and Penalva (2010) for references.)

its magnitude will vary directly with the intensity (and sign) of the output change dQ. This in turn depends on the curvature of the supply function $s''(p+x)$ around the point of the mean-preserving spread x. In particular, if the supply function is convex around this point, total output will increase and the general equilibrium effect will imply a transfer of utility from firms to consumers. A transfer in the opposite direction would occur if the supply function were concave at this point.

This endogeneity of Q, and the resulting nonlinearity, makes our problem different from the usual Bayesian persuasion models; therefore, we cannot use directly the methods developed in the literature. But, as we show below, the problem has indeed a linear structure when constrained to a given level of aggregate output Q, and this can be used to provide results on optimal disclosure. Consider the constrained optimization problem:

$$U(Q) = \max_{G \in \mathcal{G}} (1-\gamma) \int \pi(P(Q)+z)\, dG(z) + \gamma \int_0^Q (P(q) - P(Q))\, dq$$

$$\text{subject to: } Q = \int S(P(Q)+z)\, dG(z),$$

where both the objective function and the constraint are linear in G. The corresponding Lagrangian function is given by

$$\mathcal{L}(Q,\lambda) = \max_{G \in \mathcal{G}} (1-\gamma) \int \pi(P(Q)+z)\, dG(z) + \gamma \int_0^Q (P(q)-P(Q))\, dq \qquad (3)$$

$$-\lambda\left(Q - \int S(P(Q)+z)\, dG(z)\right).$$

As usual, the value $\lambda(Q)$ can be obtained by differentiating (3) with respect to Q,

$$\lambda = \frac{(1-2\gamma)QP'(Q)}{1 - P'(Q)\int S'(P(Q)+z)\, dG(z)}.$$

It follows that λ is positive if and only if $\gamma > 1/2$. This captures the intuitive idea, discussed above, that increases in total output represent, at the margin, a transfer from firms to consumers. Letting

$$V(z) = (1-\gamma)\pi(P(Q)+z) + \lambda S(P(Q)+z), \qquad (4)$$

16

the optimal information structure solves

$$\max_{G \in \mathcal{G}} \int V(z) \, dG(z) .$$

As in [Kolotilin](2018), it follows from Jensen's inequality that full revelation is optimal when $V(z)$ is convex, while no revelation is optimal when it is concave. Since the first term in (4) is convex, we can easily derive the following sufficient conditions for full revelation.

Proposition 2. *Full revelation is always optimal in the following cases:*

1. *$\gamma = 1/2$;*

2. *$\gamma < 1/2$ and S is concave; and*

3. *$\gamma > 1/2$ and S is convex.*

In addition, when $\gamma = 1$, full revelation is optimal only if S is convex.

In all of these cases the implied function $V(z)$ is convex, after factoring in the corresponding sign of the multiplier λ. The first case confirms our previous result that full revelation is optimal when the planner maximizes total surplus. The second result follows intuitively from the fact that when S is concave, improved information decreases output, implying a transfer from consumers to firms, which is desirable as $\gamma < 1/2$. Similarly, the last result follows from the fact that when S is convex, improved information increases output, implying a transfer from firms to consumers, which is desirable as $\gamma > 1/2$.

Sufficient conditions for no revelation of information are harder to obtain. Because the first term in (4) is convex, the conditions needed for $V(z)$ to be concave seem to be stronger. In the extreme case when $\gamma = 1$, the sufficient conditions will hold when the supply function S is concave, which, as we found before, is the case where consumers are better off with no information. More generally, when $\gamma > 1/2$, the supply function has to be sufficiently concave relative to the profit function for no information to be optimal, while if $\gamma < 1/2$ the supply function has to be sufficiently convex.

When considering the question of information provision using certification criteria, an issue that often arises is how hard should the test be? As an example, eBay's increase in the requirements

to qualify as eBay Top Rated Seller was an attempt to make the test harder to pass. An easy test allows creating differentiation at the lower end, while a harder one, at the upper end. So, where is information revelation more valuable? Our previous analysis suggests that more differentiation of firm qualities is of greater value in regions where the degree of convexity of $V(z)$ is stronger. In particular, when $V''(z)$ is increasing (resp. decreasing) we should expect full disclosure (pooling) starting from a point z^* and pooling (full disclosure) in the region below this point. These correspond to situations when $V(z)$ is concave-convex (resp. convex-concave). The next proposition provides the conditions under which these properties hold.

Proposition 3. *Full disclosure up to some threshold z^* and complete pooling above is optimal in the following cases:*

1. *$\gamma > 1/2$ and S''/S' is decreasing,*

2. *$\gamma < 1/2$ and S''/S' is increasing.*

Complete pooling up to some threshold z^* and full disclosure above is optimal in the following cases:

1. *$\gamma > 1/2$ and S''/S' is increasing,*

2. *$\gamma < 1/2$ and S''/S' is decreasing.*

The intuition for these results is as follows. A small increase in spread around z has a direct positive impact on expected profits that is proportional to $\pi''(P(Q)+z)$, the curvature of the profit function around this point. Likewise, it has an impact on total output and a transfer from producers to consumers that is proportional to $S''(P(Q)+z)$. This transfer is positive if S is convex at this point, and negative otherwise. The ratio $S''(z)/\pi''(z)$ measures the transfer relative to the profit increase resulting from this small increase in spread. The higher $S''(z)$ is relative to $\pi''(z)$ (lower in absolute value), the smaller the transfer (loss) is relative to the direct profit gain. In this case, it is optimal to provide information disclosure for higher values of this ratio. Our intuitive argument suggests that when $S''(z)/\pi''(z)$ is increasing (resp. decreasing), disclosure should occur in an upper interval (resp. lower interval). Note that this is precisely the case where $V(z)$ is concave-convex (resp. convex-concave).

	S convex	S concave
$\gamma > 1/2$	Full disclosure	$\frac{S'}{S''}$ increasing: Full disclosure then pooling
		$\frac{S'}{S''}$ decreasing: Pooling then full disclosure
$\gamma < 1/2$	$\frac{S'}{S''}$ increasing: Pooling then full disclosure	Full disclosure
	$\frac{S'}{S''}$ decreasing: Full disclosure then pooling	

Figure 2: Optimal Pattern

The results from this section are summarized in Figure 2.

We end this section considering the effect of asymmetries in the weights of consumers and producers. For those cases where full information is not optimal, we find that the region of pooling increases with the strength of the bias in the planner's preference for one or the other group.

Proposition 4. *Consider an increase in γ. If $\gamma > 1/2$, then the pooling region increases with γ, while if $\gamma < 1/2$ the pooling region decreases with γ.*

The intuition for this result is as follows. When $\gamma > 1/2$, it must be the case that S is concave in the pooling region; otherwise, there would be full disclosure, as stated in Corollary 2. Thus pooling takes place to mitigate the reduction in output from improved information and its negative impact on consumer surplus. The larger the weight of consumers, the larger this pooling region will be. When $\gamma < 1/2$, it must be that the supply function is convex in the pooling region, and pooling occurs precisely to mitigate the increase in output and its negative impact on producers. The lower the weight of producers (higher γ), the smaller this pooling region will be.

3.3 Buyer's Prior Information

In this section we extend our results to the case where buyers have non-degenerate priors, given as follows. There is a finite partition of sellers into N groups with respective shares $\alpha_j, j = 1, ..., N$. For all sellers in a group, buyers share symmetric information given by a Dirac prior on mean quality z_j^0. This could represent, for example, identical realizations for a finite set of ratings. For each of these groups, the planner's information can be represented by a distribution F_j of expected qualities across these sellers, with mean z_j^0. Any partial revelation of information can be represented by a distribution $G_j \epsilon \mathcal{G}_j$, where $\mathcal{G}_j$ is the set of garblings or mean-preserving contractions of F_j. This information structure implies a distribution $G = \sum_j \alpha_j G_j$ over expected qualities of sellers which refines the information of consumers up to the information held by the planner. Let $\mathcal{G}$ denote the set of distributions that can be obtained this way. The optimal problem is identical to (2), optimized over this set of distributions.

The constrained optimization problem we specified in (3) can be adapted to this case. We solve for the optimal disclosure policy G_j within each element of the information partition of buyers, holding fixed the vector of total output Q_j for each. Since the only connection between all of these planning problems is through aggregate output, holding it fixed makes the problem separable. Moreover, as total output is the sum of the output Q_j of all partitions, the multipliers λ_j are identical. In consequence, all properties derived above translate to each element of the partition. In particular, Corollaries 2 and 3 as well as Proposition 4 hold.

4 Coarse Ratings

Most rating systems are coarse, ranking sellers into a small number of categories. For example, in the case of Yelp, the partition involves five stars, including the possibility of half-stars. In the case of eBay, the partition includes two groups: the badged and unbadged. In the case of California restaurants, the partition involves three elements: A, B, and C. In addition, hundreds of governmental or non-profit certification agencies use a pass-fail or tiered signal for their certification method. This section considers the question of optimal information design when the number of

ratings the market designer can employ is limited. This restriction can be motivated not only by its wide use but also by its cost-effectiveness, as giving very precise information might be difficult or costly, and simple rankings might be easier to interpret. Moreover, as we find, most of the gains from optimal information provision, as given in the previous section, can be achieved with a very limited number of ratings.

In this section we focus on simple ratings that partition the set of sellers into N groups. We consider as an objective the maximization of total unweighted surplus ($\gamma = 1/2$ in the previous section). The cases of consumer and producer surplus are discussed in Section 5.2. To simplify our analysis, we assume that consumers have no information other than that provided by the certifier. The timing is as follows: First, the certifier observes some signals for each firm that are correlated with the firm's quality; Second, the certifier assigns a rating to each firm and makes it common knowledge to all participants in the market. Based on these ratings all market participants can infer the average quality of sellers, thus sharing a common posterior with support at the corresponding N conditional quality means.[17] Third, market equilibrium outcomes are determined for this distribution of expected qualities.

Following our earlier discussion on information structures, the certifier's information can be summarized by a distribution of posterior mean qualities that, in order to avoid further notation, we denote by $F(z)$. This is the basis on which the certifier classifies firms into rating bins. To simplify the exposition, we refer to the expected value z as the quality of the firm.[18] We assume F is differentiable on its support with density $f(z)$.

A threshold partition totally orders firms into N quality intervals. As a corollary to Proposition 1, we establish the superiority of threshold partitions.[19]

Corollary 1. *The optimal rating is given by a threshold partition.*

Given this corollary, the design of an optimal rating system reduces to finding the vector of opti-

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[17]Certifiers usually provide users with guidelines to interpret their ratings or users learn their meaning over time.

[18]The *true* distribution of qualities plays no role, as any information provision that respects the information of the certifier is a garbling of the certifier's posterior F.

[19]Dworczak and Martini (2019) provide conditions under which the optimal signal structure implies a monotone partition in a Bayesian persuasion setting. Our setting is different and our result straightforward. We can find a connection only in the case of linear supply, where our problem can be mapped into their formulation with the additional constraint of a fixed number of ratings.

mal thresholds, $\boldsymbol{z} = (z_1....z_{N-1})$, that divide sellers into the N partitions, $\{[z_0, z_1], [z_1, z_2], ..., [z_{N-1}, z_N]\}$, where z_0 and z_N are the lower and upper supports of the distribution of expected qualities given the planner's information ($-\infty$ or $+\infty$ if unbounded), respectively.

4.1 A Necessary Condition

In this section we derive a simple and intuitive necessary condition to characterize these optimal thresholds. Let $M_k = m(z_{k-1}, z_k), k = 1, ..., N$ denote the conditional means of quality z in the intervals $[z_{k-1}, z_k]$. Let $Q(\boldsymbol{z})$ denote the unique equilibrium total quantity at the optimal threshold vector. The prices for sellers in partition $[z_{k-1}, z_k]$ are denoted by $p_k = P(Q(\boldsymbol{z})) + M_k$, and quantities by $q_k = s(p_k)$. Total surplus is given by

$$W(\boldsymbol{z}) \;\; = \;\; \int_0^{Q(\boldsymbol{z})} P(x)\, dx + \sum_{k=1}^{N} [F(z_k) - F(z_{k-1})][M_k q_k - c(q_k)]. \tag{5}$$

Taking first order conditions with respect to z_k proves the following necessary condition:

Lemma 2. *Let the thresholds $\boldsymbol{z} = (z_1, ..., z_{N-1})$ maximize (5). Then*

$$(P(Q(\boldsymbol{z})) + z_k)(q_{k+1} - q_k) = c(q_{k+1}) - c(q_k) \tag{6}$$

for all z_k.

Condition (6) has an intuitive interpretation. Consider a marginal firm with quality at the threshold between two adjacent intervals. For this threshold to be optimal, the planner should be indifferent between pooling this marginal firm with those in the lower or upper interval. The left hand side shows the marginal value obtained by increasing the quantity of the marginal firm with quality z_k, from q_k to q_{k+1}; this would result from a marginal change in this threshold. The right hand side shows the difference in cost. This condition highlights the relevance of the supply behavior of firms, in particular, the curvature of the supply function, as it impacts both the response of output to changes in prices and its impact on cost.

Figure 3 provides a graphical representation of this necessary condition and its connection to
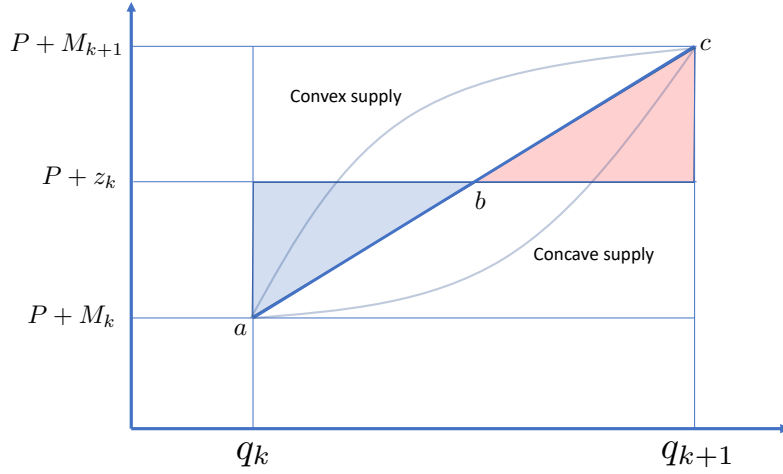
Figure 3: Optimal Pattern

the supply function. Three cases are considered: a linear supply function given by the solid diagonal line, an upper convex supply (concave marginal cost function), and a lower concave supply (convex marginal cost function). The area below the marginal cost function between q_k and q_{k+1} equals the right hand side of (6), while the area under the line $P + z_k$ equals the left hand side. The difference between these two areas is

$$\int_{q_k}^{q_{k+1}} (P + z_k - C'(q))\, dq,$$

which equals zero if and only if condition (6) holds. In the linear case, the integrand is positive up to point b and negative thereafter. Point $P + z_k$ is such that the regions from a to b and from b to c have the same areas. It is immediate that in the linear case, the corresponding value of $z_k = (M_{k+1} + M_k)/2$. It also follows easily that for the convex supply case, $P + z_k$ must be higher, so $z_k > (M_{k+1} + M_k)/2$, while the converse holds for the concave supply case.

Proposition 5. *Let the thresholds $z = (z_1, ..., z_{N-1})$ maximize (5), and denote by $M_k = E(z|z_{k-1} \leq z \leq z_k)$ the corresponding conditional means. Then*

1. *$z_k = (M_k + M_{k+1})/2$ if the supply function $S(p)$ is linear;*

2. *$z_k \geq (M_k + M_{k+1})/2$ if the supply function is convex, with strict inequality if it is not linear*

23

in the interval $[M_k, M_{k+1}]$*; and*

3. $z_k \leq (M_k + M_{k+1})/2$ *if the supply function is convex, with strict inequality if it is not linear in the interval* $[M_k, M_{k+1}]$.

A simple characterization for the solution in the linear supply and conditions for uniqueness are provided in the following proposition.[20]

Proposition 6. *If the supply function is linear, the optimal thresholds* $\boldsymbol{z} = (z_1, ..., z_N)$ *are the ones that minimize*

$$\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (z - M_k)^2 \, dF(z). \tag{7}$$

If in addition F *has log-concave density, the solution to this minimization problem is unique.*

The optimal thresholds for the linear supply case are the ones that minimize the sum of the variance of qualities within partitions. This objective coincides with the popular $k-$means criteria for clustering as introduced by MacQueen et al. (1967), commonly used in the machine learning and statistics literature. This makes estimating the optimal thresholds a trivial task, as many software programs incorporate algorithms to solve this problem.

Following from the above observations, the linear supply case seems to be a natural reference point. It is easy to compute, providing the market designer a good place to start. Moreover, Proposition 5 suggests that the thresholds for the linear case can be lower (resp. upper) bounds for the case of convex (resp. concave) supply, which is formally proved in Section 5.1. In addition, all results obtained for the linear supply case apply identically to the canonical case of Cournot competition with constant marginal cost and arbitrary demand function, as shown in Section 5.4. The remainder of this section will focus on the linear supply case. Section 4.2 considers the performance of simple rating systems. Section 4.3 considers the role of the distribution of firm qualities F and in particular its skewness.

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[20]The analysis for the nonlinear supply cases is provided in Section 5.1

4.2 Value Loss Due to Coarse Ratings

In this section, we study the performance of simple rating mechanisms. We first derive a theoretical bound for the case of two-tier ratings. Next, we explore numerically the performance of simple ratings for a large class of widely used distribution functions.

Given that in the case of linear supply considered here total quantity and consumer surplus are invariant to the information structure, without loss of generality we consider the gap in producer surplus. Profits for a firm with expected quality z are equal to $p(z)^2/2 = (P(Q) + z)^2/2$. Therefore, for any distribution G of expected quality, total profits are

$$
\begin{aligned}
\Pi &= \frac{1}{2} \int (P(Q) + z)^2 \, dG(z) \\
&= \frac{1}{2} P(Q)^2 + P(Q)\bar{z} + \frac{1}{2} \int z^2 dG(z).
\end{aligned}
$$

The first two terms do not depend on G and thus on the partition. For the full information case, the distribution of means $G = F$, so $\Pi = \frac{1}{2}P(Q)^2 + P(Q)\bar{z} + \frac{1}{2}\int z^2 dF(z)$. Therefore, the surplus gap with respect to the full information case is $\Delta\Pi = \frac{1}{2}\left(\int z^2 dF(z) - \int z^2 dG(z)\right)$ for any distribution of expected quality G. In particular, the total gap with respect to the no-information case is $\overline{\Delta\Pi} = \frac{1}{2}\left(\int z^2 dF(z) - \bar{z}^2\right)$. For a threshold partition $(z_1, ..., z_{N-1})$, where G has N mass points at the conditional means $M_1, ..., M_N$

$$
\begin{aligned}
\Delta\Pi &= \frac{1}{2} \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left(z^2 - M_k^2\right) dF(z) \\
&= \frac{1}{2} \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left[(z - M_k)^2\right] dF(z).
\end{aligned}
\tag{8}
$$

This equation corresponds again to the loss function used in $k-means$ clustering, given that at the optimal thresholds, as defined earlier, the expected values M_k are precisely the centroids of the corresponding intervals $[z_{k-1}, z_k]$. We are interested in seeing how much of the total surplus is captured by 8 or simply the following ratio:

$$G \equiv \frac{\overline{\Delta\Pi} - \Delta\Pi}{\overline{\Delta\Pi}} = 1 - \frac{\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (z - M_k)^2 \, dF(z)}{\int z^2 dF(z) - \bar{z}^2}$$

$$= \frac{\sum_{k=1}^{N} (F(z_k) - F(z_{k-1})) (M_k - \bar{z})^2}{\int (z - \bar{z})^2 \, dF(z)},$$

which is the ratio of the variance between the conditional mean qualities and the total variance. Intuitively, this bound is a measure of the relative importance of the variance between the means of the partitions, separated by their ratings, and the variance that remains in each pool. This connection to variance decomposition is used below to derive a theoretical bound on ratings' performance.

Theoretical Bounds   The simplest coarse rating scheme is a two-tier certification, widely used in many settings. The next proposition provides a useful bound for the gains from certification that builds on the variance decomposition described above. The corollary that follows gives sufficient conditions so that a two-tier rating achieves at least half of the surplus of full information.

Proposition 7. *The relative performance of a two-tier setting satisfies*

$$G \geq \frac{1}{1 + \max\left\{cv_1^2, cv_2^2\right\}},$$

where cv_1 is the coefficient of variation of $z - \bar{z}$ conditional on $z < \bar{z}$ and cv_2 the coefficient of variation of $z - \bar{z}$ conditional on $z \geq \bar{z}$.

Corollary 2. *Suppose that the distribution F has an increasing hazard rate and a decreasing reverse hazard rate. Then a two-tier rating achieves at least half of the surplus of full information.*

Proof. From a well-known result from Stoyan and Daley (1983), pp 16–19, the conditions of this corollary imply that $cv_1 < 1$ and $cv_2 < 1$. Using the bound in Proposition 7 completes the proof. □

The conditions given in the corollary are satisfied by a large class of distributions that include all those with log-concave densities, such as uniform, normal, exponential and double exponential,

Table 1: Optimal Thresholds

Distribution	Case	Mean/Median	z^*	$1 - F(z^*)$	Share of Surplus Gap Closed			
					$n = 2$	$n = 3$	$n = 5$	$n = 10$
Pareto	$\alpha = 3$	1.19	2.73	0.05	0.46	0.68	0.84	0.94
	$\alpha = 4$	1.12	1.84	0.09	0.54	0.74	0.89	0.97
Exponential	all	1.45		0.20	0.65	0.82	0.93	0.98
$F(z) = z^\alpha$	$\alpha = 0.5$	1.32	0.41	0.36	0.77	0.90	0.97	0.99
$z \in [0, 1]$	$\alpha = 2$	0.94	0.62	0.62	0.72	0.87	0.95	0.99
Log-normal	$\sigma = 0.25$	1.03	1.09	0.36	0.63	0.81	0.92	0.98
$(\mu = 0)$	$\sigma = 1$	1.64	4.25	0.07	0.55	0.75	0.89	0.97

Note: The above calculations correspond to the linear supply case.

logistic, extreme value, and many others with some restriction on parameters (e.g., power function $F(z) = z^c$ for $c \geq 1$.) Related bounds for two-sided matching problems can be found in McAfee (2002); Hoppe et al. (2011); Shao (2016). The results of Wilson (1989) imply that the losses from $N-$ratings are of order $1/N^2$.

Numerical Results We examine now numerical results for a variety of distribution functions that are often used in the economics literature. Table 1 reports the share of the total surplus gap that is closed with partitions of different sizes n. As can be seen from the calculations, a one-threshold (certification) partition closes from near 50% to almost 80% of the surplus gap, depending on the underlying distribution of qualities. The gains are diminishing as the number of thresholds increases. Even though total surplus increases with the number of tiers, our numerical results suggest that most gains are attained with a small number of ratings. As a result, the market designer should weigh in the cost of having a more complicated information structure against the diminishing return of having more tiers.

These results show that a very simple rating system consisting of a single certification threshold or few tiers can achieve a considerable share of the gains from full information. This suggests that the added cost or complexity of a more elaborate design might not be compensated by the gains from it. While we do not model the cost of providing more detailed information for the market designer or consumer's cost of analyzing and understanding detailed information, our results suggest that small costs could justify simple rating schemes and their popularity among market

designers in practice.

4.3 Skewness and Optimal Thresholds

The optimal thresholds, as depicted in Equation 7, will depend on the distribution of sellers' quality, i.e., F distribution. In this section, we study how skewness in the distribution of qualities impacts this optimal choice. In particular, we show that in the simple case of a two-tier certification, the optimal threshold is skewed in the same direction as the distribution of qualities. Then, we extend this result, providing general comparative statics for the vector of thresholds with respect to an appropriately defined skewness ordering.

Before proceeding to the analysis, we would like to provide some intuition behind our results. Consider the case of one certification threshold z^*. The following trade-off appears when deciding how strictly to draw the line separating the upper and lower segments. When putting z^* in the upper group, there is an upward distortion of the supply of the firm at z^*, which is a function of the distance $M_H - z^*$. This distance also measures the extent to which the firm at z^* gains from being pooled with higher-quality firms. When putting z^* in the lower group, there is a downward distortion of the supply of the firm at z^*, which is a function of $M_L - z^*$. This distance also measures the extent to which the firm at z^* loses from being pooled with lower-quality firms. Right skewness (resp. left skewness) of the distribution $F(z)$ will increase (resp. decrease) the upward distortion and decrease (resp. increase) the downward distortion, making it optimal to have more restrictive (resp. less restrictive) certification standards.

The condition given in Proposition 6 implies

$$z^* = \frac{1}{2}\left(M_L\left(z^*\right) + M_H\left(z^*\right)\right),\tag{9}$$

which can be used to relate this threshold to properties of the distribution. Consider first the case of a symmetric distribution, i.e., where the median, z_{median}, equals the mean, $\bar{z}$. Since for any z^*, $F(z^*)M_L + (1 - F(z^*))M_H = \bar{z}$, setting the threshold $z^* = \bar{z} = z_{median}$ would satisfy the above condition.

The same reasoning suggests that when F is skewed, the optimal threshold will also be skewed
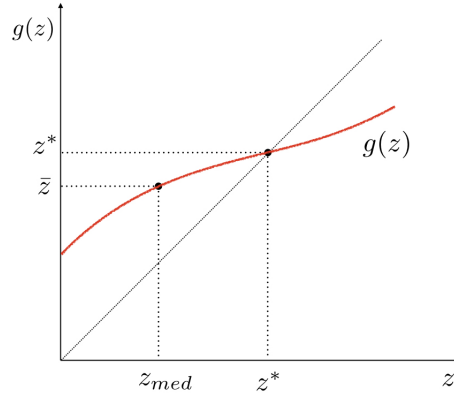
Figure 4: z^* When Mean>Median

relative to the mean in the same direction. This can be easily proved, as follows. Consider the case of a right skewed distribution where $\bar{z} > z_{median}$. Let $M_L(.)$ and $M_H(.)$ denote functions equal to the conditional average of quality of sellers below and above any value within the range of qualities, respectively. Furthermore, denote $g(z) = \frac{1}{2}(M_L(z) + M_H(z))$. Following Proposition 5, the optimal threshold is a fixed point of this function. When $z \to z_{max}$ (or as $z \to \infty$ in the case of unbounded support), $g(z) \to \frac{1}{2}\bar{z} + \frac{1}{2}z < z$, and when $z \to z_{min}$ (or as $z \to -\infty$ in the case of unbounded support), $g(z) \to \frac{1}{2}z_{min} + \frac{1}{2}\bar{z} > z$. For $z = z_{median}$, $g(z) = \bar{z} > z$. Since the function $g(z)$ is increasing and continuous, the unique fixed point z^* must be to the right of z_{median} and, as a consequence, $z^* > \bar{z}$, as illustrated in Figure 4. The result for the case of left skewness can be shown similarly.

Table 1 shows the optimal threshold for a series of distributions, as well as the corresponding fraction of certified sellers. All distributions in our example are skewed to the right except for one, so according to our argument, $z^* > \bar{z} > z_{median}$ and it is optimal to have a smaller share of sellers certified. This is shown in the fifth column of Table 1. As an example, for the Pareto distributions only a small fraction should get certified, 5% when the power parameter is 3 and 9% when the power parameter is 4.[21] For the exponential distribution, only 20% of sellers should be certified regardless of the hazard rate.

Now, we extend our findings to the case of multiple signals under a stronger skewness order. This skewness order, the convex (concave) order, was originally proposed by Van Zwet (1964).

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[21]When $\alpha \leq 2$, the value of z^* is undefined, as total surplus is strictly increasing in z^* in all the support.

Definition. Distribution $\tilde{F}$ is more skewed to the right than F if $\tilde{F}^{-1}(F(x))$ is convex; equivalently, there exists an increasing convex function $g(x)$ such that $\tilde{F}(g(x)) = F(x)$.[22]

We can think of this ordering as stretching to the right the quality scale with the transformation $g(x)$. As an example, if F is a uniform distribution in $[0,1]$ and $g(x) = x^2$, then $\tilde{F}(x^2) = x$ or, equivalently, $\tilde{F}(x) = x^{1/2}$.

Proposition 8. *Suppose the supply function is linear. Let F be a distribution with log-concave density and $\tilde{F}$ a distribution such that $\tilde{F}(g(z)) = F(z)$, where g is a strictly convex increasing function. Let $\{l_k\}$ be the optimal thresholds for F and $\{g(z_k)\}$ the optimal thresholds for $\tilde{F}$. Then $z_k > l_k$ for all k.*

This proposition implies that for all k, $\tilde{F}(g(z_k)) = F(z_k) > F(l_k)$, so the percentiles defined by the two optimal thresholds are ordered. In particular, for a two-tier certification rating, the share of certified firms should be lower for distribution $\tilde{F}$. An example is given in Table 1 for the case of power distributions $F(z) = z^\alpha$. It is easily shown that the distribution with $\alpha = 0.5$ is more skewed to the right than the one with $\alpha = 2$.[23] Consistently with the previous proposition, the share of certified sellers is lower when $\alpha = 0.5$.

5   Further Characterizations and Extensions

This section provides some additional results and extensions. Section 5.1 considers non-linear supply functions. Section 5.2 considers the optimal ratings for consumers and producers, highlighting the conflicting interests of the two groups. Section 5.3 considers the case of vertical differentiation, where buyers differ in their preference for quality, and the matching between goods' quality and consumer's type becomes important. Section 5.4 establishes the equivalence between Cournot competition and the case of linear supply. Finally, Section 5.5 considers the role of entry.

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[22] Note that this definition implies that $F^{-1}(F(x)) = g^{-1}(x)$ is concave.
[23] Take $g(x) = x^4$.

5.1 Non-linear Supply Functions

In this section, we generalize our results in Section 4.1 to the case of a non-linear supply function. The role of curvature can be conveniently illustrated comparing the two polar cases of perfectly inelastic and perfectly elastic supply. When supply is perfectly inelastic (i.e., producers can produce either zero or one unit or equally face a constant marginal cost up to a fixed capacity), quality ratings cannot reallocate output to higher-quality producers, except in the extreme case where there is no production in the absence of information. However, ratings can potentially serve to filter out the very low-quality producers that no consumer would buy from at a positive price, so the optimal threshold will be at this low end. At the other extreme, when all firms face a constant marginal cost, only the highest-quality firm should serve the market, so the optimal threshold would be at the other end. More generally, the following lemma and proposition show that the optimal thresholds in the case of a convex (resp. concave) supply function are pointwise higher (resp. lower) than those in the linear case. As an example, in the case of a simple certification rating with two groups, more elastic supply leads to a higher threshold and lower share of certified sellers, as illustrated in the following lemma.

Lemma 3. *If the supply function is concave (resp. convex), then z_k is lower (resp. higher) than $(M_k + M_{k+1})/2$.*

While the second part of this lemma gives the criteria for local deviations for a single threshold, convex to the right and concave to the left, starting at those obtained for the linear case, it does not imply an ordering of the whole vector of thresholds. The following proposition gives the conditions for the total ordering.

Proposition 9. *Suppose the quality distribution $F(z)$ has a log-concave density. Let $\left(z_1^L, ..., z_{N-1}^L\right)$ be the optimal thresholds for the linear case. The optimal vector of thresholds $(z_1, ..., z_{N-1})$ for a convex (resp. concave) supply function is pointwise higher (resp. lower) than $\left(z_1^L, ..., z_{N-1}^L\right)$.*

The formula in Equation (7) gives a simple characterization for the optimal thresholds in the linear supply case that depends only on the distribution of qualities, and, by the previous proposition, provides a lower (resp. upper) bound when the supply function is convex (resp. concave). Thus,

the problem of finding the optimal thresholds becomes a manageable task, as one can start with the simple solution of optimal thresholds in the linear case and then change them in the mentioned direction when dealing with non-linear supply functions.

5.2 Consumer and Producer Surplus

We have focused on total surplus as the objective function. However, it is easy to show that at the optimal thresholds, there is generically a conflict of interest between consumers and producers, and that the optimal choice balances off these conflicting interests. The difference lies in the general equilibrium effect: consumers' surplus increases with total output, while profits decrease. These two opposing effects balance each other exactly at the optimal thresholds. In which direction would consumers like the threshold to move? In particular, would consumers prefer stricter or less strict criteria for certification? The answer depends again on the properties of the supply function.

In the case of concave supply, total output decreases with information, so the optimal thresholds are at the extremes of the distribution. When supply is linear, total quantity is independent of the amount of information (see Proposition 1), so consumer surplus is the same for any threshold. This implies that the optimal threshold is also the one that maximizes profits. More generally, total output will vary as the thresholds change depending on the properties of the supply function. The following proposition provides sufficient conditions that determine the direction of change of output (and consumer surplus) at the optimal thresholds. The direction of change of producer surplus has the opposite sign.

Proposition 10. *Let* $z = (z_1, ..., z_{N-1})$ *be the thresholds that maximize total surplus. If* $s''(p) / s'(p)$ *is decreasing (resp. increasing) in* p*, then* $dQ(z) / dz_k$ *and* $dCS(z) / dz_k$ *are negative (resp. positive) at* z.

To illustrate the above results, consider a simple example. Suppose the supply function $s(p) = p^\theta$ (cost function $c(q) = q^{1+\theta} / (1 + \theta)$). Therefore, $s''(p) / s'(p) = (\theta - 1) / p$. For $\theta > 1$, this expression is decreasing in p. Applying the corollary, this decreasing condition implies that starting at the surplus maximizing thresholds, consumers would prefer lower thresholds, while producers would prefer higher ones. Therefore, if the planner puts more weight on the consumer side, it

should lower thresholds, and if it puts more weight on the sellers, it should increase the thresholds. The reverse occurs when $\theta < 1$.

Note that the results in Proposition 10 are consistent with Corollary 3 if we compare the implied directions of preferences with the order of pooling and separating regions. For example, Corollary 3 finds that when $s''(p)/s'(p)$ is decreasing, consumers prefer lower thresholds, so there is a larger pooling region in the higher-quality segment. In comparison, for the same case, Corollary 3 shows that the information structure that maximizes consumer surplus has full separation below some threshold and pooling above.

5.3 Heterogeneous Preference for Quality

In this extension, we consider a demand system where agents have heterogeneous preference for quality, and firms have inelastic supply. While by construction, improvements in information do not increase total quantity, they contribute to welfare by increasing the correlation between average firm quality and consumer preference for quality. The optimal threshold is defined by a slightly modified formula that weighs differences in the firms' quality gap in each interval by the respective gap in consumers' preferences. As a result, skewness in consumers' preferences for quality has similar implications to the ones observed for skewness in producers' quality.

We examine briefly the determination of optimal thresholds when consumers differ in their preferences for quality for the case of certification, i.e., $N = 2$. Suppose consumers' preferences are given by the utility function $u = \theta z + \theta_0 - p$ for a good of quality z, a la Mussa and Rosen (1978). Consumers differ in their preference for quality θ and for the value they assign the inside vs. outside good θ_0, which is distributed in the population according to some joint distribution $\Psi(\theta, \theta_0)$. As earlier, firm qualities z are distributed according to the cdf $F(z)$. For simplicity, we restrict our analysis to a partition of sellers into two groups defined by threshold z^* with qualities z_L and z_H, respectively. Given prices p_L and p_H, consumers will be split into three groups: those that do not consume and those that consume either the H or L product, with demands $D_H(p_L, p_H)$ and $D_L(p_L, p_H)$, respectively. Prices p_L and p_H will be equilibrium prices provided that $D_H(p_L, p_H) = (1 - F(z^*))\, q(p_H)$ and $D_L(p_L, p_H) = F(z^*)\, q(p_L)$. As in our previous case, there is a unique

equilibrium under fairly general conditions.

Lemma 4. *The optimal choice of threshold z^* satisfies the following first-order necessary condition:*

$$\Pi\left(p_H\right) - \Pi\left(p_L\right) \;=\; \left(z^* - z_L\right)\theta_L q_L + \left(z_H - z^*\right)\theta_H q_H, \tag{10}$$

where θ_L is the average preference for quality of consumers who purchase the L product, and θ_H of those who purchase the H product.

This formula has an intuitive explanation. The first term is the loss of profits of those firms that transition from the H to the L group, when z^* is marginally increasing. The second term measures the effect of the increase in the averages z_L and z_H as z^* is increased, valued at the quality preference of the average consumer in each group, and weighted by their respective market sizes.

Vertical Differentiation with Inelastic Supply

To establish further results, we consider the canonical model of vertical differentiation where consumers differ only in their preference for quality θ and where firms supply inelastically one unit of output.[24] Given equilibrium prices p_L and p_H, all consumers above a threshold θ^* buy an H product, while all those between $\underline{\theta}$ and θ^* buy an L product, where $\underline{\theta}z_L = p_L$ and $\theta^*\left(z_H - z_L\right) = p_H - p_L$. Substituting in Equation (10) gives the condition

$$\left(z^* - z_L\right)\left(\theta^* - \theta_L\right) \;=\; \left(z_H - z^*\right)\left(\theta_H - \theta^*\right).$$

Notice that this equation is a modified version of Equation (15), where the gaps between z^* and the respective means are weighted by the corresponding preference gaps. This equation highlights the role of the complementarities between average quality and preference for quality in the determination of the optimal threshold. In particular, when both distributions are symmetric, this also implies that the optimal threshold z^* (and also θ^*) will equal the corresponding mean. Moreover,

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[24]This case can be reinterpreted as a one-to-one matching environment with surplus function θz.

when z and θ have the same distribution, the optimal threshold is also given by our baseline condition, as given in Equation (15).[25] As an example, if both have uniform distributions, then when $\theta^* = z^* = 1/2$, this condition will hold.

5.4 Cournot Competition

Throughout the paper, we have assumed that the firms are price takers. In this section, we extend our analysis to the case of Cournot competition among firms with constant marginal cost. There is a total of n firms (per consumer), and given threshold z^*, a fraction $F(z^*)$ in the first group and $(1 - F(z^*))$ in the second. The demand structure is the same as in the competitive case considered above. Assume firms face a constant marginal cost c regardless of their type. The equilibrium conditions are

$$MR_H = P'(Q)q_h + P(Q) + z_H = c \tag{11}$$

$$MR_L = P'(Q)q_l + P(Q) + z_L = c. \tag{12}$$

Multiplying each equation by the number of firms in the respective group and adding up, we get

$$P'(Q)Q + nP(Q) + n\bar{z} = nc,$$

where $\bar{z}$ is the mean quality for the n firms. Interestingly, this equation determines Q independently of the signal threshold z^*, as in the case of perfect competition with linear supply.

Another implication of the invariance of total output is that consumer surplus does not change, as in the case of linear supply with z^*. This fact occurs because price increases capture exactly the change in average quality in each group. It follows that optimal thresholds solve the maximization problem (5), so they are identical to those obtained above for the linear case. [26]

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[25]It is interesting to note that when all consumers have the same preference for quality and supply is inelastic, welfare is independent of z^*, as the average product quality is not affected by its choice.

[26]We have considered here quantity competition. For a model of price competition with partially informed consumers, see Moscarini and Ottaviani (2001).

5.5 Entry

In our previous analysis we did not consider explicitly the effect of changes in z^* on entry. Many of our results extend to settings where the distribution of qualities of firms is not affected by entry. We discuss here two scenarios: one where entrants are ex ante differentiated and one where they are ex ante homogeneous.

Consider first the case of differentiated entrants. Our analysis extends without modification to the following scenario. Suppose there is a mass n of entrants that are differentiated in qualities z and fixed (or entry) costs f. Assume qualities are independent from fixed costs and are given by distribution F and Φ, respectively. For a given threshold partition z^*, we can define the aggregate supply functions S_L and S_H as follows. Let $S_H(p) = s(p) N_H(p)$, where $N_H(p) = n(1 - F(z^*)) \Phi(\pi(p))$. This supply function combines the effect of prices on the intensive and extensive margin. We can define similarly $S_L(p)$. Our analysis remains unchanged if we substitute $s(p)$ by $\hat{s}(p) = s(p) n \Phi(f(p))$, so total supplies are $S_L(p) = F(z^*) \hat{s}(p)$ and $S_H(p) = \hat{s}(p)(1 - F(z^*))$.[27]

For the homogeneous case, assume there is a set N of potential entrants that draw their qualities independently from distribution F upon entry, after paying an entry cost f, which is distributed according to cdf $\Phi(f)$. For fixed output, improved information results in a mean preserving spread of expected qualities and thus prices. Given that profit functions are convex in prices, this results in an increase in expected profits and a consequent increase in entry. In the case of linear supply, where in the absence of entry, total output does not change, additional entry results in an increase in total output and thus consumer surplus. In the case of concave supply, we have seen that total output decreases. This increases profits over what is produced by the mean preserving spread of average qualities, thus inducing entry, mitigating, if not totally undoing, the drop in total output that would result in the absence of entry. Finally, note that if all potential entrants were to have the same entry cost, all surplus gains from improved information would accrue to consumers, as expected, and average profits would remain unchanged. The above results apply in particular to

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[27]The properties of these modified supply functions will now depend both on the individual supply functions and the distribution of fixed costs. There exist assumptions on the latter that will guarantee that the modified supply functions are linear, convex, or concave when each of these properties holds for the original supply functions.

the effect of introducing a certification mechanism in a market where there is none.

6   Final Remarks

This paper considered the optimal design of quality ratings in markets with adverse selection. We first study the problem of optimal rating design for a planner with a flexible objective function. We find that better information has opposing welfare effects on consumers and producers that could lead to limited disclosure. For example, in regions where the supply function is concave, pooling can mitigate the reduction in output from improved information and its negative impact on consumer surplus. Where the supply function is convex, pooling decreases total output and increases prices, which might have a positive impact on producers. For those cases where full information is not optimal, we find that the region of pooling increases with the asymmetry in the weights of the two groups in the objective function of the planner.

Ratings reallocate demand across producers, impacting not only the average quality of goods consumed but also average cost. The optimal thresholds in a discrete rating system optimize this trade-off. Optimal ratings thus depend on the characteristics of the market, given by the distribution of producers' quality, the elasticity of supply, and consumers' preferences. We find that the optimal thresholds in the case of a convex (resp. concave) supply function are pointwise higher (resp. lower) than those in the linear case. Intuitively, in the case of a simple certification rating with two groups, more elastic supply leads to a higher threshold and lower share of certified sellers. We also find that skewness in the distribution of firm qualities matters for optimal ratings, which move in the direction of the skew.

We have given a simple characterization for the optimal thresholds in the case of linear supply, or Cournot competition with constant marginal cost, as the solution to standard clustering problems. Our results thus provide a straightforward and easy-to-compute method for the design of rating systems. This method is used to derive bounds on the performance of the rating system as a function of the number of categories. We first theoretically show that a simple certification mechanism, or a two-tier rating, is enough to reach half of the benefits of the best rating mechanism in case of log-concave densities. As an example, we find that for the exponential family of

distributions, 65% of the surplus gains from full information can be achieved with only two categories. The large gains in surplus with a very simple threshold mechanism suggest that the added cost of a more complex one might not be compensated by the gains from it. This could explain the popularity of these simple schemes among market designers.

Our preliminary analysis of the vertical differentiation model suggests that preferences for quality can be an important factor in determining optimal thresholds. One might conjecture that, in parallel to our results on the supply side, more convexity (resp. concavity) in the distribution of consumer types increases (resp. decreases) the gains from more assortative matching, thus leading to higher thresholds. This question and further extensions on the demand side, such as including scope for horizontal differentiation, are subject for future research. Other extensions worth considering are a more detailed modeling of entry, following results obtained in the empirical literature by Hui et al. (2021). Finally, we have abstracted from moral hazard considerations, which can be important in some settings; exploring their impact on the design of optimal ratings is left to future research.

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7 Appendix. Proofs

Proof of Proposition 1

Proof. Suppose the supply function $S(p)$ is convex and, by way of contradiction, $Q_1 < Q_2$. Let $p_1(z) = P(Q_1) + z$ denote the equilibrium price for a good of expected quality z, and define similarly $p_2(z)$. It follows immediately that $p_1(z) > p_2(z)$, since P is strictly decreasing. Let G_1 and G_2 denote the distribution of expected qualities under signal structures t_1 and t_2, respectively. By definition of integral precision, it follows that G_1 second order stochastically dominates G_2, so

$$
\begin{aligned}
Q_1 &= \int S(p_1(z)) \, dG_1(z) \geq S(p_2(z)) \, dG_1(z) \\
&\geq \int S(p_2(z)) \, dG_2(z) = Q_2,
\end{aligned}
$$

where the second inequality follows from convexity of $S(p)$. The above contradicts the original hypothesis, proving that $Q_1 \geq Q_2$. The proof is similar for concave $S(p)$.

To show that total surplus increases with better information, we show that there exists a correspondence between competitive equilibria and allocations that maximize total surplus. Given a distribution of mean qualities $G(z)$, the problem of maximizing total surplus solves

$$
S = \max_{q(z)} \int_0^Q P(x) \, dx + \int [zq(z) - C(q(z))] \, dG(z)
$$

subject to

$$
Q = \int q(z) \, dG(z).
$$

The first order conditions for the choice of $q(z)$ are

$$
z - C'(q(z)) + \lambda = 0 \tag{13}
$$

and this holds for all points in the support of G, where the Lagrange multiplier of the constraint $\lambda = P(Q)$. Substituting in (13) and letting $p(z) = P(Q) + z$ implies $p(z) = C'(q(z))$, which is the condition defining the profit maximizing output $q(z)$ in the competitive equilibrium. Hence

the allocation $q(z)$ and the prices $p(z)$ are the ones that correspond to the unique competitive equilibrium.

Consider now a distribution of expected qualities $\tilde{G}$ corresponding to a better information system than G so it is a mean-preserving spread of G. Following the characterization in Rothschild and Stiglitz (1970), there exists a garbling of signals that generates G from $\tilde{G}$. This means that a social planner could ignore the additional information contained in $\tilde{G}$ and reproduce the quantity-weighted distribution of average qualities corresponding to the optimal allocation under G and thus the same value. While this allocation is feasible under $\tilde{G}$, it is not optimal. This follows from the easily verified property that the unique competitive equilibrium (which as argued is also the optimal allocation) differs across these two information structures. $\qquad\square$

Proof of Proposition 3

Proof. Following the first condition given in Proposition 3 part (i) in Kolotilin (2018), full disclosure up to some threshold z^* and complete pooling above is optimal when $V''(z)$ changes sign from positive to negative. Note that $V''(z)$ has the same sign as

$$\frac{V''(z)}{\pi''(z)} = (1 - \gamma) + \lambda \frac{S''(z)}{\pi''(z)}.$$

So when either of the two first conditions given in the proposition holds, then $V''(z)$ can be always positive, always negative, or switch sign from positive to negative. If it is always positive, full disclosure is optimal, and if it is always negative, no disclosure is optimal. In these two cases z^* is at an extreme. When $V''(z)$ changes sign, z^* is an interior point. In this case, there is an interval with full disclosure followed by no disclosure.

The proof of the second part is analogous to the first one, using instead the second condition in Proposition 3 part (i) in Kolotilin (2018). $\qquad\square$

Proof of Proposition 4

To prove this proposition, we first need to state the following two lemmas.

Lemma 5. *1) Let $Q(z)$ be the equilibrium output for an upper interval disclosure policy with threshold z. Let $m_L(z)$ denote the conditional mean below z (the pooling interval), $p_L = P(Q(z) + m_L)$ and $p = P(Q(z) + z)$. Then $Q'(z)$ has the same sign as*

$$S'(p_L)(p - p_L) - (S(p) - S(p_L)).$$

2) Let $Q(z)$ be the equilibrium output for a lower interval disclosure policy with threshold z. Let $m_H(z)$ denote the conditional mean above z (the pooling interval), $p_H = P(Q(z) + m_H)$ and $p = P(Q(z) + z)$. Then $Q'(z)$ has the same sign as

$$S'(p_H)(p_H - p) - (S(p_H) - S(p)).$$

Proof. Consider the upper interval disclosure with threshold z. Equilibrium output $Q(z)$ is the solution to

$$Q(z) = F(z) S(P(Q(z)) + m_L(z)) + \int_z S(P(Q(z)) + s) \, dF(s),$$

where $m_L(z)$ is the conditional mean below the threshold z (the pooling interval). Differentiating with respect to z,

$$Q'(z) = f(z) \frac{S'(P(Q(z)) + m_L(z))(z - m_L(z)) - (S(P(Q(z)) + z) - S(P(Q(z)) + m_L(z)))}{1 - P'(Q(z))\left(F(z) S'(P(Q(z)) + m_L(z)) + \int_z S'(P(Q(z)) + s) \, dF(s)\right)}.$$

The denominator is positive, so the sign of $Q'(z)$ is equal to the sign of the numerator:

$$S'(P(Q(z)) + m_L(z))(z - m_L(z)) - (S(P(Q(z)) + z) - S(P(Q(z)) + m_L(z))).$$

The proof of the second part follows similar calculations. ☐

The following lemma gives the conditions that determine the sign of the above expressions.

Lemma 6. *Consider an optimal disclosure policy that is given by a threshold with pooling on one side of the threshold (above or below) and complete separation on the other side. Then a marginal increase*

in the pooling region will increase total output (resp. decrease total output) when $\gamma > 1/2$ (resp. when $\gamma < 1/2$).

Proof. Consider first the case where z corresponds to the threshold of a lower disclosure interval. Let $m_H(z)$ denote the mean above z (the pooling interval), $p_H = P(Q(z) + m_H)$ and $p = P(Q(z) + z)$. Following Kolotilin (2018), $V(m_H(z)) - V(z) - (m_H(z) - z)V'(m_H(z)) = 0$. Since $V(s) = \pi(P(Q(z)) + s) + \lambda S(P(Q(z)) + s)$ and the first term is convex, for this equality to hold it is necessary that

$$\lambda\left[S(P(Q(z)) + m_H(z)) - S(P(Q(z)) + z)\right] > \lambda\left[(m_H(z) - z)S'(P(Q(z)) + m_H(z))\right].$$

For $\gamma > 1/2$, $\lambda > 0$, so this implies that $S(p_H) - S(p) > (p_H - p)S'(p_H)$, and by Lemma 5, it follows that $Q'(z) < 0$. An increase in the pooling region corresponds to a decrease in z, so total output increases. The reverse is obviously true when $\gamma < 1/2$.

Consider now the case where z corresponds to the threshold of an upper disclosure interval. Let $m_L(z)$ denote the mean above z (the pooling interval), $p_L = P(Q(z) + m_L)$ and $p = P(Q(z) + z)$. Following Kolotilin (2018), $V(z) - V(m_L(z)) - (z - m_L(z))V'(m_L(z)) = 0$. Since $V(s) = \pi(P(Q(z)) + s) + \lambda S(P(Q(z)) + s)$ and the first term is convex, for the equality to hold it is necessary that

$$\lambda\left[S(P(Q(z)) + m_H(z)) - S(P(Q(z)) + z)\right] < \lambda\left[(m_H(z) - z)S'(P(Q(z)) + m_H(z))\right].$$

For $\gamma > 1/2$, $\lambda > 0$, so this implies that $S(p_H) - S(p) < (p_H - p)S'(p_H)$, and by Lemma 5, it follows that $Q'(z) > 0$. An increase in the pooling region corresponds to an increase in z, so total output increases. The reverse is obviously true when $\gamma < 1/2$. $\qquad\square$

Proof of Proposition 4

Proof. Consider first the setting where the optimal policy is a left disclosure interval, as in Proposition 3.

$$W\left(z,\gamma\right) = (1-\gamma)\left[\int_0^z \pi\left(P\left(Q\left(z\right)\right)+s\right)dF\left(s\right)+\left(1-F\left(z\right)\right)\pi\left(P\left(Q\left(z\right)\right)+m_H\right)\right]$$
$$+\gamma\int_0^{Q(z)}\left(p-P\left(Q\left(z\right)\right)\right)dz,$$

where $Q\left(z\right)$ is the equilibrium output under this policy.

$$\frac{\partial W}{\partial z} = (1-\gamma)f\left(z\right)\left(\pi\left(z\right)-\pi\left(m_H\right)\right)+(1-\gamma)f\left(z\right)\left(m_H-z\right)\pi'\left(m_H\right)+(1-2\gamma)Q\left(z\right)P'\left(Q\left(z\right)\right)Q'\left(z\right)$$

Taking derivative with respect to γ,

$$\frac{\partial^2 W}{\partial z\partial\gamma} = -f\left(z\right)\left[\left(\pi\left(z\right)-\pi\left(m_H\right)\right)+\left(m_H-z\right)\pi'\left(m_H\right)\right]-2Q\left(z\right)P'\left(Q\left(z\right)\right)Q'\left(z\right)$$
$$= -\frac{(1-\gamma)}{1-\gamma}f\left(z\right)\left[\left(\pi\left(z\right)-\pi\left(m_H\right)\right)+\left(m_H-z\right)\pi'\left(m_H\right)\right]-\frac{2\left(1-\gamma\right)}{1-\gamma}Q\left(z\right)P'\left(Q\left(z\right)\right)Q'\left(z\right)$$
$$= 0-\frac{1}{1-\gamma}Q\left(z\right)P'\left(Q\left(z\right)\right)Q'\left(z\right).$$

When $\gamma > 1/2$, Lemma 6 implies that $Q'\left(z\right) < 0$, so the cross partial is negative and the pooling region increases with γ. When $\gamma < 1/2$, $Q'\left(z\right) > 0$, so the cross partial is positive and the pooling region decreases with γ.

Consider next the case where the optimal policy is a right disclosure interval, as in Proposition 3:

$$W\left(z,\gamma\right) = (1-\gamma)\left[F\left(z\right)\pi\left(P\left(Q\left(z\right)\right)+m_L\right)+\int_z\pi\left(P\left(Q\left(z\right)\right)+s\right)dF\left(s\right)\right]$$
$$+\gamma\int_0^{Q(z)}\left(p-P\left(Q\left(z\right)\right)\right)dz,$$

where $Q(z)$ is the equilibrium output under this policy.

$$\frac{\partial W}{\partial z} = (1-\gamma) f(z) (\pi(m_L) - \pi(z)) + (1-\gamma) f(z) (z - m_L) \pi'(m_L) + (1-2\gamma) Q(z) P'(Q(z)) Q'(z)$$

Taking derivative with respect to γ,

$$\begin{aligned}
\frac{\partial^2 W}{\partial z \partial \gamma} &= -f(z) [(\pi(m_L) - \pi(z)) - (z - m_L) \pi'(m_L)] + -2Q(z) P'(Q(z)) Q'(z) \\
&= -\frac{1}{1-\gamma} Q(z) P'(Q(z)) Q'(z).
\end{aligned}$$

When $\gamma > 1/2$, 6 implies that $Q'(z) > 0$, so the cross partial is positive and the pooling region increases with γ. When $\gamma < 1/2$, $Q'(z) < 0$, so the cross partial is negative and the pooling region decreases with γ. $\qquad \square$

Proof of Corollary 1

Proof. Consider a partition of the set of sellers into sets $S_1, ..., S_N$. Suppose there are two sets S_k, S_{k+1} that are not totally ordered in quality with means $M_k \leq M_{k+1}$ and mass G_k and G_{k+1}. By reordering elements of these two sets, one can substitute S_k and S_{k+1} with two new disjoint sets S'_k and S'_{k+1} of equal measures to the original ones, where $S_k \cup S_{k+1} = S'_k \cup S'_{k+1}$ and $S'_k < S'_{k+1}$, element-wise. By construction, $M'_k \leq M_k \leq M_{k+1} \leq M'_{k+1}$ and $G'_k M'_k + G'_{k+1} M'_{k+1} = G_k M_k + G_{k+1} M_{k+1}$. This corresponds to a mean preserving spread of the original distribution of means and thus gives higher surplus. $\qquad \square$

Proof of Lemma 2

To totally differentiate Equation (5) with respect to z_k, first note that by the envelope condition, we can ignore the effect on the output choices $q_1, ..., q_N$. In particular, this implies that $\partial Q / \partial z_k = f(z_k)(q_k - q_{k+1})$. Since $M_k = \int_{z_{k-1}}^{z_k} z dF(z) / (F(z_k) - F(z_{k-1}))$, it follows that

$$\frac{\partial (F(z_k) - F(z_{k-1})) M_k}{\partial z_k} = f(z_k) z_k, \quad \frac{\partial (F(z_{k+1}) - F(z_k)) M_{k+1}}{\partial z_k} = -f(z_k) z_k.$$

47

The result now follows by totally differentiating (5) and setting it equal to zero.

Proof of Proposition 5

To prove this proposition we need an intermediate step, which is proven using the following lemma.

Lemma 7. *The optimal thresholds satisfy the following condition:*

$$\frac{z_k - M_k}{M_{k+1} - M_k} s\left(p_k\right) + \frac{M_{k+1} - z_k}{M_{k+1} - M_k} s\left(p_{k+1}\right) = \frac{\int_{p_k}^{p_{k+1}} s\left(p\right) dp}{p_{k+1} - p_k}, \tag{14}$$

where M_k and M_{k+1} are the conditional mean qualities for the two groups, and p_k and p_{k+1} the equilibrium prices.

Proof. First note that

$$
\begin{aligned}
\left(P\left(Q\right) + z_k\right)\left(q_{k+1} - q_k\right) &= \left(P\left(Q\right) + M_{k+1} - M_{k+1} + z_k\right) q_{k+1} \\
&\quad - \left(P\left(Q\right) + M_k - M_k + z_k\right) q_k \\
&= p_{k+1} q_{k+1} - p_k q_k - \left(M_{k+1} - z_k\right) q_{k+1} - \left(z_k - M_k\right) q_k.
\end{aligned}
$$

Substituting in (6) and rearranging gives

$$\left(M_{k+1} - z_k\right) q_{k+1} + \left(z_k - M_k\right) q_k = \pi_{k+1} - \pi_k.$$

Equation (14) follows by substituting $\pi_{k+1} - \pi_k = \int_{p_k}^{p_{k+1}} s\left(p\right) dp$, using $q_{k+1} = s\left(p_{k+1}\right)$ and $q_k = s\left(p_k\right)$, and dividing through the left hand side by $\left(M_{k+1} - M_k\right)$ and the right hand side by the equivalent value $p_{k+1} - p_k$. $\square$

We use the expression found in Lemma 7. Equation (14) equates the expected value of $s\left(p\right)$ under two lotteries. The left hand side lottery has weights $\alpha = \left(z_k - M_k\right) / \left(M_{k+1} - M_k\right)$ on price p_k and $\left(1 - \alpha\right)$ on price p_{k+1}. The second lottery is uniform between these two extreme prices.

When s is linear, it must be the case that $\alpha = 1/2$ and this implies that

$$z_k - M_k = M_{k+1} - z_k. \tag{15}$$

When s is convex, $\alpha > 1/2$ so $z_k - M_k > M_{k+1} - z_k$, so the optimal threshold is above the one defined by equation (15), while the reverse occurs when s is concave. This concludes the proof.

Proof of Proposition 6

Without loss of generality let $S(p) = p$, so the cost function $c(q) = \frac{1}{2}q^2$. Consider now the objective function (5) for this case:

$$W(\boldsymbol{z}) = \int_0^Q P(x)\,dx + \sum_{k=1}^N [F(z_k) - F(z_{k-1})] \left[M_k(P + M_k) - \frac{1}{2}\left((P + M_k)^2\right) \right] \tag{16}$$

$$= \int_0^Q P(x)\,dx + \sum_{k=1}^N [F(z_k) - F(z_{k-1})] \left[\frac{1}{2}M_k^2 - \frac{1}{2}P^2 \right]. \tag{17}$$

After suppressing the terms that are unaffected by the partition, maximizing this expression is equivalent to maximizing

$$\sum_{k=1}^N [F(z_k) - F(z_{k-1})](M_k - \bar{z})^2, \tag{18}$$

where $\bar{z} = \sum_{k=1}^N [F(z_k) - F(z_{k-1})] M_k$ is the mean firm quality, which is independent of the partition. The above expression is the variance between partitions. Since total variance is fixed, maximizing (18) is equivalent to minimizing (7). Uniqueness of the thresholds is guaranteed when the distribution has log-concave density, as shown in Mease and Nair (2006).

Proof of Proposition 7.

Proof. Let $\bar{M}_1$ and $\bar{M}_2$ be the conditional mean of z below and above the mean $\bar{z}$, respectively. By the variance decomposition,

$$
\begin{aligned}
\int (z - \bar{z})^2 \, dF(z) &= \int^{\bar{z}} \left(z - \bar{M}_1\right)^2 dF(z) + \int_{\bar{z}} \left(z - \bar{M}_2\right)^2 dF(z) \\
&\quad + F(\bar{z}) \left(\bar{M}_1 - \bar{z}\right)^2 + (1 - F(\bar{z})) \left(\bar{M}_2 - \bar{z}\right)^2 \\
&= F(\bar{z}) \left(cv_1^2 + 1\right) \left(\bar{M}_1 - \bar{z}\right)^2 + (1 - F(\bar{z})) \left(cv_2^2 + 1\right) \left(\bar{M}_2 - \bar{z}\right)^2 \\
&\leq \left(\max\left\{cv_1^2, cv_2^2\right\} + 1\right) \left(F(\bar{z}) \left(\bar{M}_1 - \bar{z}\right)^2 + (1 - F(\bar{z})) \left(\bar{M}_2 - \bar{z}\right)^2\right),
\end{aligned}
$$

where the second equality follows from

$$
cv_1 = \frac{\int^{\bar{z}} \left((z - \bar{z}) - \left(\bar{M}_1 - \bar{z}\right)\right)^2 dF(z)}{F(\bar{z}) \left(\bar{M}_1 - \bar{z}\right)^2}
$$

and similarly for cv_2. From the above inequality,

$$
\frac{F(\bar{z}) \left(\bar{M}_1 - \bar{z}\right)^2 + (1 - F(\bar{z})) \left(\bar{M}_2 - \bar{z}\right)}{\int (z - \bar{z})^2 \, dF(z)} \geq \frac{1}{1 + \max\left\{cv_1^2, cv_2^2\right\}}.
$$

This gain corresponds to setting $z^* = \bar{z}$, so it is a lower bound to the gains under the optimal threshold. $\square$

Proof of Proposition 8.

To prove this proposition, we first need to show the following lemma.

Lemma 8. *Let $g(z_1), ..., g(z_{N-1})$ be the optimal thresholds for $\tilde{F}$. Let $M_k = m(z_{k-1}, z_k) = E_F(z_{k-1} \leq z \leq z_k)$. Then $z_k - M_k > M_{k+1} - z_k$.*

Proof. Let $\tilde{M}_k = E_{\tilde{F}}(g(\tilde{z}_{k-1}) \leq z \leq g(\tilde{z}_k))$. Note that by strict convexity of g, $\tilde{M}_k > g(M_k)$. It follows that

$$
\begin{aligned}
z - M_k \ &> \ z_k - g^{-1}\left(\tilde{M}_k\right) \\
&= \ g^{-1}\left(g\left(z_k\right)\right) - g^{-1}\left(\tilde{M}_k\right) \\
&= \ g^{-1}\left(\tilde{M}_{k+1}\right) - g^{-1}\left(g\left(z_k\right)\right) \\
&> \ M_{k+1} - z_k.
\end{aligned}
$$

$\square$

To prove the proposition, let the vector $\{l_k\}$ be the optimal thresholds for F and $\{z_k\}$ the optimal thresholds for $\tilde{F}$. Equation (21) follows from the necessary condition for optimal thresholds, and (23) follows from the previous lemma.

Proof of Proposition 9

We use the following properties of distributions with log-concave densities (see Lemma 1 in Mease and Nair (2006)):

$$
\mathbb{E}\left(z | s \le z \le s + d\right) - s \text{ is decreasing in } s \text{ for } d > 0 \text{ and} \tag{19}
$$

$$
s - \mathbb{E}\left(z | s - d \le z \le s\right) \text{ is increasing in } s \text{ for } d > 0, \tag{20}
$$

and these properties are preserved when conditioning on intervals.

Lemma 9. *Suppose F is a distribution with log-concave density and let $m\left(a, b\right) = E_F\left(z | a \le z \le b\right).$ Suppose the vector of thresholds $\{l_k\}_{k=1}^{N-1}$ satisfies*

$$
l_k - m\left(l_{k-1}, l_k\right) = m\left(l_k, l_{k+1}\right) - l_k \tag{21}
$$

and let $z_1, ..., z_{N-1}$ be a vector such that

$$
z_k - m\left(z_{k-1}, z_k\right) > m\left(z_k, z_{k+1}\right) - z_k. \tag{22}
$$

51

Then $z_k > l_k$ for all k.

To prove Lemma 9 we use first the following:

Claim. Under the assumptions of Lemma 9, suppose that for some k, $z_k < l_k$ and $z_{k+1} - z_k \geq l_{k+1} - l_k$. Then $z_{k-1} < l_{k-1}$ and $z_k - z_{k-1} \geq l_k - l_{k-1}$.

Proof. Note that

$$
\begin{aligned}
z_k - m\left(z_{k-1}, z_k\right) \;>\; & m\left(z_k, z_{k+1}\right) - z_k && (23) \\
\geq\; & m\left(z_k, z_k + l_{k+1} - l_k\right) - z_k \\
\geq\; & m\left(l_k, l_{k+1}\right) - l_k \\
=\; & l_k - m\left(l_{k-1}, l_k\right).
\end{aligned}
$$

The first inequality follows from (22), the second one from monotonicity of m, the third from (19), and the last from (21). Now consider $k-1$. We will show that $z_k - z_{k-1} \geq l_k - l_{k-1}$.

Suppose, by way of contradiction, that $z_k - z_{k-1} < l_k - l_{k-1}$. Then

$$
\begin{aligned}
z_k - m\left(z_{k-1}, z_k\right) \;\leq\; & l_k - m\left(l_k - \left(z_k - z_{k-1}\right), l_k\right) \\
\leq\; & l_k - m\left(l_{k-1}, l_k\right)
\end{aligned}
$$

where the first inequality follows from condition (20) and the second one from the monotonicity of m. This inequality contradicts (23), proving that $z_k - z_{k-1} \geq l_k - l_{k-1}$. Given that $z_k < l_k$, this also guarantees that $z_{k-1} < l_{k-1}$. $\qquad\square$

We now prove Lemma 9. Let h denote the highest k for which $z_k < l_k$. By the definition of h, $z_{h+1} - z_h > l_{h+1} - l_h$. Using inductively the previous claim, it follows that the same is true for all $k = 1, ..., h$. For $k = 1$, the claim would imply that $z_0 < l_0$, which cannot be true if the distribution had a lower bound, since in that case both z_0 and l_0 should equal this lower bound. For unbounded support, an argument similar to the one used in the claim can be used to generate a contradiction. This completes the proof.

Proof of Proposition 9.

Let $\{l_k\}$ denote the optimal thresholds for the linear supply function and $\{z_k\}$ those for the convex supply function. Lemma 3 and Equations (21) and (23) hold, so Lemma 9 proves the proposition.

Proof of Proposition 10

To prove this proposition, we need to show the following lemma first:

Lemma 10. *The term $dQ\left(\boldsymbol{z}\right)/dz_k$ has the same sign as*

$$\frac{z_k - M_k}{M_{k+1} - M_k} s'\left(p_k\right) + \frac{M_{k+1} - z_k}{M_{k+1} - M_k} s'\left(p_{k+1}\right) - \frac{\int_{p_k}^{p_{k+1}} s'\left(p\right) dp}{p_{k+1} - p_k}. \tag{24}$$

Proof. Total output is

$$Q = \sum_{k=1}^{N} \left(F\left(z_k\right) - F\left(z_{k-1}\right)\right) s\left(p_k\right),$$

where $p_k = P\left(Q\right) + M_k$. Differentiating with respect to z_k and using

$$\begin{aligned}
\left(F\left(z_k\right) - F\left(z_{k-1}\right)\right) \frac{\partial M_k}{\partial z_k} &= f\left(z_k\right)\left(z_k - M_k\right) \\
\left(F\left(z_{k+1}\right) - F\left(z_k\right)\right) \frac{\partial M_{k+1}}{\partial z_k} &= f\left(z_k\right)\left(M_{k+1} - z_k\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Q}{\partial z_k} &= f\left(z_k\right)\left(s\left(p_k\right) - s\left(p_{k+1}\right)\right) \\
&+ f\left(z_k\right)\left[s'\left(p_k\right)\left(M_k - z_k\right) + s'\left(p_{k+1}\right)\left(M_{k+1} - z_k\right)\right] \\
&+ \sum_{k=1}^{N}\left(F\left(z_k\right) - F\left(z_{k-1}\right)\right) s'\left(p_k\right) P'\left(Q\right) \frac{\partial Q}{\partial z_k},
\end{aligned}$$

we get

$$\frac{\partial Q}{\partial z_k} = \frac{f\left(z_k\right)\left[s\left(p_k\right) - s\left(p_{k+1}\right) + s'\left(p_k\right)\left(M_k - z_k\right) + s'\left(p_{k+1}\right)\left(M_{k+1} - z_k\right)\right]}{1 - \sum_{k=1}^{N}\left(F\left(z_k\right) - F\left(z_{k-1}\right)\right) s'\left(p_k\right) P'\left(Q\right)}.$$

The denominator is positive since $s'(p_k) > 0$ and $P'(Q) < 0$, so $\partial Q / \partial z_k$ has the same sign as

$$s(p_k) - s(p_{k+1}) + s'(p_k)(M_k - z_k) + s'(p_{k+1})(M_{k+1} - z_k),$$

and since $s(p_k) - s(p_{k+1}) = -\int_{p_k}^{p_{k+1}} s'(p)\, dp$ and $p_{k+1} - p_k = M_{k+1} - M_k$, Equation (24) follows.

$\square$

Proof of Proposition 10

Letting $\alpha(z_k) = \frac{M_{k+1} - z_k}{M_{k+1} - M_k}$, we can rewrite Equation (14) as

$$s(p_k) + \alpha(z_k)(s(p_{k+1}) - s(p_k)) = s(p_k) + \frac{\int_{p_k}^{p_{k+1}} s(p) - s(p_k)\, dp}{p_{k+1} - p_k},$$

so

$$\alpha(z_k) = \frac{\int_{p_k}^{p_{k+1}} \frac{s(p) - s(p_k)}{s(p_{k+1}) - s(p_k)}\, dp}{p_{k+1} - p_k}. \tag{25}$$

To evaluate dQ/dz_k at the optimal thresholds $z_1, ..., z_k$ we rewrite Equation (24) in a similar fashion using the expression for $\alpha(z_k)$ given by Equation (25).

$$
\begin{aligned}
\frac{dQ}{dz_k} &= \alpha(z_k)(s'(p_{k+1}) - s'(p_k)) - \frac{\int_{p_k}^{p_{k+1}} (s'(p) - s'(p_k))\, dp}{p_{k+1} - p_k} \\
&= \frac{\int_{p_k}^{p_{k+1}} \frac{s(p) - s(p_k)(s'(p_{k+1}) - s'(p_k))}{s(p_{k+1}) - s(p_k)}\, dp - \int_{p_k}^{p_{k+1}} (s'(p) - s'(p_k))\, dp}{p_{k+1} - p_k},
\end{aligned}
$$

so a sufficient condition for dQ/dz_k to be positive (negative) is that

$$\frac{(s(p) - s(p_k))(s'(p_{k+1}) - s'(p_k))}{s(p_{k+1}) - s(p_k)} - (s'(p) - s'(p_k)) > 0\ (< 0),$$

or, equivalently,

$$\frac{s'(p_{k+1}) - s'(p_k)}{s(p_{k+1}) - s(p_k)} - \frac{s'(p) - s'(p_k)}{s(p) - s(p_k)} > 0\ (< 0). \tag{26}$$

A sufficient condition for this equation to hold is that

$$\frac{s'\left(p\right)-s'\left(p_{k}\right)}{s\left(p\right)-s\left(p_{k}\right)} \tag{27}$$

increasing (resp. decreasing) in p (for all $p > p_k$). The derivative of (27) with respect to p has the sign of

$$s''\left(p\right)\left(s\left(p\right)-s\left(p_{k}\right)\right)-s'\left(p\right)\left(s'\left(p\right)-s'\left(p_{k}\right)\right)$$
$$=s''\left(p\right)\int_{p_{k}}^{p}s'\left(x\right)dx-s'\left(p\right)\int_{p_{k}}^{p}\left(s''\left(x\right)\right)dx,$$

which in turn has the sign of

$$\frac{s''\left(p\right)}{s'\left(p\right)}-\frac{\int_{p_{k}}^{p}\frac{\left(s''\left(x\right)\right)}{s'\left(x\right)}s'\left(x\right)dx}{\int_{p_{k}}^{p}s'\left(x\right)dx}.$$

The second term is a weighted average of the coefficient of absolute risk aversion of s for values between p_k and p. So, if $s''\left(x\right)/s'\left(x\right)$ is increasing (resp. decreasing) in x, then this difference will be positive (resp. negative).

Proof of Lemma 4

Let

$$U\left(Q_{H},Q_{L},z^{*}\right) \;=\; \max_{A_{L},A_{H}}\int_{A_{L}}u\left(\theta,z_{L}\left(z^{*}\right)\right)d\Psi\left(\theta\right)+\int_{A_{H}}u\left(\theta,z_{H}\left(z^{*}\right)\right)d\Psi\left(\theta\right) \tag{28}$$
$$\text{subject to }\int_{A_{L}}d\Psi\left(\theta\right)=Q_{L}\text{and }\int_{A_{H}}d\Psi\left(\theta\right)=Q_{H}.$$

Given this problem, we can write the general problem as

$$V\left(z^{*}\right)=\max_{q_{L},q_{H}}U\left(q_{L}F\left(z^{*}\right),q_{H}\left(1-F\left(z^{*}\right)\right),z^{*}\right)-F\left(z^{*}\right)c\left(q_{L}\right)-\left(1-F\left(z^{*}\right)\right)c\left(q_{H}\right).$$

By the envelope theorem,

$$
\begin{aligned}
\frac{\partial V\left(z^{*}\right)}{\partial z^{*}} = \ & \frac{\partial U}{\partial Q_{L}} q_{L} f\left(z^{*}\right) - \frac{\partial U}{\partial Q_{H}} q_{H} f\left(z^{*}\right) + \frac{\partial U}{\partial z^{*}} \\
& - f\left(z^{*}\right) c\left(q_{L}\right) + f\left(z^{*}\right) c\left(q_{H}\right).
\end{aligned}
\tag{29}
$$

The first two derivatives are the respective multipliers p_{L} and p_{H} of the constraints in (28). To evaluate the last term, first we note that

$$
\frac{\partial z_{L}}{\partial z^{*}} = f\left(z^{*}\right) \frac{z^{*} - z_{L}\left(z^{*}\right)}{F\left(z^{*}\right)}
\tag{30}
$$

and

$$
\frac{\partial z_{H}}{\partial z^{*}} = f\left(z^{*}\right) \frac{z_{H}\left(z^{*}\right) - z^{*}}{1 - F\left(z^{*}\right)}.
\tag{31}
$$

Also,

$$
\begin{aligned}
\frac{\partial U}{\partial z^{*}} = \ & \left(\int_{(\theta_{1},\theta_{0}) \epsilon A_{L}} \theta_{1} dG\right)\left(\frac{\partial z_{L}}{\partial z^{*}}\right) \\
& + \left(\int_{(\theta_{1},\theta_{0}) \epsilon A_{H}} \theta_{1} dG\right)\left(\frac{\partial z_{H}}{\partial z^{*}}\right).
\end{aligned}
\tag{32}
$$

Finally, note that the measure of the set A_{L} is $Q_{L} = q_{L} F\left(z^{*}\right)$, and the measure of the set A_{H} is $Q_{H} = q_{H}\left(1 - F\left(z^{*}\right)\right)$. Dividing and multiplying (32) by these respective measures and substituting (30) and (31), we get

$$
\begin{aligned}
\frac{\partial U}{\partial z^{*}} = \ & f\left(z^{*}\right)\left(z^{*} - z_{L}\right) E\left(\theta_{1} \mid (\theta_{0}, \theta_{1}) \epsilon A_{L}\right) q_{L} \\
& + f\left(z^{*}\right)\left(z_{H} - z^{*}\right) E\left(\theta_{1} \mid (\theta_{0}, \theta_{1}) \epsilon A_{H}\right) q_{H}.
\end{aligned}
\tag{33}
$$

Substituting (33) in (29) we obtain

$$
\begin{aligned}
\frac{1}{f\left(z^{*}\right)} \frac{\partial V}{\partial z^{*}} = \ & p_{L} q_{L} - c\left(q_{L}\right) - \left[p_{H} q_{H} - c\left(q_{H}\right)\right] \\
& + \left(z^{*} - z_{L}\right) E\left(\theta_{1} \mid (\theta_{0}, \theta_{1}) \epsilon A_{L}\right) q_{L} \\
& + \left(z_{H} - z^{*}\right) E\left(\theta_{1} \mid (\theta_{0}, \theta_{1}) \epsilon A_{H}\right) q_{H}.
\end{aligned}
$$

Noting that the multipliers p_L and p_H are also the equilibrium prices, we can rewrite the first order condition for the optimal z^* as

$$
\begin{aligned}
\Pi\left(p_H\right) - \Pi\left(p_L\right) \; = \; & \left(z^* - z_L\right) E\left(\theta_1 \middle| \left(\theta_0, \theta_1\right) \epsilon A_L\right) q_L \\
+ \; & \left(z_H - z^*\right) E\left(\theta_1 \middle| \left(\theta_0, \theta_1\right) \epsilon A_H\right) q_H .
\end{aligned}
\tag{34}
$$