A Mediator Approach to Mechanism Design with Limited Commitment*

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Abstract

We study the role of information structures in mechanism design problems with limited commitment. In each period, a principal offers a "spot" contract to a privately informed agent without committing to future spot contracts, and the agent responds to the contract. In contrast to the classical approach in which the information structure is fixed, we allow for all admissible information structures. We represent the information structure as a fictitious mediator and re-interpret the model as a mechanism design problem by the mediator with commitment. The mediator collects the agent's private information and then, in each period, privately recommends the principal's spot contract and the agent's response in an incentive-compatible manner (both in truth-telling and obedience). We provide several examples to identify why new equilibrium outcomes can arise once we allow for general information structures. We next develop a durable-good monopoly application. We show that trading outcomes and welfare consequences can substantially differ from those in the classical model with a fixed information structure. In the seller-optimal mechanism, the seller offers a discounted price to the high-valuation buyer only in the initial period, followed by the high, surplus-extracting price until some endogenous deadline, when the buyer's information is revealed and hence fully extracted. As a result, the Coase conjecture fails: even in the limiting case of perfect patience, the seller makes a positive surplus, and the trading outcome is not the first best. We also characterize mediated and unmediated implementation of the seller-optimal outcome.

Keywords: Mechanism Design; Limited Commitment; Information Structures; Communication Equilibrium; Durable-Good Monopoly; Coase Conjecture.

JEL Classification: C7; D4; D8.

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1 Introduction

We provide a systematic analysis of information structures in mechanism design problems with limited commitment. A durable-good monopoly problem by a limited-committed seller is a classical application and also central in this paper: in each period, a seller (he) who owns a good and a buyer (she) may trade; if they trade, the game ends; otherwise, the game continues to the next period. The buyer's willingness to pay for the good is her private information and does not vary over time. With full commitment, the seller's optimal long-term mechanism takes the form of a perfectly rigid posted price (see, for example, Baron and Besanko, 1984). That is, the seller sets a price for the good at the beginning of the interaction, and the buyer buys in the initial period (resp., never buys) if her willingness to pay for the good is above (resp., below) that price. Accordingly, trade is inefficient, and both parties earn some ex ante surplus. With limited commitment, the celebrated *Coase conjecture* (Coase, 1972) argues that the seller's rent is diluted relative to the full-commitment case; in particular, in the limiting case of perfect patience, the buyer earns the entire trade surplus. The Coase conjecture is first formally shown by Stokey (1981), Fudenberg, Levine, and Tirole (1985), and Gul, Sonnenschein, and Wilson (1986) for the case in which the seller can only offer posted-price mechanisms. Skreta (2006) shows that this conclusion is robust even if the seller's feasible set of contracts is larger.

Some recent contributions observe that the standard model of mechanism design with limited commitment (such as that in the above papers) imposes an implicit assumption on the players' information structure. Thus, they study how the conclusion changes under alternative, but a specific class of, informational assumptions. For example, Doval and Skreta (2021a,b) consider the case in which, in each period, the seller can garble the information input by the buyer (namely, the report of her willingness-to-pay) in that period. On the one hand, such garbling can be costly for the seller because the allocation in the current period becomes less customized than without garbling. On the other hand, garbling can be beneficial because the next-period seller has more limited information and so a smaller set of deviations.[1] Indeed, they show that, with a fixed patience parameter, the seller's expected payoff can be strictly greater than that in the classical case in which information cannot be garbled. However, in the limiting case of perfect pa-

[1]See Bester and Strausz (2001, 2007) as earlier papers that point out this trade-off. This trade-off implies the failure of the revelation principle in the sense that it can be *with* loss of generality to focus on direct mechanisms in which the agent reports her type truthfully to the principal in each period.

tience, the Coase conjecture survives, and the buyer earns the entire trade surplus. Brzustowski, Georgiadis, and Szentes (2021) consider a more sophisticated class of contracts in which the seller sets up a long-term contract—subject to the constraint that he would not want to revise it in the future, reflecting the limited commitment nature of the problem. The contract determines the entire sequence of allocations as a function of the buyer's initial type report, but without revealing any information to the seller than that revealed through each period's trading outcome. In particular, the buyer's report can affect allocations in the far future, simultaneously keeping the seller in the dark. Brzustowski et al. (2021) show that with these "smart contracts", the Coase conjecture now fails: even in the limiting case of perfect patience, the seller can guarantee himself a non-vanishing expected revenue.[2]

As such, the recent literature points out that the information structure plays a crucial role in the predictions of mechanism design with limited commitment and, in particular, in a durable-good monopoly setting. Although each of the above papers considers some specific classes of information structures and studies their implications, a more systematic study of the entire class of admissible (in the sense formalized below) information structures seems important. For example, whereas Brzustowski et al. (2021) show that keeping the buyer's type report undisclosed to a limited-committed seller can be beneficial for the seller's ex ante expected revenue, is this the only way in which the seller deviates from the Coase conjecture? What is the optimal information structure from the seller's (or the buyer's or the society's) viewpoint? What about the efficiency properties? The goal of our paper is precisely this systematic analysis of information structures in mechanism design with limited commitment.

We begin by formalizing a general model of mechanism design with limited commitment in which any information structure that satisfies the following conditions is considered *admissible*. First, a player's private information in the game remains private whenever the player would like to do so. In the durable-good monopoly application, this means that the buyer's willingness to pay for the good is her private information in any admissible information structure.[3] Second, each period's realized allocation is publicly observed in that period. In the durable-good

[2]The above discussion is about the "gap case", in which the Coase conjecture holds in any equilibrium in the classical model of price-posting sellers. In the "no-gap case," the Coase conjecture fails even with posted prices if buyers use non-stationary strategies (Ausubel and Deneckere, 1989).

[3]Without this restriction, the seller's optimal information structure is trivially the one in which he *knows* the buyer's willingness to pay.

monopoly application, observability of whether trade happens and, if so, at which price seems a natural assumption.[4] Third, the contract chosen by the principal in each period is publicly observed in that period. This assumption is natural in many applications and also reflects the idea of limited commitment. Indeed, if the information structure is such that the agent cannot observe a principal's deviation, then the commitment outcome as in (as in Baron and Besanko, 1984) can be sustained.

Otherwise, our dynamic game form is standard: in every period, the principal offers the best contract from the feasible set given his information at that point; then, after observing the chosen contract, the agent chooses a message to input to the contract. The allocation of that period is publicly observed, and the game moves to the next period.

After introducing our general model, we provide three examples that show how the set of equilibrium outcomes expands once we allow for general information structures in mechanism design settings with limited commitment. Our examples identify three reasons for why this can be the case: delayed disclosure of stored information; punishing a principal's future self who deviates to a non-equilibrium contract by deleting any planned disclosure of stored information; punishing a principal's current or future self who deviates to a non-equilibrium contract by selecting a suboptimal continuation equilibrium.[5]

Next, we move to the characterization of the set of all equilibrium outcomes under all possible admissible information structures. To do so, we build on the notion of *(sequential) communication equilibrium* of Forges (1986), Myerson (1986), and Sugaya and Wolitzky (2021). They consider mediated communication in (finite) extensive-form games and establish various versions of *revelation principle* results depending on the equilibrium concepts of interest. Roughly, these results state that any equilibrium outcome given any information structure is attainable as an equilibrium outcome with the *canonical* information structure. In the canonical information structure, in each period: (i) a *(fictitious) mediator* (he) privately asks each player's private information and then privately recommends each player's action for the period; and (ii) players truthfully report their private information to the mediator and obey the mediator's recommendation. In a sense, we apply the concepts of (sequential) communication equilibrium to the environment in which one of the players is a limited-committed principal.

[4]In other applications, alternative assumptions might be more appropriate. For example, in a repeated auction environment, only the winner might surely be able to observe the winning price of that period. We believe that our framework is flexible in such dimensions.

[5]Of course, additional reasons may be possible.

However, precisely the point that the principal is one of the players is the main challenge we face. To see this, observe that a naive application of the revelation principle results would imply that, in the canonical information structure, the mediator recommends one of the feasible contracts to the principal, and the principal prefers obeying to deviating to any other contract. This statement, however, is not so useful in characterizing possible outcomes because the set of feasible contracts is too large to be tractably handled. First, recall that the revelation principle is not applicable for the principal's contract space. Potentially, the contract the mediator should recommend to the principal in each period might be a very complex indirect contract. Second, the set of obedience constraints could be large for the same reason, and there is not much guidance as to which constraints would be relevant.[6]

To circumvent this difficulty, we propose an alternative indirect approach. First, we consider an auxiliary game in which the mediator—not the principal—proposes an allocation in each period after he privately asks each player's information at the beginning of that period. This outcome-based approach avoids the first complication related to the seller's contract recommendations because the set of allocations is much more well-structured than the unrestricted set of contracts.

Second, we identify necessary conditions on outcomes to form an equilibrium of the game. In particular, for the principal's obedience, we focus on a small subset of contracts as potential deviations. In the durable-good monopoly application, we focus only on the seller's deviations to the constant contact that ends the game by allocating the good to all buyers types at a price equal to the smallest buyer's valuation. We ignore all the other obedience constraints. This result, formalized by Theorem 1, can be interpreted as a *relaxed revelation principle*. The theorem allows us to characterize candidate equilibrium outcomes by solving an upper-bound problem that is a much simpler relaxation of the original problem.

Of course, the converse need not be true. In particular, the choice of deviation contracts that implies the converse is not necessarily trivial in general. Thus, as a third step, we identify sufficient conditions on outcomes to form an equilibrium of the game. Theorem 2 shows that if an outcome satisfies the seller's obedience with respect to the sequence of contracts corresponding to the Coasean outcome of the game, then we can ignore all his other obedience constraints. Theorem 2 allows us to focus on a lower-bound problem that is simple enough to provide *implementation* results. First, Theorems 6 and 6 show that the seller-optimal outcome that solves the lower-bound problem is attainable in the original problem in which the mediator recommends a contract rather than an allocation, and all

[6]To the best of our knowledge, this challenge is novel in the literature.

the obedience constraints are considered. We call it the *mediated implementation* result. Next, in Section 4.3.2, we show that the seller-optimal outcome that solves the lower-bound problem is also attainable without the mediator as long as, in each period, the seller himself can garble the information provided to each of his future selves (not only to his next-period self, as in Doval and Skreta, 2021a,b) about the buyer's report at that period. We call this the *unmediated implementation* result. Which of the two implementation results is most useful may depend on the context.

Our characterization of the seller-optimal information structure in the durable-good monopoly application is summarized by Theorems 3, 4, and 5 and uncovers the following results of economic substance. First, in general, the seller finds it beneficial if the information input by the buyer arrives precisely (i.e., without garbling) but with delay. More formally, the optimal information structure specifies the time at which the buyer's private information report is made public; until then, no information—except whether a trade has happened or not—is revealed. This timing depends on the model's parameters, such as the distribution of the buyer type and the discount factor. That delayed information disclosure is beneficial to the seller is consistent with Brzustowski et al. (2021). However, our result shows that the seller can do strictly better than in Brzustowski et al. (2021)—even in the limiting case of perfect patience—and, in addition, that is the best possible he can do. Our result also implies that the seller's optimal information structure always makes the trade outcome inefficient, as the low-value buyer can trade with a significant delay (even in the limiting case of perfect patience).

Intuitively, delayed (but precise) information disclosure makes the seller's bargaining power stronger. Relative to the case in which no such information arrives, the seller's incentive to offer a more aggressive price is higher because even if the buyer did not buy at that aggressive price, full extraction would be possible once the time comes. Furthermore, once the seller of some period becomes aggressive, then the seller of the previous period can also be more aggressive because the buyer has less continuation payoff conditional on no trading. In this sense, the aggressiveness of each period's seller is a strategic complement to each other. Indeed, except for the initial period, the seller continues to offer the commitment price every period until the time of revelation. This price pattern is completely different from the classical Coase-conjecture pattern of decreasing prices: our case may rather be interpreted as the initial "fire sale" followed by the rigid high price.

1.1 Related literature

Our paper contributes to the literature on mechanism design with limited commitment. The failure of the revelation principle is well-known since the seminal contributions of Laffont and Tirole (1988) and Bester and Strausz (2001), further developed in Bester and Strausz (2000) and Bester and Strausz (2007). The mediator approach based on Forges (1986) and Myerson (1986) is first proposed by Bester and Strausz (2007) in the context of mechanism design with limited commitment. In Bester and Strausz (2007), the game has two periods, and the mechanism design is only in the initial period; in the second period, the principal just selects an action from a given set. Therefore, the main challenge we face in our long-horizon setting—namely, how to handle potentially large sets of contracts for the principal (both on-path and off-path)—does not arise. This challenge also makes our problem a non-trivial application of the (sequential) communication equilibrium notion of Forges (1986), Myerson (1986), and Sugaya and Wolitzky (2021). As we briefly discuss in the conclusion, we hope that our approach would be useful in other potential applications of (sequential) communication equilibrium in which some players' action spaces may be complicated.

Several papers study durable-good monopoly (or, equivalently, bargaining with one-sided incomplete information) as a representative application.[7] As explained above, classically, the literature of durable-good monopoly implicitly assumes that the seller's action is simply a price offer, but the (limited-committed) mechanism-design perspective adds by allowing for more general contracts (see the above discussion about Skreta, 2006; Doval and Skreta, 2021a,b; Brzustowski et al., 2021). In contrast to these papers, which consider specific classes of contracts and/or information structures, we propose a more systematic treatment of them.

Our paper also relates to the literature that checks the robustness of the Coase conjecture or documents its failure. For instance, a monopolist could relax its commitment problem and increase its profit by renting the good rather than selling it (Bulow, 1982), by introducing best-price provisions (Butz, 1990), or by introducing new updated versions of the durable good over time (Levinthal and Purohit, 1989; Waldman, 1993, 1996; Choi, 1994; Fudenberg and Tirole, 1998; Lee and Lee, 1998). Other studies have analyzed environments which preclude the market from fully deteriorating. These include environments with capacity constraints (Kahn, 1986; McAfee and Wiseman, 2008), with arrival of new traders (Sobel,

[7] Other than durable-good monopoly, Liu, Mierendorff, Shi, and Zhong (2019) consider posted prices in dynamic auction, and Gerardi and Maestri (2020) consider repeated production. Strulovici (2017) and Maestri (2017) study renegotiation with limited commitment.

1991; Inderst, 2008; Fuchs and Skrzypacz, 2010) or information (Daley and Green, 2019; Duraj, 2020; Lomys, 2021; Laiho and Salmi, 2021), with time-varying costs (Ortner, 2017), in which buyers' valuations are subject to idiosyncratic stochastic shocks (Biehl, 2001; Deb, 2014; Garrett, 2016), in which buyers can exercise an outside option (Board and Pycia, 2014), in which goods depreciate over time (Bond and Samuelson, 1987), and in which demand is discrete (Bagnoli, Salant, and Swierzbinski, 1989; von der Fehr and Kuhn, 1995; Montez, 2013). In contrast to these papers, we consider the same basic game as that in the classical seminal contributions. Instead, we examine alternative information structures.

1.2 Road Map

In Section 2, we present the general model, which illustrates the mediator approach to mechanism design with limited commitment. In Section 3, we provide several examples to motivate our approach to the topic. In Section 4, we study durable-good monopoly as a representative application of the mediator approach to mechanism design with limited commitment. In Section 5, we conclude. Omitted proofs and additional details are in the Appendices.

2 Model

In this section, we present the general model, which illustrates the mediator approach to mechanism design with limited commitment.

Primitives. There are three players: a *principal* (he, player P), an *agent* (she, player A), and a *mediator* (he, player M). *Time* is discrete and periods are indexed by $t \in \mathcal{T} := \{0, \dots, T\}$, where $2 \leq T \leq \infty$. Let $\mathcal{T}_0 := \mathcal{T} \setminus \{0\}$ and $\mathcal{T}_1 := \mathcal{T} \setminus \{0, 1\}$. At the beginning of period $t = 0$, the agent observes her *private information (type)* $\theta \in \Theta$, which is distributed according to a full-support probability distribution μ. The agent's type does not change over time. Each period $t \in \mathcal{T}_0$, as a result of the interaction between the players, an *allocation* $a_t \in A_t$ is determined; the set of admissible allocations may be time dependent—hence, its dependence on t. Each allocation set A_t contains the element $\varnothing$, corresponding to the non-participation allocation (or, equivalently, to the agent's outside option). For all $t \in \mathcal{T}_0$, let $A^t := \times_{\tau=1}^{t} A_\tau$ be the set of all allocation sequences of the form $a^t := (a_\tau)_{\tau=1}^{t}$. For all $t \in \mathcal{T}_1$, there is a correspondence $\mathcal{A}_t \colon A^{t-1} \rightrightarrows A_t$ such that, for all $a^{t-1} \in A^{t-1}$, $\mathcal{A}_t(a^{t-1}) \subseteq A_t$ describes the set of all allocations that the principal can offer in period t given the allocations he has offered through period $t-1$. This allows for the case in which past allocations restrict what the principal can offer the agent in the future. We assume that $\varnothing \in \mathcal{A}_t(a^{t-1})$ for all $t \in \mathcal{T}_1$ and $a^{t-1} \in A^{t-1}$.

The set of all *payoff-relevant pure outcomes* of the game is $\Theta \times A^T$. The principal's (resp., agent's) *preferences* are represented by a bounded payoff function $U_P \colon \Theta \times A^T \to \mathbb{R}$ (resp., $U_A \colon \Theta \times A^T \to \mathbb{R}$). The mediator is indifferent over payoff-relevant pure outcomes—that is, his preferences are represented by a constant payoff function $U_M \colon \Theta \times A^T \to \mathbb{R}$. All players are expected-payoff maximizers. Thus, the mediator can commit to any strategy.

In each period $t \in \mathcal{T}_0$, the principal offers a *spot contract* $C_t := (M_t, \alpha_t)$ to the agent, where: (i) M_t is a set of input messages to the contract; (ii) $\alpha_t \colon M_t \to \Delta(A_t)$ is an allocation rule. We write $\alpha_t(m_t)$ for the probability distribution on A_t when the input message to the contract is m_t and $\alpha_t(a_t \,|\, m_t)$ for the probability of allocation a_t when the input message to the contract is m_t. We endow the principal with a class of input message sets $\mathcal{M} := \{M^i\}_{i \in \mathcal{I}}$, where $\mathcal{I}$ is an index set; each M^i contains element $\varnothing$, interpreted as non-participation message. For all $t \in \mathcal{T}_0$, let $\mathcal{C}_t := \{C_t := (M_t, \alpha_t) \in \mathcal{M} \times \Delta(A_t)^{M_t} : \alpha_t(\varnothing \,|\, \varnothing) = 1, \alpha_t(\varnothing \,|\, m_t) = 0 \text{ for all } m_t \neq \varnothing\}$ be the set of all admissible spot contracts in period t. Let $\mathcal{C} := \{\mathcal{C}_t\}_{t \in \mathcal{T}_0}$ be the set of all admissible spot contracts. Finally, for all $t \in \mathcal{T}_0$, let $\mathcal{M}^t := \times_{\tau=1}^{t} \mathcal{M}$ and $\mathcal{C}^t := \times_{\tau=1}^{t} \mathcal{C}_t$. Hereafter, we refer to spot contract simply as contracts.

In period $t = 0$, the agent has a set of possible *private reports* to send to the mediator, denoted by R, with $|R| \geq |\Theta|$. In each period $t \in \mathcal{T}_0$, the principal (resp., the agent) has a set of possible *private signals* to receive from the mediator, denoted by $S_{P,t}$ (resp., $S_{A,t}$). For all $t \in \mathcal{T}_0$, let $S_P^t := \times_{\tau=1}^{t} S_{P,t}$ and $S_A^t := \times_{\tau=1}^{t} S_{A,t}$.

Timing. The timing of events within period $t = 0$ is the following:

0.1 The agent privately observes her type $\theta \in \Theta$;

0.2 The agent sends a private report $r \in R$ to the mediator.

The timing of events within each period $t \in \mathcal{T}_0$ is the following:

t.1 The mediator sends a private signal $s_{P,t} \in S_{P,t}$ to the principal;

t.2 The principal offers a contract $C_t \in \mathcal{C}_t$ to the agent with the property that $\sum_{a^t \in \mathcal{A}_t(a^{t-1})} \alpha_t(a_t \,|\, m_t) = 1$ for all $t \in \mathcal{T}_1$, $a^{t-1} \in A^{t-1}$, and $m_t \in M_t$;

t.3 After both the mediator and the agent observe C_t, the mediator sends a private signal $s_{A,t} \in S_{A,t}$ to the agent;

$t.4$ The agent sends input message $m_t \in M_t$ to the contract, where m_t is observed by the mediator, but not by the principal;

$t.5$ An allocation a_t is drawn from $\alpha_t(m_t)$, the allocation a_t is publicly observed, and the game proceeds to the next period.

For all $t \in \mathcal{T}$ and $n \in \{1, \ldots, 5\}$, we refer to $t.n$ as "stage $t.n$" of the game.

Histories. For all $t \in \mathcal{T}$, and $n \in \{1, \ldots, 5\}$, we denote by $h^{t.n}$ a history at the beginning of stage $t.n$ and by $H^{t.n}$ the set of all possible such histories. Then, we have $H^{0.1} = \{\emptyset\}$, $H^{0.2} = \Theta$, $H^{1.1} = \Theta \times R$, $H^{1.2} = \Theta \times R \times S_P^1$, $H^{1.3} = \Theta \times R \times S_P^1 \times \mathcal{C}^1$, $H^{1.4} = \Theta \times R \times S_P^1 \times \mathcal{C}^1 \times S_A^1$, $H^{1.5} = \Theta \times R \times S_P^1 \times \mathcal{C}^1 \times S_A^1 \times \mathcal{M}$, and, for all $t \in \mathcal{T}_1$,

$$\begin{aligned}
H^{t.1} &= \Theta \times R \times S_P^{t-1} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.2} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.3} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
H^{t.4} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t} \times \mathcal{M}^{t-1} \times A^{t-1}, \\
\text{and} \qquad H^{t.5} &= \Theta \times R \times S_P^{t} \times \mathcal{C}^{t} \times S_A^{t} \times \mathcal{M}^{t} \times A^{t-1}.
\end{aligned}$$

For all $t \in \mathcal{T}_0$, we denote by h^t a history at the end of period t and by H^t the sets of all possible such histories, with $H^t = \Theta \times R \times S_P^t \times \mathcal{C}^t \times S_A^t \times \mathcal{M}^t \times A^t$. We denote by H^T the set of all terminal histories of the game.

Information Sets. For all $i \in \{P, A, M\}$, $t \in \mathcal{T}$, and $n \in \{1, \ldots, 5\}$, let $h_i^{t.n}$ denote player i's information set at the beginning of stage $t.n$ and by $H_i^{t.n}$ the set of all possible such information sets. In particular, we have that: $H_P^{1.2}$ is the projection of $H^{1.2}$ on S_P^1 and, for all $t \in \mathcal{T}_1$, $H_P^{t.2}$ is the projection of $H^{t.2}$ on $S_P^t \times \mathcal{C}^{t-1} \times A^{t-1}$; $H_A^{0.2} = H^{0.2}$, $H_A^{1.4}$ is the projection of $H^{1.4}$ on $\Theta \times R \times \mathcal{C}^1 \times S_A^1$, and, for all $t \in \mathcal{T}_1$, $H_A^{t.4}$ is the projection of $H^{t.4}$ on $\Theta \times R \times \mathcal{C}^t \times S_A^t \times \mathcal{M}^{t-1} \times A^{t-1}$; $H_M^{1.1}$ is the projection of $H^{1.1}$ on R, $H_M^{1.3}$ is the projection of $H^{1.3}$ on $R \times S_P^1 \times \mathcal{C}^1$, and, for all $t \in \mathcal{T}_1$, $H_M^{t.1}$ is the projection of $H^{t.1}$ on $R \times S_P^{t-1} \times \mathcal{C}^{t-1} \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}$ and $H_M^{t.3}$ is the projection of $H^{t.3}$ on $R \times S_P^t \times \mathcal{C}^t \times S_A^{t-1} \times \mathcal{M}^{t-1} \times A^{t-1}$.

Mediated Contract-Selection Game. The above defines an extensive-form game, which we dub the *mediated contract-selection game* (hereafter, MCS game) and denote by $\mathcal{G}$. Game $\mathcal{G}$ is common knowledge among the players.

Strategies. A behavioral strategy for the principal is a collection of functions $\sigma_P := (\sigma_P^{t.2})_{t=1}^T$, where $\sigma_P^{t.2}\colon H_P^{t.2} \to \Delta(\mathcal{C}_t)$. A behavioral strategy for the agent is a collection of functions $\sigma_A := \big(\sigma_A^{0.2}, (\sigma_A^{t.4})_{t=1}^T\big)$, where $\sigma_A^{0.2}\colon H_A^{0.2} \to \Delta(R)$ and $\sigma_A^{t.4}\colon H_A^{t.4} \to \Delta(M_t)$. A behavioral strategy for the mediator is a collection of functions $\sigma_M := \big(\sigma_M^{t.1}, \sigma_M^{t.3}\big)_{t=1}^T$, where $\sigma_M^{t.1}\colon H_M^{t.1} \to \Delta\big(S_t^P\big)$ and $\sigma_M^{t.3}\colon H_M^{t.3} \to \Delta\big(S_t^A\big)$. A profile of behavioral strategies is $\sigma := (\sigma_P, \sigma_A, \sigma_M)$.

A prior μ and a profile of behavioral strategies σ induce a probability distribution over payoff-relevant pure outcomes. We extend players' payoff functions from payoff-relevant pure outcomes $(\theta, a^T) \in \Theta \times A^T$ to outcomes $\nu \in \Delta(\Theta \times A^T)$ in the usual way. For $i \in \{P, A\}$, we denote by $U_i(\nu)$ player i's ex ante expected payoff at the beginning of the MCS game under outcome ν.

Beliefs. A principal's belief is a collection of functions $\beta_P := (\beta_P^{t.2})_{t=1}^T$, where $\beta_P^{t.2}\colon H_P^{t.2} \to \Delta(H^{t.2})$. Similarly, an agent's belief is a collection of functions $\beta_A := \big(\beta_A^{0.2}, (\beta_A^{t.4})_{t=1}^T\big)$, where $\beta_A^{0.2}\colon H_A^{0.2} \to \Delta(H^{0.2})$ and $\beta_A^{t.4}\colon H_A^{t.4} \to \Delta(H^{t.4})$. Since the mediator is indifferent over payoff-relevant pure outcomes, there are no optimality conditions on the mediator's strategy, and hence no need to introduce beliefs for the mediator. A belief system is a pair $\beta := (\beta_P, \beta_A)$.

Solution Concept. We refer to a profile of behavioral strategies and a belief system (σ, β) as an assessment. The equilibrium notion we adopt is *weak perfect Bayesian equilibrium* (hereafter, wPBE), which is defined as follows.

Definition 1. An assessment (σ, β) is a weak perfect Bayesian equilibrium of $\mathcal{G}$ if:

1. σ_P is sequentially rational given (σ_M, σ_A) and β;
2. σ_A is sequentially rational given (σ_M, σ_P) and β; and
3. β is on-path consistent given σ.

We say that $\nu \in \Delta(\Theta \times A^T)$ is a *wPBE outcome* of $\mathcal{G}$ if there is a wPBE (σ, β) of $\mathcal{G}$ that (together with μ) induces ν. We denote by $\mathcal{E}$ (resp., $\mathcal{O}$) the set of all wPBEs (resp., wPBE outcomes) of $\mathcal{G}$.

Notation and Terminology. A contract $C_t \in \mathcal{C}_t$ is *constant* if, for some $a_t \in A_t$, we have $\alpha_t(a_t \,|\, m_t) = 1$ for all $m_t \in M_t \setminus \{\varnothing\}$. A contract $C_t \in \mathcal{C}_t$ is *direct* if $M_t = \{\varnothing\} \cup \Theta$. Suppose $\mathcal{E} \neq \emptyset$ and that $\arg\max_{\nu \in \mathcal{O}} U_P(\nu) \neq \emptyset$; we say that $\nu^* \in \Delta(\Theta \times A^T)$ is a principal-optimal wPBE outcome of $\mathcal{G}$ if $\nu^* \in \arg\max_{\nu \in \mathcal{O}} U_P(\nu)$.

Discussion. Our approach to mechanism design with limited commitment builds on the notion of (sequential) communication equilibrium in multistage games with communication (Forges, 1986; Myerson, 1986; Sugaya and Wolitzky, 2021, hereafter FMSW). As in the literature on communication equilibrium, we interpret the mediator as a fictitious player who has commitment power and designs the entire information structure (or communication protocol) of the game. We follow this approach because, by the revelation-principle results in FMSW (several versions depending on the specific solution concept), any admissible information structure of the underlying multistage game without a mediator and its equilibrium outcome can be represented by the mediated version of that game and its corresponding equilibrium outcome. An information structure is admissible if it satisfies the two following conditions. First, the agent's type remains private unless the agent wants to disclose it. Second, the exogenously given basic game—hence, the extensive form and the principal's limited commitment—is respected; in particular, in each period, the contract chosen by the principal and the realized allocation are publicly observed.

3 Motivating Examples

In this section, we provide three examples to motivate the mediator approach to mechanism design with limited commitment. The examples show that the set of all wPBE outcomes under our approach can be larger than that under the alternative approaches in the literature. We identify three reasons why this can be the case.

1. Information Storing and Delayed Disclosure. In the MCS game, the mediator can store a given period's information—namely, the agent's report to the mediator—and disclose it to the principal in future periods. Example 1 shows that the principal's ex ante expected wPBE payoff when information storing and delayed disclosure are possible can be greater than when a given period's information can only be either (noisily) disclosed in the current period or never disclosed.

2. Deletion of Stored Information. In the MCS game, because of the possibility of information storing and delayed disclosure, the information provided to the principal's future selves can depend on their contract choices. In particular, the mediator can punish a principal's future self who deviates to a non-equilibrium contract by deleting any planned disclosure of stored information. Example 2 shows that the deletion of stored information can serve as an effective threat to prevent the principal's future selves from deviating to non-equilibrium contracts. As a result, the principal's ex ante expected wPBE payoff can be greater than when such punishments are not possible.

3. Equilibrium Multiplicity and Equilibrium Selection. Principal-optimal wPBE outcomes may necessarily require suboptimal continuation equilibrium selection from the viewpoint of some of the principal's future selves. Such suboptimal equilibrium selection is possible in the MCS game. In particular, the mediator can punish a principal's current or future self who deviates to a non-equilibrium contract by suboptimal equilibrium selection. Example 3 shows that suboptimal continuation equilibrium selection can serve as an effective threat to prevent the principal from deviating to non-equilibrium contracts. As a result, the principal's ex ante expected wPBE payoff can be greater than when such punishments are not possible.

3.1 Example 1

Let $\mathcal{T} = \{0, 1, 2, 3\}$. The principal is a seller who owns one unit of a durable, indivisible good to which he assigns value 0 (normalization). The agent is a buyer whose private type $\theta \in \Theta = \{1, 2\}$ corresponds to her valuation for the good. Let $\mu = \frac{9}{10}$ be the probability that $\theta = 2$ at $t = 0$. If the agent participates in period t, an allocation for the period is a pair $(x_t, p_t) \in \{0, 1\} \times \mathbb{R}$, where x_t indicates whether the good is traded ($x_t = 1$) or not ($x_t = 0$) and p_t is a transfer from the agent to the principal. If $x_t = 1$ for some $t < 3$, by convention, allocation $(x_\tau, p_\tau) = (0, 0)$ is implemented for all $\tau \in \mathcal{T}$ with $\tau > t$. Thus, the allocation set in period t is $A_t = \{\varnothing\} \cup (\{0, 1\} \times \mathbb{R})$. If $a_t = \varnothing$, the principal's and the agent's flow payoffs are 0; if $a_t = (x_t, p_t)$, flow payoffs are $\theta x_t - p_t$ for the agent and p_t for the principal. The principal and the agent share a common discount factor $\delta = \frac{1}{2}$. The principal's and the agent's payoffs, $U_P(\theta, a)$ and $U_A(\theta, a)$, are the discounted sum of flow payoffs.

Information Storing and Delayed Disclosure. The following events correspond to a principal-optimal wPBE outcome the MCS game (see Section 4 for the complete equilibrium characterization):

- *Period* $t = 0$. The agent truthfully reports her type θ to the mediator.
- *Period* $t = 1$. The mediator recommends to the principal to offer a menu with two options: (i) trade with probability $\frac{17}{18}$ at price $\frac{59}{34}$; (ii) do not trade. The mediator recommends to type $\theta = 2$ to choose option (i) and to type

$\theta = 1$ to choose option (ii). The principal and the agent obey the mediator's recommendation.

- *Period* $t = 2$. Conditional on no trade before, the mediator recommends to the principal to offer a menu with two options: (i) trade with probability 1 at price 2; (ii) do not trade. The mediator recommends to type $\theta = 2$ to choose option (i) and to type $\theta = 1$ to choose option (ii). The principal and the agent obey the mediator's recommendation.
- *Period* $t = 3$. Conditional on no trade before, the mediator fully discloses the agent's type report θ to the principal. The principal offers to trade at price θ and type θ trades with probability 1 at price θ.

The mediator's key role in this principal-optimal wPBE of the MCS game is to store the agent's type report in period $t = 0$ and fully disclose it in period $t = 3$ to the principal. Note that the mediator's recommendation to the principal in periods $t = 1$ and $t = 2$ does not depend on the agent's report.

To better understand the scope of information storing and delayed disclosure, we contrast the previous principal-optimal wPBE outcome the MCS game with a principal-optimal wPBE outcome of the game in which a given period's information can only be either (noisily) disclosed in the current period or never disclosed. In particular, the following events correspond to a principal-optimal wPBE outcome of the BS/DS game (see Doval and Skreta (2021b) for the complete equilibrium characterization):

- *Period* $t = 1$. Type $\theta = 2$ trades with probability $\frac{8}{9}$ at price $\frac{7}{4}$, while type $\theta = 1$ does not trade.
- *Period* $t = 2$. Conditional on no trade before, type $\theta = 2$ trades with probability 1 at price $\frac{3}{2}$, while type $\theta = 1$ does not trade.
- *Period* $t = 3$. Conditional on no trade before, both types trades with probability 1 at price 1.

Let $x_{\ell,t}$ (resp., $x_{h,t}$) denote the probability with which type $\theta = 1$ (resp., $\theta = 2$) trades in period t; moreover, let $p_{\ell,t}$ (resp., $p_{h,t}$) denote the transfer from type $\theta = 1$ (resp., $\theta = 2$) to the principal conditional on trade in period t. Table 1 summarizes the two principal-optimal wPBE outcomes of the MCS game and the BS/DS game described above.

In both wPBE outcomes, type $\theta = 2$ trades gradually over time (i.e., with positive probability over multiple periods), whereas type $\theta = 1$ trades only in the last period, when the principal becomes sufficiently pessimistic. Crucially,

Table 1: Summary of the Equilibrium Outcomes

	Period $t=1$				Period $t=2$				Period $t=3$			
	$x_{\ell,1}$	$p_{\ell,1}$	$x_{h,1}$	$p_{h,1}$	$x_{\ell,2}$	$p_{\ell,2}$	$x_{h,2}$	$p_{h,2}$	$x_{\ell,3}$	$p_{\ell,3}$	$x_{h,3}$	$p_{h,3}$
MCS Game	0	0	$\frac{17}{18}$	$\frac{59}{34}$	0	0	$\frac{1}{18}$	2	1	1	0	0
BS/DS Game	0	0	$\frac{8}{9}$	$\frac{7}{4}$	0	0	$\frac{1}{9}$	$\frac{3}{2}$	1	1	0	0

however, price paths and trade probabilities are different in the two wPBEs: first, the price path is not decreasing over time in the MCS game, whereas it is so in the BS/DS game; second, the probability of trade in period $t = 1$ (resp., $t = 2$) is greater (resp., less) in the MCS game than in the BS/DS game.

The intuition behind the different equilibrium dynamics is the following. In the MCS game, in period $t = 3$, the principal can fully extract the surplus from the transaction thanks to the mediator's disclosure of the agent's type report (made in period $t = 0$). Such surplus extraction is not possible without information storing.

In period $t = 2$, the principal offers price 2 in the MCS game and price $\frac{3}{2}$ in the BS/DS game. In the MCS game, the principal has no incentive to lower the price below 2 because he knows that, even if type $\theta = 2$ does not buy the good, he can still sell it at price 2 (and fully extract the surplus) in period $t = 3$ thanks to the mediator's disclosure of the agent's type report. The principal's (posterior) belief on $\theta = 2$ at the beginning of period $t = 2$ is less than $\frac{1}{2}$, and so the ability of storing information and delaying its disclosure to the principal is crucial to sustain a price of 2; without such ability, sequential rationality would require the principal to post a price equal to 1 whenever his (posterior) belief on $\theta = 2$ at the beginning of period $t = 2$ is less than $\frac{1}{2}$. In other words, information storing and its delayed disclosure increases the principal's bargaining power in period $t = 2$.

In period $t = 1$, in the MCS game, the price is lower but the probability of trade is greater than in the BS/DS game. The probability of trade in the BS/DS game must be small enough for the principal's (posterior) belief on $\theta = 2$ at the beginning of period $t = 2$ to be no less than $\frac{1}{2}$ and sell only to type $\theta = 2$ at a price greater than 1; otherwise, sequential rationality would require the principal to sell to both types with probability 1 at price 1.

The principal's ex ante expected payoff in a principal-optimal wPBE outcome of the MCS game, denoted by U_P^*, is

$$U_P^* = \mu\left(\sum_{t=1}^{3} \delta^{t-1} x_{h,t} p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{3} \delta^{t-1} x_{\ell,t} p_{\ell,t}\right)$$

$$= \frac{9}{10}\left[\frac{17}{18}\frac{59}{34} + \frac{1}{2}\frac{1}{18}2\right] + \frac{1}{10}\frac{1}{4}1$$
$$= \frac{558}{360}.$$

The principal's ex ante expected payoff in a principal-optimal wPBE outcome of the BS/DS game, denoted by $\widehat{U}_P^*$, is

$$\widehat{U}_P^* = \mu\left(\sum_{t=1}^{3} \delta^{t-1} x_{h,t} p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{3} \delta^{t-1} x_{\ell,t} p_{\ell,t}\right)$$
$$= \frac{9}{10}\left[\frac{8}{9}\frac{7}{4} + \frac{1}{2}\frac{1}{9}\frac{3}{2}\right] + \frac{1}{10}\frac{1}{4}1$$
$$= \frac{540}{360}.$$

Thus, we have

$$U_P^* - \widehat{U}_P^* = \frac{1}{20} > 0,$$

showing that information storing and delayed disclosure are beneficial from the principal's ex ante viewpoint.

3.2 Example 2

Let $\mathcal{T} = \{0, 1, 2, 3\}$. The agent's type θ is uniformly distributed over $\Theta = \{-1, 1\}$. The allocation sets are $A_1 = A_3 = \{\varnothing, -1, 1\}$ and $A_2 = \{\varnothing, e, n\}$, where $a_2 = e$ is interpreted as "extract" and $a_2 = n$ is interpreted as "not extract". Non-participation in any period t is an irreversible option for the agent (i.e., it implies non-participation in all future periods). The principal's payoff is

$$U_P(\theta, a) = \begin{cases} 0 & \text{if } a_1 = \varnothing \text{ or } a_2 = \varnothing \\ \mathbb{1}_{\{a_2 = e\}} & \text{if } a_1 \neq \varnothing, a_2 \neq \varnothing, \text{ and } a_3 = \varnothing \\ \mathbb{1}_{\{a_2 = e\}} - \theta a_3 \lambda & \text{otherwise} \end{cases}$$

and the agent's payoff is

$$U_A(\theta, a) = \begin{cases} 0 & \text{if } a_1 = \varnothing \\ \theta a_1 \lambda - K & \text{if } a_1 \neq \varnothing \text{ and } a_2 = \varnothing \\ -1 - K & \text{if } a_1 \neq \varnothing, a_2 = e, \text{ and } a_3 = \varnothing \\ \theta a_1 \lambda - K & \text{if } a_1 \neq \varnothing, a_2 = n, \text{ and } a_3 = \varnothing \\ -1 & \text{if } a_1 \neq \varnothing, a_2 = e, \text{ and } a_3 \neq \varnothing \\ \theta a_1 \lambda + \theta a_3 & \text{if } a_1 \neq \varnothing, a_2 = n, \text{ and } a_3 \neq \varnothing \end{cases},$$

where $\lambda \in (1, 2)$ and $K > 0$ is large enough for participation in periods 2 and 3 to be optimal for the agent.

Deletion of Stored Information. If the principal could commit to the allocation sequence $(a_1, a_2, a_3) = (\theta, n, -\theta)$, the agent would be truthful and the principal's payoff would be $\lambda > 0$. However, without commitment power and without the possibility of storing information, the principal's ex ante expected payoff in any wPBE of the resulting game is 0. To establish this point, it suffices to show that, in any wPBE, the agent never participate to any contract in period $t = 1$. By contradiction, suppose the agent participates in period $t = 1$. Then, in period $t = 2$, the principal offers a constant contract which allocates $a_2 = e$ (and the agent participates), because it increases the principal's payoff without affecting the agent's incentive constraint in period $t = 3$; thus, the agent's flow payoff in period $t = 2$ is -1. In period $t = 3$, the principal's expected flow payoff is non-negative because he can always offer any constant contract guaranteeing 0 expected flow payoff; thus, the agent's flow payoff in period $t = 3$ is non-positive. Therefore, the agent's expected payoff from participating in period $t = 1$ is -1. This is a contradiction because the agent can always guarantee herself a payoff of 0 by not participating in period $t = 1$.

The next claim shows that the MCS game has a wPBE in which the principal's ex ante expected payoff is λ. As in Example 1, the mediator again stores the agent's type report in period $t = 0$ and fully disclose it in period $t = 3$ to the principal, but his role is different. In particular, the mediator's role is now to prevent the principal deviation to the constant contract allocating $a_2 = e$ in period $t = 2$; this is achieved by the mediator's threat of not disclosing the agent's type report in period $t = 3$ if the principal deviates in period $t = 2$. The next claim and its proof formalize these ideas.

Claim 1. The MCS game has a wPBE in which the principal's ex ante expected payoff is λ.

Proof. We begin by describing a candidate wPBE of the MCS game. On-path events are the following:

- *Period* $t = 0$. The agent truthfully report her type θ to the mediator.
- *Period* $t = 1$. The mediator sends signal $s_{P,1} = \theta$ with probability α and signal $s_{P,1} = -\theta$ with probability $1 - \alpha$ to the principal, where $(1 + \lambda)/2\lambda < \alpha < (2\lambda - 1)/2\lambda$. The principal offers a constant contract which allocates $a_1 = s_{P,1}$. The agent participates to the contract.
- *Period* $t = 2$. The principal offers a constant contract which allocates $a_2 = n$. The agent participates to the contract.

- *Period* $t = 3$. The mediator fully discloses the agent's type report θ (made in period $t = 0$) to the principal. The principal offers a constant contract which allocates $a_3 = -\theta$. The agent participates to the contract.

Off-path events are the following:

- If the principal observes any off-path event by period $t = 2$, then he offers a constant contract which allocates $a_2 = e$.
- If the principal deviates in period $t = 1$ or $t = 2$ (which is observed by the mediator by assumption), then the mediator does not reveal the agent's type report θ (made in period $t = 0$) to the principal in period $t = 3$.
- If the principal observes any off-path event by period $t = 3$, then he offers any best responding contract in period $t = 3$.

The principal's ex ante expected payoff in the candidate wPBE is λ. Thus, it remains to show that the candidate wPBE is indeed a wPBE of the MCS game. We begin with the agent's incentives. The agent's participation in periods $t = 2$ and $t = 3$ is optimal. In period $t = 0$, the agent's expected continuation payoff from truthfully reporting her type to the mediator is $\lambda(\alpha - (1 - \alpha)) - 1$, whereas her expected continuation payoff from lying to the mediator is $\lambda(-\alpha + (1-\alpha)) + 1$; since the former is greater than the latter for $\alpha > (1 + \lambda)/2\lambda$, the agent finds it optimal to be truthful. In period $t = 1$, the agent's expected continuation payoff from participating to the mechanism is $\lambda(\alpha - (1-\alpha)) - 1$, whereas the agent's expected continuation payoff from non-participation is 0; since the former is greater than the latter for $\alpha > (1 + \lambda)/2\lambda$, the agent finds it optimal to participate.

Next, we consider the principal's incentive. First, in period $t = 3$, the principal has no incentive to deviate since the allocation $a_3 = -\theta$ is the best one from the period-3 principal's viewpoint. The principal's continuation payoff from following the candidate equilibrium strategy in period $t = 1$ and $t = 2$ is λ. If the principal deviates in period $t = 1$ or $t = 2$, then the mediator does not reveal the agent's type report to the principal in period $t = 3$. Accordingly, at the beginning of period $t = 3$, the principal has only the partial information $s_{P,1}$ about the agent's type θ. Thus, the principal in period $t = 3$ can do no better than offering a constant contract which allocates $a_3 = s_{P,1}$. Therefore, the principal's expected continuation payoff from deviating in period $t = 1$ or $t = 2$ is at most $1 + \lambda(2\alpha - 1)$. Since $\lambda > 1 + \lambda(2\alpha - 1)$ for $\alpha < (2\lambda - 1)/2\lambda$, the principal has no incentive to deviate from the candidate equilibrium strategy in periods $t = 1$ and $t = 2$. ■

3.3 Example 3

Let $\mathcal{T} = \{0, 1, 2\}$. There are two agents, denoted by $i = 1, 2$.[8] In each period $t \in \mathcal{T}_0$, the principal allocates $a_t = (a_{1,t}, a_{2,t}) \in A_t = \{\varnothing, -1, 1\}^2$, where $a_{i,t} = \varnothing$ if and only if agent i does not participate in period t. Each agent i has a type $\theta_i \in \Theta = \{-1, 1\}$, equally likely and independent across players. The principal's payoff is

$$U_P((\theta_1, \theta_2), a) = \begin{cases} 0 & \text{if } a_{1,1} = \varnothing \text{ or } a_{2,1} = \varnothing \\ \sum_i a_{i,1}\theta_i & \text{if } a_{1,1} \neq \varnothing, a_{2,1} \neq \varnothing, \text{ and } a_{1,2} = \varnothing \text{ or } a_{2,2} = \varnothing \\ \sum_i (a_{i,1}\theta_i + a_{i,2}\varepsilon) & \text{otherwise} \end{cases},$$

where $\varepsilon \in (0, 1)$. Agent i's payoff is

$$U_i(\theta_i, a) = \begin{cases} 0 & \text{if } a_{i,1} = \varnothing \\ 1 & \text{if } a_{i,1} \neq \varnothing, \text{ and } a_{1,2} = \varnothing \text{ or } a_{2,2} = \varnothing \\ 2 - a_{i,1}\theta_i & \text{otherwise} \end{cases}.$$

Each agent finds it optimal to participate in the contract in period $t = 1$. The best continuation events from the period-2 principal's viewpoint is such that both agents participate in period $t = 2$ and the allocation is $(a_{1,2}, a_{2,2}) = (1, 1)$; the worst continuation event from the period-2 principal's viewpoint is such that (at least) one agent does not participate in period $t = 2$. If (at least) one agent does not participate in period $t = 2$, the principal and agent i have the same preferences regarding $a_{i,1}$ for $i = 1, 2$; in contrast, if both agents participate in period $t = 2$, the principal and agent i have opposite preferences regarding $a_{i,1}$ for $i = 1, 2$.

Equilibrium Multiplicity and Equilibrium Selection. The MCS game has a wPBE in which the principal's ex ante expected payoff is 1; such wPBE requires the worst continuation equilibrium selection from the period-2 principal's viewpoint. The next claim and its proof formalize these ideas.

Claim 2. The MCS game has a wPBE in which the principal's ex ante expected payoff is 2.

Proof. Regardless of what happens in period $t = 1$, a continuation equilibrium in period $t = 2$ is as follows: neither agent participates in whatever contract is offered by the principal (if agent $-i$ does not participate, then agent i's participation is payoff irrelevant and so it is a best response for i not to participate); for $i = 1, 2$, the principal offers any constant contract if agent i participates. Given this, each

[8] Although the main model only considers a single agent, a simple extension with two agents should not introduce any confusion.

agent i's continuation payoff in period $t = 1$ (upon participation, which is optimal for each agent) is 1; thus, for $i = 1, 2$, the principal offers a direct contract with allocation rule $\alpha_{i,1}(a_{i,1} = m_{i,1} \,|\, m_{i,1}) = 1$ for $i = 1, 2$, and agents' truth-telling is incentive compatible. As a result, the principal's ex ante expected payoff in this wPBE is 2. ■

The next claim shows that if the best continuation equilibrium from the period-2 principal viewpoint is selected, then the principal's ex ante expected payoff in a wPBE cannot exceed 2ε. Since $\varepsilon \in (0, 1)$, such payoff is less than that in the wPBE of the MCS game characterized by Claim 2; thus, the principal-optimal wPBE outcome necessarily requires suboptimal continuation equilibrium selection from the period-2 principal viewpoint.

Claim 3. If the best continuation equilibrium from the period-2 principal's viewpoint is selected, then the principal's ex ante expected wPBE payoff is at most 2ε.

Proof. Regardless of what happens in period $t = 1$, the best continuation equilibrium from the period-2 principal's viewpoint is as follows: for $i = 1, 2$, the principal offers a constant contract which allocates $a_{i,2} = 1$ if agent i participates; both agents participate. Given this, each agent i's continuation payoff in period $t = 1$ (upon participation, which is optimal for each agent) is $2 - a_{i,1}\theta_i$; thus, for $i = 1, 2$, the principal finds it optimal to offer any constant contract in period $t = 1$. As a result, the principal's ex ante expected payoff in this wPBE is 2ε.■

4 Durable-Good Monopoly

In this section, we study durable-good monopoly as a representative application of the mediator approach to mechanism design with limited commitment. The setting is as in Example 1, with the following generalizations: the time horizon is infinite ($T = \infty$); the seller and the buyer share a common discount factor $\delta \in (0, 1)$; the buyer's private valuation for the good is $\theta \in \Theta = \{\theta_\ell, \theta_h\}$, where $0 < \theta_\ell < \theta_h$; the probability that $\theta = \theta_h$ at $t = 0$ is $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$.[9] Hereafter, $\mathcal{G}$ refers to the durable-good monopoly MCS game and $\mathcal{E}$ (resp., $\mathcal{O}$) to all its wPBEs (resp., wPBE outcomes).

[9]If $\mu \in \left(0, \frac{\theta_\ell}{\theta_h}\right]$, then a fully committed seller would post a price equal to θ_ℓ in order to trade with both buyer types in period $t = 1$; this is also a seller-optimal wPBE outcome of the durable-good monopoly MCS game.

The rest of this section is organized as follows. In Section 4.1, we present our indirect approach to characterize the set $\mathcal{O}$. In Section 4.2, we build on the indirect approach to characterize a seller-optimal wPBE outcome of $\mathcal{G}$. In Section 4.3, we discuss both mediated and unmediated implementation of the seller-optimal wPBE outcome of $\mathcal{G}$ characterized in Section 4.2.

4.1 Indirect Approach

To characterize the set $\mathcal{O}$, we could rely on (some versions of) the revelation principle in FMSW. Informally, when applied to our setting, the FMSW revelation principle says that it is without loss to assume the following:

(i) In period $t = 0$, the buyer's report to the mediator is her type, that is, $R = \Theta$.

(ii) In each period $t \in \mathcal{T}_0$: the mediator's signal to the seller is a recommendation of an admissible contract, that is, $S_{P,t} = \mathcal{C}_t$; the mediator's signal to the buyer is a recommendation of an input message to the contract, that is, $S_{A,t} = M_t$.

(iii) In period $t = 0$, the buyer truthfully reports her type to the mediator; in each period $t \in \mathcal{T}_0$, the seller and the buyer obey the mediator's recommendation.

The revelation principle in FMSW simplifies the characterization of the set $\mathcal{O}$ relative to the original setting in which any indirect reports and signals are allowed. However, its literal application is still challenging. In particular, when the set $\mathcal{C}$ is large, there are two challenges: (i) identify the sequence of contracts that the seller offers on the equilibrium path; and (ii) verify seller's obedience to the on-path contracts for all possible deviations to off-path contracts. We circumvent these issues by following an indirect approach based on the insights we develop in the remaining part of this section.

4.1.1 Outcome-Based Approach

We take an outcome-based approach. We represent any (equilibrium and non-equilibrium) outcome of $\mathcal{G}$ by a sequence of trade probabilities and expected transfers

$$(\boldsymbol{x}, \boldsymbol{p}) := (x_{\ell,t}, x_{h,t}, p_{\ell,t}, p_{h,t})_{t=1}^{\infty},$$

where $x_{\ell,t}$ (resp., $x_{h,t}$) denotes the probability with which the seller trades with type θ_ℓ (resp., θ_h) in period t, and $p_{\ell,t}$ (resp., $p_{h,t}$) denotes the expected transfer from type θ_ℓ (resp., θ_h) to the seller in period t. To form an outcome of $\mathcal{G}$, a sequence of trade probabilities and expected transfers $(\boldsymbol{x}, \boldsymbol{p})$ must satisfy the obvious feasibility constraints: $x_{k,t} \geq 0$ for all $k \in \{\ell, h\}$ and $t \in \mathcal{T}_0$, and $\sum_{t=1}^{\infty} x_{k,t} \leq 1$ for all $k \in \{\ell, h\}$.

Given an outcome $(\boldsymbol{x}, \boldsymbol{p})$:

- For any $k \in \{\ell, h\}$, we denote by $x_{k,t}(\boldsymbol{x}, \boldsymbol{p})$ the probability with which type θ_k trades in period t in outcome $(\boldsymbol{x}, \boldsymbol{p})$ conditional on not having traded in the previous periods.
- For any $k \in \{\ell, h\}$, we denote by $p_{k,t}(\boldsymbol{x}, \boldsymbol{p})$ the expected transfer from type θ_k to the principal in period t in outcome $(\boldsymbol{x}, \boldsymbol{p})$ conditional on not having traded in the previous periods.
- We denote by $\mathcal{C}(\boldsymbol{x}, \boldsymbol{p}) := (C_t(\boldsymbol{x}, \boldsymbol{p}))_{t=1}^{\infty}$ sequence of direct contracts corresponding to outcome $(\boldsymbol{x}, \boldsymbol{p})$. That is, $C_t(\boldsymbol{x}, \boldsymbol{p})$ is the direct contract allocating
$$(x_t, p_t) = \begin{cases} (1, p_{k,t}(\boldsymbol{x}, \boldsymbol{p})) & \text{with probability } x_{k,t}(\boldsymbol{x}, \boldsymbol{p}) \\ (0, p_{k,t}(\boldsymbol{x}, \boldsymbol{p})) & \text{with probability } 1 - x_{k,t}(\boldsymbol{x}, \boldsymbol{p}) \end{cases}$$
if the input message to the direct contract is θ_k for all $k \in \{\ell, h\}$.

Thus, one can think of the mediator's strategy as directly recommending outcomes—or, equivalently, direct contracts to the seller and input messages (i.e., type reports) to the direct contract to the buyer—instead of sending signals to the seller and the buyer.

4.1.2 Upper-Bound Problem

Let C^ℓ denote the constant contract allocating $(x, p) = (1, \theta_\ell)$. To form a wPBE outcome of $\mathcal{G}$, an outcome $(\boldsymbol{x}, \boldsymbol{p})$ must be such that: (i) the buyer's incentive compatibility constraint when sending a report to the mediator in period $t = 0$ is satisfied; (ii) in each period $t \in \mathcal{T}_0$, the buyer's expected continuation payoff is non-negative; (iii) in each period $t \in \mathcal{T}_0$, the seller's expected continuation payoff must be no less than that from ending the game by offering the constant contract C^ℓ in period t. Requirements (i) and (ii) are obvious. To understand requirement (iii), it is enough to note that, in any wPBE of $\mathcal{G}$, if the seller offers contract C^ℓ, both buyer types accept the contract, and the game ends.

The next theorem formalizes the previous discussion by identifying necessary conditions on outcomes $(\boldsymbol{x}, \boldsymbol{p})$ to form a wPBE outcome of $\mathcal{G}$.

Theorem 1 (Upper-Bound Problem)**.** Consider any $(\boldsymbol{x}, \boldsymbol{p}) \in \mathcal{O}$. Then, there exists $(\sigma, \beta) \in \mathcal{E}$ that induces $(\boldsymbol{x}, \boldsymbol{p})$ and satisfies the following properties:

1. For all $\theta, \theta' \in \Theta$, at history $h_A^{0,2} = \{\theta\}$, the buyer's expected continuation payoff by playing $\sigma_A^{0,2}(\theta)$ is at least as large as that by playing $\sigma_A^{0,2}(\theta')$.
2. For all $t \in \mathcal{T}_0$ and $h_A^{t,4} \in H_A^{t,4}$, the buyer's expected continuation payoff is non-negative.
3. For all $t \in \mathcal{T}_0$ and $h_P^{t,2} \in H_P^{t,2}$, the seller's expected continuation payoff is not less than that from ending the game by offering the constant contract C^ℓ in period t.

The outcome-based approach and Theorem 1 together can be interpreted as a *relaxed revelation principle*. They allow us to characterize a superset of the set $\mathcal{O}$—hence, candidate elements of $\mathcal{O}$—by solving a problem that is a much simpler relaxation of the original problem. First, the space of all outcomes (i.e., trade probabilities and expected transfers) is much more well-structured than the unrestricted set of all admissible contracts $\mathcal{C}$. Second, the set of incentive compatibility and obedience constraints is much smaller; in particular, for the seller's obedience, we can focus only on deviations to a single contract (i.e., the constant contract C^ℓ), ignoring all the other obedience constraints.

The outcome-based approach and the relaxed revelation principle can be useful not only for mechanism design problems with limited commitment—and, in particular, the current durable-good monopoly application—but also more generally (see the discussion in Section 5).

4.1.3 Lower-Bound Problem

For all $\tilde{\mu} \in (0,1)$, let $(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu}))$ denote the Coasean outcome of $\mathcal{G}$ when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. That is, outcome $(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu}))$ corresponds to the sequence of trade probabilities and expected transfers that would arise in the equilibrium of $\mathcal{G}$ in which the mediator never communicates with the seller and the buyer—except, possibly, for selecting the appropriate continuation equilibrium—when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. Moreover, let $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu})) := (C_t(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu})))_{t=1}^{\infty}$ be the corresponding sequence of direct contracts.

The next theorem identifies sufficient conditions on outcomes $(\boldsymbol{x}, \boldsymbol{p})$ to form a wPBE outcome of $\mathcal{G}$.

Theorem 2 (Lower-Bound Problem)**.** Consider an outcome $(\boldsymbol{x}, \boldsymbol{p})$ of $\mathcal{G}$ and let

(σ, β) be an assessment of $\mathcal{G}$ that induces $(\boldsymbol{x}, \boldsymbol{p})$ and such that β is on-path consistent given σ. Suppose that the following properties hold:

1. For all $\theta, \theta' \in \Theta$, at history $h_A^{0,2} = \{\theta\}$, the buyer's expected continuation payoff by playing $\sigma_A^{0,2}(\theta)$ is at least as large as that by playing $\sigma_A^{0,2}(\theta')$.
2. For all $t \in \mathcal{T}_0$ and $\theta, \theta' \in \Theta$, at history $h_A^{t,4}$, the buyer's expected continuation payoff by playing $\sigma_A^{t,4}(\theta)$ is at least as large as that by playing $\sigma_A^{t,4}(\theta')$.
3. For all $t \in \mathcal{T}_0$ and $h_A^{t,4} \in H_A^{t,4}$, the buyer's expected continuation payoff is non-negative.
4. For all $t \in \mathcal{T}_0$ and $h_P^{t,2} \in H_P^{t,2}$, the seller's expected continuation payoff is not less than that he would obtain by offering the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$ thereafter, where μ_t is the seller's belief on $\theta = \theta_h$ at $h_P^{t,2}$ derived from (σ, β).

Then, $(\sigma, \beta) \in \mathcal{E}$ and $(\boldsymbol{x}, \boldsymbol{p}) \in \mathcal{O}$.

The outcome-based approach and Theorem 2 together can be interpreted as *mediated implementation in direct contracts*. Parts 1 and 3 of Theorem 2 correspond to the necessary conditions in parts 1 and 2 of Theorem 1; part 2 of Theorem 2 requires that the buyer's incentive compatibility constraint when sending an input message to the contract in each period $t \in \mathcal{T}_0$ is satisfied. When parts 1–3 of Theorem 2 hold, the buyer's sequential rationality follows. Part 4 of Theorem 2 tells us that if an outcome satisfies seller's obedience with respect to the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$, then we can ignore all his other obedience constraints. To see why it suffices to focus on such deviations, consider the following mediator's strategy at any $t \in \mathcal{T}_0$:

- At stage $t.1$, the mediator recommends the direct contract $C_t(\boldsymbol{x}, \boldsymbol{p})$ to the seller.
- At stage $t.3$: (i) if the seller offered contract $C_t(\boldsymbol{x}, \boldsymbol{p})$ at stage $t.2$, the mediator recommends the buyer to participate and to truthfully report her type; (ii) if the seller offered contract $C_t \neq C_t(\boldsymbol{x}, \boldsymbol{p})$ at stage $t.2$, the mediator remains silent forever after—except, possibly, for selecting the appropriate continuation equilibrium.

Given the mediator's strategy, the seller's most profitable deviation in period t is to the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_t), \boldsymbol{p}^{\boldsymbol{c}}(\mu_t))$.[10] This guess-and-verify approach

[10] Skreta (2006) shows that posted prices are optimal in any wPBE of the contract selection game without a mediator. Thus, by Gul et al. (1986), the Coasean outcome arises in the (essentially) unique sequential equilibrium of the contract selection game without a mediator.

works in the current durable-good monopoly setting; more sophisticated guesses about the binding obedience constraints may be necessary in other applications.

4.2 Seller-Optimal wPBE Outcomes

In this section, we are interested in characterizing seller-optimal wPBE outcomes of $\mathcal{G}$. We denote by U_P^* the seller's ex ante expected payoff in a seller-optimal wPBE outcome of $\mathcal{G}$. Moreover, for all $n \geq 2$, we denote by $\mathbb{R}^{n\infty}$ the set of all sequences with values in $\mathbb{R}^n$.

4.2.1 Upper-Bound Problem

Because of Theorem 1, to find a candidate for a seller-optimal wPBE outcome of $\mathcal{G}$, it suffices to solve the following linear program:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \ \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right)+(1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P1}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \text{for all } t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{\ell,t}-p_{\ell,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{h,t}-p_{h,t}) \tag{ICℓ}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t}-p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t}-p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t}-p_{\ell,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t}-p_{h,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRh}$$

$$\begin{aligned}&\mu\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{h,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) \\ &\quad +(1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right) \geq \theta_\ell \quad \text{for all } \tau \in \mathcal{T}_0.\end{aligned} \tag{Oℓ}$$

The inequalities in (F1)–(F3) are the feasibility constraints. Inequalities (ICℓ) and (ICh) are the buyer's incentive compatibility constraints at $t = 0$ (corresponding to part 1 of Theorem 1). The inequalities (IRℓ) and (IRh) are the buyer's individual rationality (or participation) constraints for all $\tau \in \mathcal{T}_0$ (corresponding to part 2 of Theorem 1). Finally, the inequalities in (Oℓ) are the seller's obedience constraints for all $\tau \in \mathcal{T}_0$ with respect to the deviation that consists in ending the game by offering contract C^ℓ (corresponding to part 3 of Theorem 1).

In Appendix B, by using duality arguments, we solve the linear program (P1) and establish the following result.

Theorem 3 (Candidate for a Seller-Optimal wPBE Outcome)**.** The following characterizes a candidate for a seller-optimal wPBE outcome of $\mathcal{G}$. For all $\delta \in (0,1)$ and $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, there exists a positive integer $T(\delta,\mu)$ such that:

(a) The game ends in period $t = T(\delta,\mu)$;

(b) Type θ_ℓ trades only (and with probability 1) in period $t = T(\delta,\mu)$ at price θ_ℓ;

(c) Type θ_h trades with positive probability in all periods $t \in \{1,\dots,T(\delta,\mu)-1\}$ at price θ_h, except for period 1, when he trades at price lower than θ_h.

(d) As $\delta \to 1$, $T(\delta,\mu) \to \infty$.

4.2.2 Lower-Bound Problem

Let $V^c(\tilde{\mu};\delta)$ denote the seller's expected payoff from the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\tilde{\mu}), \boldsymbol{p}^{\boldsymbol{c}}(\tilde{\mu}))$ when the seller's initial belief that $\theta = \theta_h$ is $\tilde{\mu}$. Moreover, let $(\boldsymbol{x},\boldsymbol{p}) \coloneqq (x_{\ell,t}, x_{h,t}, p_{\ell,t}, p_{h,t})_{t=1}^{\infty}$ be a solution to program (P1). For all $\tau \in \mathcal{T}_0$, let

$$\mu_\tau(\boldsymbol{x},\boldsymbol{p}) \coloneqq \frac{\left(1-\sum_{t=1}^{\tau-1} x_{h,t}\right)\mu}{\left(1-\sum_{t=1}^{\tau-1} x_{h,t}\right)\mu + \left(1-\sum_{t=1}^{\tau-1} x_{\ell,t}\right)(1-\mu)}$$

denote the seller's belief that $\theta = \theta_h$ at the beginning of stage $\tau.2$ under under $(\boldsymbol{x},\boldsymbol{p})$. Therefore, under outcome $(\boldsymbol{x},\boldsymbol{p})$, $V^c(\mu_\tau(\boldsymbol{x},\boldsymbol{p});\delta)$ corresponds to the seller's expected continuation payoff at stage $\tau.2$ from the sequence of contracts $\mathcal{C}(\boldsymbol{x}^{\boldsymbol{c}}(\mu_\tau(\boldsymbol{x},\boldsymbol{p})), \boldsymbol{p}^{\boldsymbol{c}}(\mu_\tau(\boldsymbol{x},\boldsymbol{p})))$. By Theorem 2, if $(\boldsymbol{x},\boldsymbol{p})$ also satisfies the following constraints

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_\ell x_{h,t} - p_{h,t}) \quad \text{for all } \tau \in \mathcal{T}_0, \tag{ICℓ'}$$

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_h x_{\ell,t} - p_{\ell,t}) \quad \text{for all } \tau \in \mathcal{T}_0, \tag{ICh'}$$

$$\mu_\tau\left(\sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{h,\tau}\right) + \tag{O}$$

$$(1-\mu_\tau)\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,\tau}\right)\geq V^c(\mu_\tau(\boldsymbol{x},\boldsymbol{p});\delta)\quad\text{for all }\tau\in\mathcal{T}_0,$$

then $(\boldsymbol{x},\boldsymbol{p})$ is a seller-optimal wPBE outcome of $\mathcal{G}$. The inequalities in (ICℓ') and (ICh') are the buyer's incentive compatibility constraints for all $\tau\in\mathcal{T}_0$ (corresponding to part 2. of Theorem 2). The inequalities in (O) are the seller's obedience constraints for all $\tau\in\mathcal{T}_0$ with respect to the deviation that consists in offering the sequence of contracts $\mathcal{C}(\boldsymbol{x^c}(\mu_\tau),\boldsymbol{p^c}(\mu_\tau))$ thereafter (corresponding to part 4. of Theorem 2).

Let $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ be the solution to program (P1) we characterize in Appendix B (and describe in Theorem 3). Appendix B also shows that $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ satisfies constraints (ICℓ') and (ICh'). In Appendix C, we show that, for all $\delta\in(0,1)$, there exists $\overline{\mu}(\delta)\in\left(\frac{\theta_\ell}{\theta_h},1\right)$ such that, for all $\mu\in\left(\frac{\theta_\ell}{\theta_h},\overline{\mu}(\delta)\right]$, $\mu_2(\boldsymbol{x}^*,\boldsymbol{p}^*)\leq\frac{\theta_\ell}{\theta_h}$, and so $\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*)\leq\frac{\theta_\ell}{\theta_h}$ for all $\tau>2$. Hence, $V^c(\mu_\tau(\boldsymbol{x}^*,\boldsymbol{p}^*);\delta)=\theta_\ell$ for all $\tau\in\mathcal{T}_1$. Therefore, since $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ satisfies constraint (Oℓ), it also satisfies constraint (O) for all $\tau\in\mathcal{T}_1$. Moreover, since $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ maximizes the seller's ex ante expected payoff, it also satisfies constraint (O) at time $\tau=1$. The next theorem follows.

Theorem 4 (A Seller-Optimal wPBE Outcome)**.** For all $\delta\in(0,1)$, there exists $\overline{\mu}(\delta)\in\left(\frac{\theta_\ell}{\theta_h},1\right)$ such that, for all $\mu\in\left(\frac{\theta_\ell}{\theta_h},\overline{\mu}(\delta)\right]$, the outcome characterized by Theorem 3 is a seller-optimal wPBE outcome of $\mathcal{G}$. Moreover, the threshold belief $\overline{\mu}(\delta)$ satisfies the following properties: (i) $\overline{\mu}(\delta)$ is decreasing in δ; (ii) $\overline{\mu}(\delta)\to1$ as $\delta\to0$; and (iii) $\overline{\mu}(\delta)$ is bounded away from $\frac{\theta_\ell}{\theta_h}$ as $\delta\to1$.

For $\mu>\overline{\mu}(\delta)$, the solution $(\boldsymbol{x}^*,\boldsymbol{p}^*)$ to program (P1) need not satisfy the sufficient condition (O) to be a seller-optimal wPBE outcome of $\mathcal{G}$, as $V^c(\mu_\tau(\boldsymbol{x},\boldsymbol{p});\delta)>\theta_\ell$ can hold for $\tau\in\mathcal{T}_1$. However, note that $V^c(\tilde{\mu};\delta)$ is increasing in $\tilde{\mu}$. Therefore, by defining $\varepsilon(\delta):=V^c(\mu;\delta)-\theta_\ell$, we have that if an outcome $(\boldsymbol{x},\boldsymbol{p})$ satisfies the following constraint,

$$\mu_\tau\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,\tau}\right)+(1-\mu_\tau)\left(\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,\tau}\right)\geq\theta_\ell+\varepsilon(\delta)\quad\text{for all }\tau\in\mathcal{T}_0,\qquad(\text{O}')$$

then it also satisfies constraint (O). Note that, by the Coase conjecture, for all $\mu\in\left(\frac{\theta_\ell}{\theta_h},1\right)$, as $\delta\to1$, $V^c(\mu;\delta)\to\theta_\ell$, and so $\varepsilon(\delta)\to0$; that is, the right-hand side of constraint (O$'$) converges to the right-hand side of constraint (Oℓ) in program (P1).

Next, consider the following linear program:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}}\ \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right)+(1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right)\qquad(\text{P2})$$

$$\text{s.t.} \quad (\text{F1}), (\text{F2}), (\text{IR}\ell), (\text{IR}h), (\text{IC}\ell'), (\text{IC}\ell'), (\text{O}').$$

Denote by $V(\text{P2};\delta)$ the optimal value of the linear program (P1a). Since constraint (O′) is more demanding than constraint (O), we have that $V(\text{P2};\delta) \leq U_P^*$ for all $\delta \in (0,1)$. In Appendix D, we characterize a solution to program (P1a), which we denote by $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$, and show that, as $\delta \to 1$, $V(\text{P2};\delta) \to U_P^*$ and $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta)) \to (\boldsymbol{x}^*, \boldsymbol{p}^*)$. The next theorem follows.

Theorem 5 (Failure of the Coase Conjecture)**.** For all $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, as $\delta \to 1$:

(a) Properties (a)–(d) in Theorem 3 approximate a seller-optimal wPBE outcome of $\mathcal{G}$;

(b) The outcome is bounded away from first-best efficiency;

(c) The seller's ex ante expected payoff is bounded away from θ_ℓ;

(d) If, in addition, $\theta_\ell \to 0$, the seller's ex ante expected payoff approaches the commitment payoff.

4.3 Implementation

4.3.1 Mediated Implementation

Suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$. Let $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ be the solution to program (P1) we characterize in Appendix B (and describe in Theorem 3) and $\mathcal{C}(\boldsymbol{x}^*, \boldsymbol{p}^*) := \{C_t(\boldsymbol{x}^*, \boldsymbol{p}^*)\}_{t=1}^{\infty}$ the corresponding sequence of direct contracts. Then, we have the following.

Theorem 6 (Mediated Implementation of $(\boldsymbol{x}^*, \boldsymbol{p}^*)$)**.** Suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, \overline{\mu}(\delta)\right]$. Then, there exists a wPBE of $\mathcal{G}$ which satisfies the following properties:

1. The two buyer types fully separate from each other at stage 0.2.
2. The seller and the buyer always obey the mediator's recommendations.
3. For all $t \in \mathcal{T}_0$:
   - At stage t.1, the mediator recommends contract $C_t(\boldsymbol{x}^*, \boldsymbol{p}^*)$ to the seller;
   - A stage t.3, the mediator recommends the buyer to participate and to truthfully report her type.

Such wPBE induces outcome $(\boldsymbol{x}^*, \boldsymbol{p}^*)$.

Any wPBE in which such properties hold induces outcome $(\boldsymbol{x}^*, \boldsymbol{p}^*)$. Since $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies the sufficient conditions in Theorem 2, properties 1.—3., Theorem 6 follows.

Next, suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$. Let $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ be the solution to program (P1a) we characterize in Appendix D and $\mathcal{C}(\boldsymbol{x}^*(\boldsymbol{\delta}), \boldsymbol{p}^*(\boldsymbol{\delta})) := \{C_t(\boldsymbol{x}^*(\boldsymbol{\delta}), \boldsymbol{p}^*(\delta))\}_{t=1}^{\infty}$ the corresponding sequence of direct contracts. Then, we have the following.

Theorem 7 (Mediated Implementation of $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$)**.** Suppose $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$. Then, there exists a wPBE of $\mathcal{G}$ which satisfies the following properties:

1. The two buyer types fully separate from each other at stage 0.2.
2. The seller and the buyer always obey the mediator's recommendations.
3. For all $t \in \mathcal{T}_0$:
   - At stage $t.1$, the mediator recommends contract $C_t(\boldsymbol{x}^*(\boldsymbol{\delta}), \boldsymbol{p}^*(\delta))$ to the seller;
   - A stage $t.3$, the mediator recommends the buyer to participate and to truthfully report her type.

Such wPBE induces outcome $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$.

Any wPBE in which such properties hold induces outcome $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$. Since the solution $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ to program (P1a) satisfies constraints (IRℓ), (IRh), (ICℓ'), and (ICℓ'), it satisfies conditions 1.—3. in Theorem 2. Moreover, since the solution $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ to program (P1a) satisfies constraint (O$'$), which is more stringent than constraint (O), it also satisfies condition 4. in Theorem 2. Thus, Theorem 7 follows.

4.3.2 Unmediated Implementation

In Section 4.3.1, by considering the mediator, we allowed for arbitrary information structures. However, it is also interesting to understand whether the seller-optimal outcome is attainable without the mediator (i.e., by considering specific information structures). In Appendix 4.3.2, inspired by Doval and Skreta (2021a,b) and Brzustowski et al. (2021), we revisit the durable-good monopoly example of Section 3.1 and show that the seller-optimal outcome is attainable as long as, in each period, the seller himself can garble the information provided to each of his future selves (not only to his next-period self, as in Doval and Skreta, 2021a,b) about the buyer's report in that period.

As in Section 3.1, suppose $\mathcal{T} = \{0, 1, 2, 3\}$, $\Theta = \{1, 2\}$, and $\mu = \frac{9}{10}$. In the mediated game, the mediator collects the buyer's private information at $t = 0$, and store that information until $t = 3$. Here instead, we let the seller design a contract with *imperfect communication* in the following sense.

Let the seller in period $t = 1$ choose a direct contract C_1 and a *signaling device* $\beta_1 : \Theta \to \Delta\left(S_1^2 \times S_1^3\right)$. In particular, the direct contract C_1 allocates

$$(x_1, p_1) = \begin{cases} \left(1, \frac{59}{36}\right) & \text{with probability } \frac{17}{18} \\ \left(0, \frac{59}{36}\right) & \text{with probability } \frac{1}{18} \end{cases}$$

if the input message is $\theta = 2$, and $(x_1, p_1) = (0, 0)$ with probability 1 if the input message is $\theta = 1$. The signaling device, instead, is such that $S_1^2 = \{s\}$, $S_1^3 = \Theta$, and β_1 sends signal

$$\left(s_1^2, s_1^3\right) = (s, \theta)$$

with probability 1 given the input message is θ. The signaling device β_1 is interpreted as follows. Although the buyer inputs her type to the contract at $t = 1$, the seller does not observe it directly, and he only observes the realized allocation. The signaling device, instead, generates a (possibly random) signal $s_1^2 \in S_1^2$ to the seller at the beginning of period $t = 2$ and another signal $s_1^3 \in S_1^3$ (only) to the seller at the beginning of period $t = 3$. Note that signals are correlated to each other and to θ. This is what Doval and Skreta (2021a,b) consider, although they only allow for s_1^2 but not s_1^3, that is, no information can be stored from period $t = 1$ to period $t = 3$ without being disclosed in period $t = 2$. In the current example, s_1^3 is fully informative, while s_1^2 is uninformative, and this combination is important to replicated the mediated outcome.

In principle, in period $t = 2$, the seller can also offer a contract with a signaling device. For simplicity, however, we let the seller to offer a direct contract C_2 without a signaling device. In particular, the direct contract C_2 allocates $(x_2, p_2) = (1, 2)$ with probability 1 if the input message is $\theta = 2$ and $(x_2, p_2) = (0, 0)$ with probability 1 if the input message is $\theta = 1$. That is, in period $t = 2$, the seller trades with the high-value buyer with probability one, extracting her full surplus.

In period $t = 3$, the seller obtains a signal s_1^3 which fully informs the buyer's true value. Therefore, the seller sets price s_1^3, extracting the buyer's full surplus.

Note that the same outcome as in the mediated case is obtain. We next explain why no one has a strict incentive to deviate. First, by the analysis in the mediated case, the buyer has no incentive to misreport her type in period $t = 1$, as long as she is truthful in period $t = 2$ (and no report is made in period $t = 3$). Since the buyer has no incentive to misreport her type in period $t = 2$ regardless of her behavior in period $t = 1$, the buyer's truthfulness in period $t = 1$ follows.

The seller, obviously, has no incentive to deviate in periods $t = 1$ and $t = 3$. At the beginning of period $t = 2$, conditional on no trade in period $t = 1$, the seller's belief that $\theta = 2$ is

$$\frac{\frac{1}{18}\frac{9}{10}}{1\frac{1}{10} + \frac{1}{18}\frac{9}{10}} = \frac{1}{3} < \frac{1}{2}$$

Therefore, even if the seller had full commitment power from that point on, the best he could do is to offer a posted-price contract with price 1, yielding payoff 1, which is lower than the on-path expected payoff.

The previous argument suggests that a similar unmediated implementation result is possible more generally. This is left for our next step.

4.4 Discussion

Our characterization of the seller-optimal information structure in the previous sections uncovers the following results of economic substance. First, in general, the seller finds it beneficial if the information input by the buyer arrives precisely (i.e., without garbling) but with delay. More formally, the optimal information structure specifies a time $T(\delta, \mu)$ at which the buyer's private information report is made public; until then, no information—except whether a trade has happened or not—is revealed. Our result also implies that the seller's optimal information structure always makes the trade outcome inefficient, as the low-value buyer can trade with a significant delay (even in the limiting case of perfect patience).

Intuitively, delayed (but precise) information disclosure makes the seller's bargaining power stronger. Relative to the case in which no such information arrives, the seller's incentive to offer a more aggressive price is higher because even if the buyer did not buy at that aggressive price, full extraction would be possible once the time comes. Furthermore, once the seller of some period becomes aggressive, then the seller of the previous period can also be more aggressive because the buyer has less continuation payoff conditional on no trading. In this sense, the aggressiveness of each period's seller is a strategic complement to each other. Indeed, except for the initial period, the seller continues to offer the commitment price every period until the time of revelation. This price pattern is completely different from the classical Coase-conjecture pattern of decreasing prices: our case may rather be interpreted as the initial "fire sale" followed by the rigid high price.

5 Conclusion

We propose a mediator approach to mechanism design with limited commitment. Our approach builds on the communication equilibrium notion in multistage games. The approach enables a systematic study of all equilibrium outcomes under all admissible information structures by representing the information structure as a fictitious mediator and re-interpreting the model as a mechanism design problem by the mediator with commitment. Using several examples, we show that new equilibrium outcomes arise when all admissible information structures are considered and identify why previous approaches fail to capture such outcomes.

In the durable-good monopoly application, we show that trading outcomes and welfare consequences can substantially differ from those in the classical model with a fixed information structure. In the seller-optimal mechanism, the seller offers a discounted price to the high-valuation buyer only in the first period, followed by the high, surplus-extracting price until an endogenous deadline, when the buyer's information is revealed and hence fully extracted. As a result, the Coase conjecture fails.

A key challenge we face is to deal with the complexity of the principal's contact space. To circumvent this challenge, we first take an outcome-based approach. Next, we propose two simpler auxiliary problems and show that the values of those problems provide an upper- and a lower-bound to the value of the original problem. We interpret the two problems as relaxed revelation principle and mediated implementation in direct contracts. In the durable-good monopoly application, those bounds suffice to characterize seller-optimal equilibrium outcomes. We also discuss the unmediated implementation of the seller-optimal outcome.

Our approach to mechanism design with limited commitment can be used to characterize robust predictions in bargaining games with one-sided incomplete information—i.e., to characterize the set of equilibrium outcomes that can arise under any admissible information structure. Moreover, our approach is potentially useful in other problems with a partially committed principal. A first example is mechanism design with an informed-principal—where the principal can perfectly commit to a mechanism, but only after observing a private signal, and so he is subject to sequential rationality given each signal realization. A second example is mechanism design with multiple principals—where each principal can perfectly commit to the mechanism he offers but cannot control other principals' mechanisms. We plan to explore these ideas in future work.

A Notation

Throughout the appendix, we use the following notation:

$$\Delta\theta := \theta_h - \theta_\ell, \tag{1}$$

$$r := \frac{\theta_h - \delta\theta_\ell}{\delta\Delta\theta}, \tag{2}$$

$$\rho := \delta r. \tag{3}$$

Note that $\Delta\theta > 0$, $r > 0$, and $\rho > 0$. Moreover, note that

$$(2) \implies 1 - r = -\frac{(1-\delta)\theta_h}{\delta\Delta\theta} \tag{4}$$

and

$$(3) \implies 1 - \rho = -\frac{(1-\delta)\theta_\ell}{\Delta\theta}. \tag{5}$$

B Proof of Theorem 3

B.1 Simplifying the Primal Linear Program (P1)

In the primal linear program (P1):

- We ignore constraint (ICℓ); we will verify in Step 1 of Section B.5 that it is satisfied by the solution to the relaxed version of the program.
- Constraints (ICh) and (IRℓ) for $\tau = 1$, together with the assumption that $\theta_h > \theta_\ell$, imply that constraint (IRh) holds for $\tau = 1$.
- That the seller's payoff in a seller-optimal wPBE must be at least θ_ℓ implies that constraint (Oℓ) holds for $\tau = 1$.

As a result, the relaxed version of program (P1) is the following:

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \quad \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P1a}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \text{for all } t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t} - p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_1 \tag{IRh}$$

$$\mu\left(\sum_{t=1}^{\tau-1} \theta_\ell x_{h,t} + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{h,t}\right) \tag{Oℓ}$$

$$+ (1-\mu)\left(\sum_{t=1}^{\tau-1} \theta_\ell x_{\ell,t} + \sum_{t=\tau}^{\infty} \delta^{t-\tau} p_{\ell,t}\right) \geq \theta_\ell \quad \text{for all } \tau \in \mathcal{T}_1.$$

B.2 The Dual Linear Program of Program (P1a)

Let $\boldsymbol{\xi} := \left(\alpha, \beta, \zeta, (\lambda_{\ell,\tau}, \lambda_{h,\tau+1}, \gamma_{\tau+1})_{t=1}^{\infty}\right) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$, where: α (resp., β) is the Lagrange multiplier associated to constraint (F2) (resp., (F3)) in program (P1a); ζ is the Lagrange multiplier associated to constraint (ICh) in program (P1a); for all $\tau \in \mathcal{T}_0$, $\lambda_{\ell,\tau}$ is the Lagrange multiplier associated to constraint (IRℓ) at time τ in program (P1a); for all $\tau \in \mathcal{T}_0$, $\lambda_{h,\tau+1}$ is the Lagrange multiplier associated to constraint (IRh) at time $\tau+1$ in program (P1a); for all $\tau \in \mathcal{T}_0$, $\gamma_{\tau+1}$ is the Lagrange multiplier associated to constraint (Oℓ) at time $\tau+1$ in program (P1a).

The dual linear program of program (P1a) is the following:

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}} \quad \alpha + \beta - \sum_{t=2}^{\infty} \theta_\ell \gamma_t \tag{P1b}$$

$$\text{s.t.} \quad \alpha \geq 0, \quad \beta \geq 0, \quad \zeta \geq 0, \quad \lambda_{\ell,t}, \lambda_{h,t+1}, \gamma_{t+1} \geq 0 \quad \text{for all } t \in \mathcal{T}_0 \tag{6}$$

$$\alpha \geq -\delta^{t-1}\theta_h \zeta + \sum_{\tau=1}^{t} \delta^{t-\tau}\theta_\ell \lambda_{\ell,\tau} + \sum_{\tau=t+1}^{\infty} (1-\mu)\theta_\ell \gamma_\tau \quad \text{for all } t \in \mathcal{T}_0, \tag{7}$$

$$\beta \geq \delta^{t-1}\theta_h \zeta + \sum_{\tau=2}^{t} \delta^{t-\tau}\theta_h \lambda_{h,\tau} + \sum_{\tau=t+1}^{\infty} \mu\theta_\ell \gamma_\tau \quad \text{for all } t \in \mathcal{T}_0, \tag{8}$$

$$\delta^{t-1}\zeta - \sum_{\tau=1}^{t} \delta^{t-\tau}\lambda_{\ell,\tau} + \sum_{\tau=2}^{t} \delta^{t-\tau}(1-\mu)\gamma_\tau + \delta^{t-1}(1-\mu) = 0 \quad \text{for all } t \in \mathcal{T}_0, \tag{9}$$

$$-\delta^{t-1}\zeta - \sum_{\tau=2}^{t} \delta^{t-\tau}\lambda_{h,\tau} + \sum_{\tau=2}^{t} \delta^{t-\tau}\mu\gamma_\tau + \delta^{t-1}\mu = 0 \quad \text{for all } t \in \mathcal{T}_0, \tag{10}$$

where constraints (9) and (10) hold with equality because $p_{\ell,t}$ and $p_{h,t}$ are unrestricted (i.e., they can be positive or negative) for all $t \in \mathcal{T}_0$ in the primal program (P1a).

B.3 A Candidate Solution to the Dual Linear Program (P1b)

In this section, we recover a candidate solution

$$\boldsymbol{\xi}^* := \left(\alpha^*, \beta^*, \zeta^*, (\lambda^*_{\ell,\tau}, \lambda^*_{h,\tau+1}, \gamma^*_{\tau+1})_{t=1}^{\infty}\right) \in \mathbb{R}^2 \times \mathbb{R}^{3\infty}$$

to program (P1b).

Step 1. Solving equations (10) at successive values of t, starting with $t = 1$, we obtain

$$\zeta^* = \mu \qquad \text{and} \qquad \lambda^*_{h,t} = \mu\gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{11}$$

Next, solving equations (9) at successive values of t, starting with $t = 1$, we obtain

$$\lambda^*_{\ell,1} = 1 \qquad \text{and} \qquad \lambda^*_{\ell,t} = (1-\mu)\gamma^*_t \quad \text{for all } t \in \mathcal{T}_1. \tag{12}$$

Step 2. Given equations (11) and (12), the dual program (P1b) simplifies as follows:

$$\min_{(\alpha,\beta,(\gamma_t)_{t=2}^{\infty})\in\mathbb{R}^2\times\mathbb{R}^\infty} \quad \alpha + \beta - \sum_{t=2}^{\infty} \theta_\ell \gamma_t \tag{P1c}$$

$$\text{s.t.} \quad \alpha \geq 0, \quad \beta \geq 0, \quad \gamma_t \geq 0 \quad \text{for all } t \in \mathcal{T}_1 \tag{13}$$

$$\alpha \geq \delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell \left[\sum_{\tau=2}^{t} \delta^{t-\tau}\gamma_\tau + \sum_{\tau=t+1}^{\infty} \gamma_\tau\right] \quad \text{for all } t \in \mathcal{T}_0, \tag{14}$$

$$\beta \geq \delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t} \delta^{t-\tau}\theta_h\gamma_\tau + \sum_{\tau=t+1}^{\infty} \theta_\ell\gamma_\tau\right] \quad \text{for all } t \in \mathcal{T}_0. \tag{15}$$

Step 3. Since program (P1c) is a minimization problem, its solutions must satisfy

$$\alpha^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\gamma^*_\tau\right]\right\}\right\} \tag{16}$$

and

$$\beta^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma^*_\tau\right]\right\}\right\}. \tag{17}$$

Step 4. We guess that there exists $T(\delta,\mu) \in \mathcal{T}_1$ such that $\gamma^*_t = 0$ for all $t > T(\delta,\mu)$. Moreover, we guess that

$$\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma^*_\tau\right] = \delta^{t}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t+1}\delta^{t+1-\tau}\theta_h\gamma^*_\tau + \sum_{\tau=t+2}^{\infty}\theta_\ell\gamma^*_\tau\right] \tag{18}$$

for all $t \in \{1, \ldots, T(\delta,\mu) - 1\}$.

Step 5. Solving equations (18) for γ^*_t at successive values of t, starting with $t = 1$, we obtain

$$\gamma^*_t = \frac{(1-\delta)\theta_h}{\theta_h - \theta_\ell}\left(\frac{\theta_h - \delta\theta_\ell}{\theta_h - \theta_\ell}\right)^{t-2} = \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2} \quad \text{for all } t \in \{2, \ldots, T(\delta,\mu)\}, \tag{19}$$

where the last equality holds by definitions (1) and (3).

Step 6. From equation (17), the guess in equations (18), and the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$, we obtain

$$\beta^* = \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \gamma_t^*\right). \tag{20}$$

Step 7. By using equation (16) and (19), definition (2), and the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$, constraint (14) becomes

$$\alpha^* = \max_{t\in\{1,\dots,T(\delta,\mu)\}}\left\{\delta^{t-1}(\theta_\ell - \mu\theta_h) + \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{T(\delta,\mu)-1}}{1-\rho}\right)\right\}, \tag{21}$$

assuming that the right-hand side of equation (21) is non-negative (which we will show to be the case in Step 9 of this section). The maximand on the right-hand side of equation of equation (21) simplifies as follows:

$$\begin{aligned}
&\delta^{t-1}(\theta_\ell - \mu\theta_h) + \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{T(\delta,\mu)-1}}{1-\rho}\right)\\
&= \delta^{t-1}(\theta_\ell - \mu\theta_h) - \delta^{t-1}(1-\mu)\theta_\ell\big(1-r^{t-1}\big) - (1-\mu)\theta_h\Big(\rho^{t-1}-\rho^{T(\delta,\mu)-1}\Big)\\
&= (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t-1}\mu\Delta\theta - \delta^{t-1}(1-\mu)\Delta\theta r^{t-1}\\
&= (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t-1}\Delta\theta\big(\mu + (1-\mu)r^{t-1}\big),
\end{aligned} \tag{22}$$

where the first equality holds by implications (4) and (5), and the third equality holds by definition (3). Thus, by equation (22), equation (21) is equivalent to

$$\alpha^* = \max_{t\in\{1,\dots,T(\delta,\mu)\}}\Big\{(1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t-1}\Delta\theta\big(\mu+(1-\mu)r^{t-1}\big)\Big\}. \tag{23}$$

Let t^* denote the value of $t \in \{1,\dots,T(\delta,\mu)\}$ that maximizes the maximand on the right-hand side of equation (23). Moreover, let $\underline{t}$ be such that

$$\delta^{\underline{t}-1}\Delta\theta\big(\mu+(1-\mu)r^{\underline{t}-1}\big) = \delta^{\underline{t}}\Delta\theta\big(\mu+(1-\mu)r^{\underline{t}}\big). \tag{24}$$

Since the maximand on the right-hand side of equation (23) is concave in t and $T(\delta,\mu)$ is yet to be determined (and so can be chosen arbitrarily large), $t^* = \inf\{t \in \mathcal{T}_0 : t \geq \underline{t}\}$.

Note that

$$\begin{aligned}
(24) &\iff \mu + (1-\mu)r^{\underline{t}-1} = \delta\big(\mu+(1-\mu)r^{\underline{t}}\big)\\
&\iff \mu(1-\delta) = -r^{\underline{t}-1}(1-\mu)(1-\delta r)\\
&\iff \mu(1-\delta) = r^{\underline{t}-1}(1-\mu)\frac{(1-\delta)\theta_\ell}{\Delta\theta}\\
&\iff r^{\underline{t}-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}
\end{aligned}$$

$$\iff \underline{t} = 1 + \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r}, \tag{25}$$

where the third equivalence follows from implication (5). Since $\mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right)$, $\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} > 1$; moreover, $r > 1$. Therefore, $\underline{t} > 1$, which implies that $t^* = \inf\{t \in \mathcal{T}_0 : t \geq \underline{t}\} \geq 2$.

Step 8. The choice of $T(\delta,\mu)$ is part of the choice of the Lagrange multipliers (in particular, that of $(\gamma_t^*)_{t=2}^{\infty}$). Thus, $T(\delta,\mu)$ must be chosen to minimize the objective function of program (P1b).

Let $V(\text{P1b})$ denote the optimal value of program (P1b). Note that

$$\begin{aligned}
V(\text{P1b}) &= \alpha^* + \beta^* - \sum_{t=2}^{\infty} \theta_\ell \gamma_t^* \\
&= \alpha^* + \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \gamma_t^*\right) - \sum_{t=2}^{T(\delta,\mu)} \theta_\ell\gamma_t^* \\
&= \alpha^* + \mu\theta_h - (1-\mu)\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \gamma_t^*\right) \\
&= \alpha^* + \mu\theta_h - (1-\mu)\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2}\right) \\
&= \alpha^* + \mu\theta_h - \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\left(\sum_{t=2}^{T(\delta,\mu)} \rho^{t-2}\right) \\
&= \alpha^* + \mu\theta_h - \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\frac{1-\rho^{T(\delta,\mu)-1}}{1-\rho} \\
&= \alpha^* + \mu\theta_h + \frac{(1-\mu)\theta_\ell(1-\delta)\theta_h}{\Delta\theta}\frac{\Delta\theta}{\theta_\ell(1-\delta)}\left(1-\rho^{T(\delta,\mu)-1}\right) \\
&= \alpha^* + \mu\theta_h + (1-\mu)\theta_h\left(1-\rho^{T(\delta,\mu)-1}\right) \\
&= (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big) + \theta_h - (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} \\
&= \theta_h - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big),
\end{aligned} \tag{26}$$

where: the second equality follows from equation (20) and the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$; the fourth equality follows from equation (19); the seventh equality follows from implication (5); the second-to-last equality follows from equation (23) and the definition of t^*.

Since $V(\text{P1b})$ does not depend on $T(\delta,\mu)$, we take

$$T(\delta,\mu) = t^*. \tag{27}$$

Step 9. Summing up, a candidate solution

$$\boldsymbol{\xi}^* := (\alpha^*, \beta^*, \zeta^*, (\lambda_{\ell,\tau}^*, \lambda_{h,\tau+1}^*, \gamma_{\tau+1}^*)_{t=1}^{\infty}) \in \mathbb{R}^2 \times \mathbb{R}^{3\infty}$$

to program (P1b) is as follows. For

$$T(\delta,\mu) = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\},$$

where

$$\underline{t} = 1 + \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r},$$

we have:

$$\begin{aligned}
\alpha^* &= \delta^{T(\delta,\mu)-1}\Big[(1-\mu)\theta_\ell r^{T(\delta,\mu)-1} - \mu\Delta\theta\Big], && (28)\\
\beta^* &= \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \gamma_t^*\right) && (29)\\
\zeta^* &= \mu && (30)\\
\lambda_{\ell,t}^* &= \begin{cases} 1 & \text{if } t = 1 \\ (1-\mu)\gamma_t^* & \text{otherwise} \end{cases}, && (31)\\
\lambda_{h,t}^* &= \mu\gamma_t^*, && (32)\\
\gamma_t^* &= \begin{cases} \frac{(1-\delta)\theta_h}{\Delta\theta}\rho^{t-2} & \text{if } t \in \{2,\ldots,T(\delta,\mu)\} \\ 0 & \text{if } t > T(\delta,\mu) \end{cases}. && (33)
\end{aligned}$$

Except for α^*, all elements of $\boldsymbol{\xi}^*$ are clearly non-negative. To see that α^* is also non-negative, note that

$$\alpha^* \geq \delta^{\underline{t}-1}\Big[(1-\mu)\theta_\ell r^{\underline{t}-1} - \mu\Delta\theta\Big] = 0$$

where the inequality holds by definition of $\underline{t}$ and the equality holds by equation (25).

Hereafter, for simplicity, we take

$$T(\delta,\mu) = \underline{t} = 1 + \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r} \tag{34}$$

or, equivalently,

$$r^{T(\delta,\mu)-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}. \tag{35}$$

From equations (26), (27), (34), and (35), we obtain

$$\begin{aligned}
V(\text{P1b}) &= \theta_h - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big)\\
&= \theta_h - \delta^{T(\delta,\mu)-1}\left(\mu + \frac{\mu\Delta\theta}{\theta_\ell}\right)\\
&= \theta_h - \delta^{T(\delta,\mu)-1}\frac{\mu\Delta\theta\theta_h}{\theta_\ell}\\
&= \theta_h\left(1 - \delta^{T(\delta,\mu)-1}\frac{\mu\Delta\theta}{\theta_\ell}\right).
\end{aligned} \tag{36}$$

B.4 A Candidate Solution to the Primal Linear Program (P1a)

In this section, we recover a candidate solution

$$(\boldsymbol{x}^*, \boldsymbol{p}^*) := (x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t})_{t=1}^{\infty}$$

to program (P1a).

Step 1. Since $\lambda^*_{\ell,t} > 0$ for all $t \in \{1, \ldots, T(\delta,\mu)\}$ (see equation (31)), constraint (IRℓ) is binding for all such t. Thus,

$$p^*_{\ell,t} = \theta_\ell x^*_{\ell,t} \quad \text{for all } t \in \{1, \ldots, T(\delta,\mu)\}. \tag{37}$$

Since $\lambda^*_{h,t} > 0$ for all $t \in \{2, \ldots, T(\delta,\mu)\}$ (see equation (32)), constraint (IRh) is binding for all such t. Thus,

$$p^*_{h,t} = \theta_h x^*_{h,t} \quad \text{for all } t \in \{2, \ldots, T(\delta,\mu)\}. \tag{38}$$

Moreover, we conjecture that the solution to program (P1a) is such that

$$x^*_{\ell,t} = \begin{cases} 1 & \text{if } t = T(\delta,\mu) \\ 0 & \text{otherwise} \end{cases}. \tag{39}$$

From equation (37) and the conjecture in equation (39), we have that

$$p^*_{\ell,t} = \begin{cases} 0 & \text{if } t \in \{1, \ldots, T(\delta,\mu) - 1\} \\ \theta_\ell & \text{if } t = T(\delta,\mu) \end{cases}. \tag{40}$$

Finally, we conjecture that

$$p^*_{\ell,t} = 0 \quad \text{for all } t > T(\delta,\mu). \tag{41}$$

and

$$x^*_{h,t} = p^*_{h,t} = 0 \quad \text{for all } t \geq T(\delta,\mu). \tag{42}$$

Step 2. Since $\zeta^* > 0$ (see equation (30)), constraint (ICh) is binding. This, together with equations (38)–(42), implies that

$$\theta_h x^*_{h,1} - p^*_{h,1} = \delta^{T(\delta,\mu)-1} \Delta\theta$$

or, equivalently,

$$p^*_{h,1} = \theta_h x^*_{h,1} - \delta^{T(\delta,\mu)-1} \Delta\theta. \tag{43}$$

Step 3. To find $x^*_{h,t}$ for all $t \in \{2, \ldots, T(\delta,\mu) - 1\}$, we use that constraint (Oℓ) is binding for all such t (as $\gamma^*_t > 0$ for all such t, see equation (33)). From constraint (Oℓ) binding at $t = T(\delta,\mu) - 1$, we obtain

$$\mu\Big[\theta_\ell\Big(1 - x^*_{h,T(\delta,\mu)-1}\Big) + \theta_h x^*_{h,T(\delta,\mu)-1}\Big] + (1-\mu)\delta\theta_\ell = \theta_\ell,$$

or, equivalently

$$x^*_{h,T(\delta,\mu)-1} = \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}.$$

Similarly, for all $t \in \{2, \ldots, T(\delta,\mu)-2\}$, we have

$$\begin{aligned} x^*_{h,t} &= \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\left(\frac{\theta_h - \delta\theta_\ell}{\Delta\theta}\right)^{T(\delta,\mu)-t-1} \\ &= \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{T(\delta,\mu)-t-1}, \end{aligned}$$

where the second equality holds by definition (3). Finally, we have

$$\begin{aligned} x^*_{h,1} &= 1 - \sum_{t=2}^{T(\delta,\mu)-1} x^*_{h,t} \\ &= 1 - \sum_{t=2}^{T(\delta,\mu)-1} \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{T(\delta,\mu)-t-1} \\ &= 1 - \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\frac{1-\rho^{T(\delta,\mu)-2}}{1-\rho} \\ &= 1 + \frac{1-\mu}{\mu}\left(1-\rho^{T(\delta,\mu)-2}\right) \\ &= \frac{1}{\mu}\left(1-(1-\mu)\rho^{T(\delta,\mu)-2}\right) \\ &= \frac{1}{\mu}\left(1-(1-\mu)\frac{1}{\rho}r^{T(\delta,\mu)-1}\delta^{T(\delta,\mu)-1}\right) \\ &= \frac{1}{\mu}\left(1-\frac{\Delta\theta}{\theta_h-\delta\theta_\ell}\frac{\mu\Delta\theta}{\theta_\ell}\delta^{T(\delta,\mu)-1}\right), \end{aligned} \tag{44}$$

where: the second-to-last equality holds by definition (3); the last equality holds by definitions (2) and (3) and equation (35).

Step 4. It remains to show that $x^*_{h,1} > 0$ or, equivalently, that

$$\frac{\Delta\theta}{\theta_h-\delta\theta_\ell}\frac{\mu\Delta\theta}{\theta_\ell}\delta^{T(\delta,\mu)-1} < 1 \quad \text{for all } \mu \in \left(\frac{\theta_\ell}{\theta_h}, 1\right). \tag{45}$$

The left-hand side of inequality (45) evaluated at $\mu = \frac{\theta_\ell}{\theta_h}$ is equal to

$$\frac{\Delta\theta}{\theta_h-\delta\theta_\ell}\frac{\Delta\theta}{\theta_h}\delta^{T(\delta,\mu)-1},$$

which is less than 1 because each term of the product is less than 1. Thus, it suffices to show that $\mu\delta^{T(\delta,\mu)-1}$ is decreasing in μ. Since

$$\frac{\partial}{\partial\mu}\left[\mu\delta^{T(\delta,\mu)-1}\right] = \delta^{T(\delta,\mu)-1}\left(1+\frac{\log\delta}{(1-\mu)\log r}\right),$$

to show that $\mu\delta^{T(\delta,\mu)-1}$ is decreasing in μ, it suffices to show that

$$1+\frac{\log\delta}{(1-\mu)\log r}\leq 0. \tag{46}$$

Since the left-hand side of inequality (46) is decreasing in μ (as $\delta\in(0,1)$, and so $\log\delta<0$), it suffices to show that inequality (46) is satisfied for $\mu=\frac{\theta_\ell}{\theta_h}$, i.e., that

$$1+\frac{\log\delta}{\frac{\Delta\theta}{\theta_h}\log r}\leq 0$$

or, equivalently,

$$\frac{\Delta\theta}{\theta_h}\log r+\log\delta\leq 0. \tag{47}$$

Since inequality (47) is satisfied with equality if $\delta=1$ (as $r=1$ if $\delta=1$, see (2)), it suffices to show that its left-hand side is increasing in δ (so that, for $\delta<1$, the left-hand side is negative). To see this, it suffices to note that

$$\begin{aligned}\frac{\partial}{\partial\delta}\left[\frac{\Delta\theta}{\theta_h}\log r+\log\delta\right]&=\frac{\Delta\theta}{\theta_h}\frac{1}{r}\frac{\partial r}{\partial\delta}+\frac{1}{\delta}\\&=-\frac{\Delta\theta}{\theta_h}\frac{\delta\Delta\theta}{\theta_h-\delta\theta_\ell}\frac{\theta_h}{\delta^2\Delta\theta}+\frac{1}{\delta}\\&\propto 1-\frac{\delta\Delta\theta}{\theta_h-\delta\theta_\ell}\\&=\frac{(1-\delta)\theta_h}{\theta_h-\delta\theta_\ell}>0,\end{aligned}$$

where the second equality holds by recalling definition (2).

Step 5. Summing up, a candidate solution

$$(\boldsymbol{x}^*,\boldsymbol{p}^*)\coloneqq(x^*_{\ell,t},x^*_{h,t},p^*_{\ell,t},p^*_{h,t})_{t=1}^{\infty}$$

to program (P1a) is as follows. For

$$T(\delta,\mu)=\inf\{t\in\mathcal{T}:t\geq\underline{t}\}\quad\text{ or }\quad T(\delta,\mu)=\sup\{t\in\mathcal{T}:t\leq\underline{t}\}, \tag{48}$$

where

$$\underline{t}=1+\frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r}, \tag{49}$$

we have:

$$x^*_{\ell,t}=\begin{cases}1 & \text{if } t=T(\delta,\mu)\\ 0 & \text{otherwise}\end{cases}, \tag{50}$$

$$x^*_{h,t} = \begin{cases} \frac{1}{\mu}\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) & \text{if } t = 1 \\ \frac{(1-\delta)(1-\mu)\theta_\ell}{\mu\Delta\theta}\rho^{T(\delta,\mu)-t-1} & \text{if } t \in \{2,\dots,T(\delta,\mu)-1\} \\ 0 & \text{if } t \geq T(\delta,\mu) \end{cases}, \tag{51}$$

$$p^*_{\ell,t} = \begin{cases} \theta_\ell x^*_{\ell,t} & \text{if } t \in \{1,\dots,T(\delta,\mu)\} \\ 0 & \text{otherwise} \end{cases}, \tag{52}$$

$$p^*_{h,t} = \begin{cases} \theta_h x^*_{h,1} - \delta^{T(\delta,\mu)-1}\Delta\theta & \text{if } t = 1 \\ \theta_h x^*_{h,t} & \text{if } t \in \{2,\dots,T(\delta,\mu)\} \\ 0 & \text{otherwise} \end{cases}. \tag{53}$$

Clearly, all elements of $\boldsymbol{x}^*$ are non-negative.

Step 6. Le $V(\text{P1a})$ be the optimal value of the primal linear program (P1a). In this step, we show that $V(\text{P1a}) = V(\text{P1b})$, so that, by weak duality, the candidate solution to program (P1a) is indeed a solution to the program.

Note that

$$\begin{aligned} V(\text{P1a}) &= \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p^*_{\ell,t}\right) \\ &= \mu\left(\sum_{t=1}^{T(\delta,\mu)-1}\delta^{t-1}p^*_{h,t}\right) + (1-\mu)\delta^{T(\delta,\mu)-1}p^*_{\ell,T(\delta,\mu)} \\ &= \mu\left[\frac{\theta_h}{\mu}\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) - \delta^{T(\delta,\mu)-1}\Delta\theta\right] \\ &+ \mu\left[\sum_{t=2}^{T(\delta,\mu)-1}\delta^{t-1}\frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\mu\Delta\theta}\rho^{T(\delta,\mu)-t-1}\right] \\ &+ (1-\mu)\delta^{T(\delta,\mu)-1}\theta_\ell \\ &= \theta_h\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) - \mu\delta^{T(\delta,\mu)-1}\Delta\theta \\ &+ \frac{(1-\delta)(1-\mu)\theta_\ell\theta_h}{\Delta\theta}\delta^{T(\delta,\mu)-2}\frac{1 - r^{T(\delta,\mu)-2}}{1-r} + (1-\mu)\delta^{T(\delta,\mu)-1}\theta_\ell \\ &= \theta_h\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) - \mu\delta^{T(\delta,\mu)-1}\Delta\theta \\ &- (1-\mu)\delta^{T(\delta,\mu)-1}\theta_\ell\left(1 - r^{T(\delta,\mu)-2}\right) + (1-\mu)\delta^{T(\delta,\mu)-1}\theta_\ell \\ &= \theta_h\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) - \mu\delta^{T(\delta,\mu)-1}\Delta\theta + (1-\mu)\theta_\ell\delta^{T(\delta,\mu)-1}r^{T(\delta,\mu)-2}, \end{aligned} \tag{54}$$

where: the second and third equalities hold by equations (50)–(53); the fourth equality holds because, by definition (3), $\delta^{t-1}\rho^{T(\delta,\mu)-t-1} = \delta^{T(\delta,\mu)-2}r^{T(\delta,\mu)-t-1}$; the second-to-last equality holds by using implication (4). Therefore,

$$
\begin{aligned}
V(\text{P1a}) - V(\text{P1b}) &= \theta_h \left(1 - \frac{\Delta\theta}{\theta_\ell} \frac{\Delta\theta}{\theta_h - \delta\theta_\ell} \mu \delta^{T(\delta,\mu)-1}\right) - \mu \delta^{T(\delta,\mu)-1}\Delta\theta + (1-\mu)\theta_\ell \delta^{T(\delta,\mu)-1} r^{T(\delta,\mu)-2} \\
&\quad - \theta_h \left(1 - \delta^{T(\delta,\mu)-1} \frac{\mu\Delta\theta}{\theta_\ell}\right) \\
&= -\frac{\Delta\theta\delta^{T(\delta,\mu)}}{\theta_h - \delta\theta_\ell} + (1-\mu)\theta_\ell \delta^{T(\delta,\mu)-1} r^{T(\delta,\mu)-2} \\
&\propto -\frac{\delta\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\Delta\theta + (1-\mu)\theta_\ell r^{T(\delta,\mu)-2} \\
&= -\frac{1}{r}\mu\Delta\theta + \frac{1}{r}(1-\mu)\theta_\ell r^{T(\delta,\mu)-1} \\
&= -\frac{1}{r}\mu\Delta\theta + \frac{1}{r}\mu\Delta\theta \\
&= 0.
\end{aligned}
$$

where: the first equality holds by equations (36) and (54); the second-to-last equality holds by equation (35).

B.5 A Solution to the Primal Linear Program (P1)

Step 1. In this step, we show that the sequence $(\boldsymbol{x}^*, \boldsymbol{p}^*) \coloneqq (x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t})_{t=1}^{\infty}$ described in Step 5 of Section B.4 satisfies constraint (ICℓ) in the primal linear program (P1), so that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is also a solution to program (P1).

By equations (50) and (52), the left-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is equal to 0. By equations (51) and (53), and since $\theta_h > \theta_\ell$, the right-hand side of constraint (ICℓ) evaluated at $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is smaller than

$$
\begin{aligned}
\theta_\ell x^*_{h,1} - p^*_{h,1} &= \theta_h x^*_{h,1} - p^*_{h,1} - \Delta\theta x^*_{h,1} \\
&= \delta^{T(\delta,\mu)-1}\Delta\theta - \Delta\theta x^*_{h,1},
\end{aligned}
$$

where the second equality holds by equation (43). Thus, to show that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies constraint (ICℓ), it suffices to show that

$$x^*_{h,1} \geq \delta^{T(\delta,\mu)-1}.$$

Note that

$$
\begin{aligned}
x^*_{h,1} \geq \delta^{T(\delta,\mu)-1} &\iff \frac{1}{\mu}\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{T(\delta,\mu)-1}\right) - \delta^{T(\delta,\mu)-1} \geq 0 \\
&\iff 1 - \left(1 + \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\right)\mu\delta^{T(\delta,\mu)-1} \geq 0, \qquad (55)
\end{aligned}
$$

where the first equivalence holds by equation (51). Since $\mu\delta^{T(\delta,\mu)-1}$ is decreasing in μ (see step 11), it suffices to show that inequality (55) is satisfied at $\mu = \frac{\theta_\ell}{\theta_h}$. By equation (34), if $\mu = \frac{\theta_\ell}{\theta_h}$, then

$T(\delta, \mu) = 1$. Therefore, it suffices to show that

$$1 - \left(1 + \frac{\Delta\theta}{\theta_\ell} \frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\right) \frac{\theta_\ell}{\theta_h} \geq 0.$$

The desired result follows by observing that

$$\begin{aligned} 1 - \left(1 + \frac{\Delta\theta}{\theta_\ell} \frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\right) \frac{\theta_\ell}{\theta_h} &\propto \theta_h - \left(\theta_\ell + \Delta\theta \frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\right) \\ &\propto 1 - \frac{\Delta\theta}{\theta_h - \delta\theta_\ell} \\ &\propto (1-\delta)\theta_\ell \\ &\geq 0. \end{aligned}$$

Step 2. At the solution $(\boldsymbol{x}^*, \boldsymbol{p}^*) \coloneqq (x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t})_{t=1}^{\infty}$ to the primal linear program (P1) described in Step 5 of Section B.4, properties (a), (b), and (c) in Theorem 3 hold true.

Step 3. From equations (48) and (49), we have that $T(\delta, \mu)$ is finite for any $\delta \in (0, 1)$, and

$$\begin{aligned} \lim_{\delta \to 1} T(\delta, \mu) &\geq \lim_{\delta \to 1} \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r} \\ &= \lim_{\delta \to 1} \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log\left(\frac{\theta_h - \delta\theta_\ell}{\delta(\theta_h - \theta_\ell)}\right)} \\ &= \infty, \end{aligned}$$

where: the inequality holds by equations (48) and (49); the first equality holds by definitions (1) and (2). Thus, also property (d) in Theorem 3 holds true at the solution to the primal linear program (P1) described in Step 5 of Section B.4. This concludes the proof of Theorem 3. ■

C Proof of Theorem 4

Let $(\boldsymbol{x}^*, \boldsymbol{p}^*) \coloneqq (x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t})_{t=1}^{\infty}$ denote the solution to program (P1) we characterize in Appendix B (and describe in Theorem 3). By Theorem 2, a sufficient condition for $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ to wPBE outcome of $\mathcal{G}$ is that $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h}$.

To begin, note that

$$\begin{aligned} \mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h} &\iff \frac{(1 - x^*_{h,1})\mu}{(1 - x^*_{h,1})\mu + (1-\mu)} \leq \frac{\theta_\ell}{\theta_h} \\ &\iff \frac{-(1-\mu)\left(1 - \rho^{T(\delta,\mu)-2}\right)}{-(1-\mu)\left(1 - \rho^{T(\delta,\mu)-2}\right) + (1-\mu)} \leq \frac{\theta_\ell}{\theta_h} \\ &\iff -\frac{1 - \rho^{T(\delta,\mu)-2}}{\rho^{T(\delta,\mu)-2}} \leq \frac{\theta_\ell}{\theta_h} \\ &\iff -\theta_h + \theta_h \rho^{T(\delta,\mu)-2} \leq \theta_\ell \rho^{T(\delta,\mu)-2} \end{aligned}$$

$$
\begin{aligned}
&\iff \frac{\Delta\theta}{\theta_h}\rho^{T(\delta,\mu)-2} \leq 1 \\
&\iff \log\left(\frac{\Delta\theta}{\theta_h}\right) + (T(\delta,\mu)-2)\log\rho \leq 0 \\
&\iff \log\left(\frac{\Delta\theta}{\theta_h}\right) + \left(\frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r} - 1\right)\log(\delta r) \leq 0 \\
&\iff \log\left(\frac{\Delta\theta}{\theta_h}\right) + [\log(\mu\Delta\theta) - \log((1-\mu)\theta_\ell) - \log r]\frac{\log(\delta r)}{\log r} \leq 0, \qquad (56)
\end{aligned}
$$

where: the second equivalence follows from the fourth equality in equation (44); the seventh inequality follows from equation (34) and definition (3).

Next, note that the left-hand side of (56) is increasing in $\frac{\log(\delta r)}{\log r}$. In turn, the first derivative of $\frac{\log(\delta r)}{\log r}$ with respect to δ,

$$
\frac{\partial}{\partial\delta}\left[\frac{\log(\delta r)}{\log r}\right] = \log\left(\frac{\theta_h - \delta\theta_\ell}{\Delta\theta}\right) + \frac{\delta\theta_\ell}{\theta_h - \delta\theta_\ell}\log\delta,
$$

approaches 0 as $\delta \to 1$. Moreover, the second derivative of $\frac{\log(\delta r)}{\log r}$ with respect to δ,

$$
\frac{\partial^2}{\partial\delta^2}\left[\frac{\log(\delta r)}{\log r}\right] = \frac{\theta_\ell\theta_h}{(\theta_h - \delta\theta_\ell)^2}\log\delta,
$$

is negative (as $\delta < 1$). Thus, $\frac{\log(\delta r)}{\log r}$ is increasing in δ. Hence, to show that $\mu_2(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq \frac{\theta_\ell}{\theta_h}$, it suffices to show the inequality in inequality (56) is satisfied for $\delta \simeq 1$.

Since

$$
\begin{aligned}
\lim_{\delta\to 1}\frac{\log(\delta r)}{\log r} &= \lim_{\delta\to 1}\frac{\frac{\partial}{\partial\delta}(\log(\delta r))}{\frac{\partial}{\partial\delta}(\log r)} \\
&= \lim_{\delta\to 1}\frac{\delta\theta_\ell - (1-\delta)\theta_h}{\delta\theta_h} \\
&= \frac{\theta_\ell}{\theta_h},
\end{aligned}
$$

where the first equality holds by de L'Hôpital's rule, for $\delta \simeq 1$ we have:

$$
\begin{aligned}
(56) &\iff \log\left(\frac{\theta_h}{\Delta\theta}\right) \geq [\log(\mu\Delta\theta) - \log((1-\mu)\theta_\ell) - \log r]\frac{\theta_\ell}{\theta_h} \\
&\iff \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}} \geq \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell r} \\
&\iff \mu \leq \frac{\left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}{\frac{\Delta\theta}{\theta_\ell r} + \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}} := \overline{\mu}(\delta).
\end{aligned}
$$

Note that $\lim_{\delta\to 0}\overline{\mu}(\delta) = 1$. Moreover, $\overline{\mu}(\delta)$ is decreasing in $\frac{\Delta\theta}{\theta_\ell r}$ which, in turn, is decreasing in r.

Moreover, r is decreasing in δ, and so $\overline{\mu}(\delta)$ is decreasing in δ. Finally, note that

$$\lim_{\delta\to 1}\overline{\mu}(\delta) = \frac{\left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}{\frac{\Delta\theta}{\theta_\ell} + \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}.$$

Since

$$\frac{\left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}}{\frac{\Delta\theta}{\theta_\ell} + \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}}} > \frac{\theta_\ell}{\theta_h} \iff \left(\frac{\theta_h}{\Delta\theta}\right)^{\frac{\theta_h}{\theta_\ell}} > 1,$$

that $\lim_{\delta\to 1}\overline{\mu}(\delta)$ is bounded away from $\frac{\theta_\ell}{\theta_h}$ follows. This completes the proof of Theorem 4. ■

D Proof of Theorem 5

Consider the following relaxation of the linear program (P2):

$$\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \quad \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right) + (1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) \tag{P2a}$$

$$\text{s.t.} \quad x_{\ell,t}, x_{h,t} \geq 0 \quad \text{for all } t \in \mathcal{T}_0 \tag{F1}$$

$$1 \geq \sum_{t=1}^{\infty} x_{\ell,t} \tag{F2}$$

$$1 \geq \sum_{t=1}^{\infty} x_{h,t} \tag{F3}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_\ell x_{h,t} - p_{h,t}) \tag{ICℓ}$$

$$\sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t} - p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t} - p_{\ell,t}) \tag{ICh}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t} - p_{\ell,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRℓ}$$

$$\sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t} - p_{h,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 \tag{IRh}$$

$$\mu\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{h,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) \tag{O$'$}$$
$$+ (1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t} + \sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right) \geq \theta_\ell + \varepsilon(\delta) \quad \text{for all } \tau \in \mathcal{T}_0.$$

The linear program (P2a) is obtained from program (P2) by ignoring the buyer's incentive compatibility constraints at all $\tau \in \mathcal{T}_1$.

In this section, we solve program (P2a) and show that its solution satisfies the omitted constraint of program (P2). The approach to solve program (P2a) mimics that to solve program (P1) in Appendix B.

D.1 Simplifying the Primal Linear Program (P2a)

In the primal linear program (P2a):

- We ignore constraints (ICℓ); we will verify in Step 1 of Section B.5 that it is satisfied by the solution to the relaxed version of the program.
- Constraints (ICh) and (IRℓ) for $\tau = 1$, together with the assumption that $\theta_h > \theta_\ell$, imply that constraint (IRh) holds for $\tau = 1$.
- That the seller's payoff in a seller-optimal wPBE must be at least $V^c(\mu; \delta)$ implies that constraint (Oℓ) holds for $\tau = 1$.

As a result, the relaxed version of program (P2a) is the following:

$$
\begin{aligned}
\max_{(\boldsymbol{x},\boldsymbol{p})\in\mathbb{R}^{4\infty}} \quad & \mu\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{h,t}\right)+(1-\mu)\left(\sum_{t=1}^{\infty}\delta^{t-1}p_{\ell,t}\right) && \text{(P2b)}\\
\text{s.t.} \quad & x_{\ell,t}, x_{h,t} \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 && \text{(F1)}\\
& 1 \geq \sum_{t=1}^{\infty} x_{\ell,t} && \text{(F2)}\\
& 1 \geq \sum_{t=1}^{\infty} x_{h,t} && \text{(F3)}\\
& \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{h,t}-p_{h,t}) \geq \sum_{t=1}^{\infty}\delta^{t-1}(\theta_h x_{\ell,t}-p_{\ell,t}) && \text{(IC}h\text{)}\\
& \sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_\ell x_{\ell,t}-p_{\ell,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_0 && \text{(IR}\ell\text{)}\\
& \sum_{t=\tau}^{\infty}\delta^{t-\tau}(\theta_h x_{h,t}-p_{h,t}) \geq 0 \quad \text{for all } \tau \in \mathcal{T}_1 && \text{(IR}h\text{)}\\
& \mu\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{h,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{h,t}\right) && \text{(O}'\text{)}\\
& \qquad +(1-\mu)\left(\sum_{t=1}^{\tau-1}\theta_\ell x_{\ell,t}+\sum_{t=\tau}^{\infty}\delta^{t-\tau}p_{\ell,t}\right) \geq \theta_\ell+\varepsilon(\delta) \quad \text{for all } \tau \in \mathcal{T}_1.
\end{aligned}
$$

D.2 The Dual Program of Program (P2b)

Let $\boldsymbol{\xi} \coloneqq (\alpha, \beta, \zeta, (\lambda_{\ell,\tau}, \lambda_{h,\tau+1}, \gamma_{\tau+1})_{t=1}^{\infty}) \in \mathbb{R}^3 \times \mathbb{R}^{3\infty}$, where: α (resp., β) is the Lagrange multiplier associated to constraint (F2) (resp., (F3)) in program (P2b); ζ is the Lagrange multiplier associated to constraint (ICh) in program (P2b); for all $\tau \in \mathcal{T}_0$, $\lambda_{\ell,\tau}$ is the Lagrange multiplier associated to constraint (IRℓ) at time τ in program (P2b); for all $\tau \in \mathcal{T}_0$, $\lambda_{h,\tau+1}$ is the Lagrange multiplier associated to constraint (IRh) at time $\tau+1$ in program (P2b); for all $\tau \in \mathcal{T}_0$, $\gamma_{\tau+1}$ is the Lagrange multiplier associated to constraint (O$'$) at time $\tau+1$ in program (P2b).

The dual linear program of program (P2b) is the following:

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^3\times\mathbb{R}^{3\infty}} \quad \alpha+\beta-\sum_{t=2}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_t \tag{P2c}$$

$$\text{s.t.}\quad \alpha\geq 0,\quad \beta\geq 0,\quad \gamma_t\geq 0 \quad \text{for all } t\in\mathcal{T}_1 \tag{57}$$

$$\alpha \geq -\delta^{t-1}\theta_h\zeta+\sum_{\tau=1}^{t}\delta^{t-\tau}\theta_\ell\lambda_{\ell,\tau}+\sum_{\tau=t+1}^{\infty}(1-\mu)\theta_\ell\gamma_\tau \quad \text{for all } t\in\mathcal{T}_0, \tag{58}$$

$$\beta \geq \delta^{t-1}\theta_h\zeta+\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\lambda_{h,\tau}+\sum_{\tau=t+1}^{\infty}\mu\theta_\ell\gamma_\tau \quad \text{for all } t\in\mathcal{T}_0, \tag{59}$$

$$\delta^{t-1}\zeta-\sum_{\tau=1}^{t}\delta^{t-\tau}\lambda_{\ell,\tau}+\sum_{\tau=2}^{t}\delta^{t-\tau}(1-\mu)\gamma_\tau+\delta^{t-1}(1-\mu)=0 \quad \text{for all } t\in\mathcal{T}_0, \tag{60}$$

$$-\delta^{t-1}\zeta-\sum_{\tau=2}^{t}\delta^{t-\tau}\lambda_{h,\tau}+\sum_{\tau=2}^{t}\delta^{t-\tau}\mu\gamma_\tau+\delta^{t-1}\mu=0 \quad \text{for all } t\in\mathcal{T}_0, \tag{61}$$

where constraints (60) and (61) hold with equality because $p_{\ell,t}$ and $p_{h,t}$ are unrestricted (i.e., they can be positive or negative) for all $t\in\mathcal{T}_0$ in the primal program (P2b).

Let

$$\boldsymbol{\xi}^* := \left(\alpha^*,\beta^*,\zeta^*,\left(\lambda^*_{\ell,\tau},\lambda^*_{h,\tau+1},\gamma^*_{\tau+1}\right)_{t=1}^{\infty}\right)\in\mathbb{R}^2\times\mathbb{R}^{3\infty}$$

denote a candidate solution to program (P2c).

D.3 A Candidate Solution to the Dual Linear Program (P2c)

Step 1. Solving equations (60) at successive values of t, we obtain

$$\zeta^*=\mu \qquad \text{and} \qquad \lambda^*_{h,t}=\mu\gamma^*_t \quad \text{for all } t\in\mathcal{T}_1. \tag{62}$$

Next, solving equations (61) at successive values of t, we obtain

$$\lambda^*_{\ell,1}=1 \qquad \text{and} \qquad \lambda^*_{\ell,t}=(1-\mu)\gamma^*_t \quad \text{for all } t\in\mathcal{T}_1. \tag{63}$$

Step 2. Given equations (62) and (63), the dual program (P1b) simplifies as follows:

$$\min_{(\alpha,\beta,(\gamma_t)_{t=2}^{\infty})\in\mathbb{R}^2\times\mathbb{R}^{\infty}} \quad \alpha+\beta-\sum_{t=2}^{\infty}(\theta_\ell+\varepsilon(\delta))\gamma_t \tag{P2d}$$

$$\text{s.t.}\quad \alpha\geq 0,\quad \beta\geq 0,\quad \gamma_t\geq 0 \quad \text{for all } t\in\mathcal{T}_1 \tag{64}$$

$$\alpha\geq\delta^{t-1}(\theta_\ell-\mu\theta_h)+(1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma_\tau+\sum_{\tau=t+1}^{\infty}\gamma_\tau\right] \quad \text{for all } t\in\mathcal{T}_0, \tag{65}$$

$$\beta\geq\delta^{t-1}\mu\theta_h+\mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau+\sum_{\tau=t+1}^{\infty}\theta_\ell\gamma_\tau\right] \quad \text{for all } t\in\mathcal{T}_0. \tag{66}$$

Step 3. Since (P2d) is a minimization problem, its solutions must satisfy

$$\alpha^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}(\theta_\ell - \mu\theta_h) + (1-\mu)\theta_\ell\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}\gamma_\tau^*\right]\right\}\right\} \tag{67}$$

and

$$\beta^* = \max\left\{0, \max_{t\in\mathcal{T}_0}\left\{\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma_\tau^*\right]\right\}\right\}. \tag{68}$$

Step 4. We guess that there exists $T(\delta,\mu) \in \mathcal{T}_1$ such that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$. Moreover, we guess that

$$\delta^{t-1}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t}\delta^{t-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+1}^{\infty}\theta_\ell\gamma_\tau^*\right] = \delta^{t}\mu\theta_h + \mu\left[\sum_{\tau=2}^{t+1}\delta^{t+1-\tau}\theta_h\gamma_\tau^* + \sum_{\tau=t+2}^{\infty}\theta_\ell\gamma_\tau^*\right] \tag{69}$$

for all $t \in \{1,\dots,T(\delta,\mu)-1\}$.

Step 5. From the guess in equation (69), we obtain that

$$\gamma_t^* = \frac{\theta_h(1-\delta)}{\theta_h-\theta_\ell}\left(\frac{\theta_h-\delta\theta_\ell}{\theta_h-\theta_\ell}\right)^{t-2} = \frac{\theta_h(1-\delta)}{\Delta\theta}\rho^{t-2} \quad \text{for all } t\in\{2,\dots,T(\delta,\mu)\}, \tag{70}$$

where the last equality holds by definitions (1) and (3).

Step 6. From equation (68), the guess in equation (69), and the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$ we obtain

$$\beta^* = \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)}\gamma_t^*\right). \tag{71}$$

Step 7. By using equations (67) and (70), definition (2), and the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$, constraint (65) becomes

$$\alpha^* = \max_{t\in\{1,\dots,T(\delta,\mu)\}}\left\{\delta^{t-1}(\theta_\ell-\mu\theta_h) + \frac{(1-\mu)\theta_\ell\theta_h(1-\delta)}{\Delta\theta}\left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{T(\delta,\mu)-1}}{1-\rho}\right)\right\}, \tag{72}$$

assuming that the right-hand side of equation (72) is non-negative (which we will show to be the case in Step 9). The maximand on the right-hand side of equation of equation (72) simplifies as follows:

$$\begin{aligned}
&\delta^{t-1}(\theta_\ell-\mu\theta_h) + \frac{(1-\mu)\theta_\ell\theta_h(1-\delta)}{\Delta\theta}\left(\delta^{t-2}\frac{1-r^{t-1}}{1-r} + \frac{\rho^{t-1}-\rho^{T(\delta,\mu)-1}}{1-\rho}\right)\\
&= \delta^{t-1}(\theta_\ell-\mu\theta_h) - \delta^{t-1}(1-\mu)\theta_\ell(1-r^{t-1}) - (1-\mu)\theta_h\left(\rho^{t-1}-\rho^{T(\delta,\mu)-1}\right)\\
&= (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t-1}\mu\Delta\theta - \delta^{t-1}(1-\mu)\Delta\theta r^{t-1}\\
&= (1-\mu)\theta_h\rho^{T(\delta,\mu)-1} - \delta^{t-1}\Delta\theta\left(\mu+(1-\mu)r^{t-1}\right),
\end{aligned} \tag{73}$$

where the first equality holds by implications (4) and (5), and the third equality holds by definition (3). Thus, by equation (73), equation (72) is equivalent to

$$\alpha^* = \max_{t\in\{1,\dots,T(\delta,\mu)\}} \Big\{(1-\mu)\theta_h \rho^{T(\delta,\mu)-1} - \delta^{t-1}\Delta\theta\big(\mu + (1-\mu)r^{t-1}\big)\Big\}. \tag{74}$$

Let t^* denote the value of $t \in \{1,\dots,T(\delta,\mu)\}$ that maximizes the maximand on the right-hand side of equation (74). Moreover, let $\underline{t}$ be such that

$$\delta^{\underline{t}-1}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}-1}\big) = \delta^{\underline{t}}\Delta\theta\big(\mu + (1-\mu)r^{\underline{t}}\big). \tag{75}$$

Since the maximand on the right-hand side of equation (74) is concave in t and $T(\delta,\mu)$ is yet to be determined (and so can be chosen arbitrarily large), $t^* = \inf\{t \in \mathcal{T}_0 : t \ge \underline{t}\}$.

Note that

$$\begin{aligned}
(75) &\iff \mu + (1-\mu)r^{\underline{t}-1} = \delta\big(\mu + (1-\mu)r^{\underline{t}}\big)\\
&\iff \mu(1-\delta) = -r^{\underline{t}-1}(1-\mu)(1-\delta r)\\
&\iff \mu(1-\delta) = r^{\underline{t}-1}(1-\mu)\frac{(1-\delta)\theta_\ell}{\Delta\theta}\\
&\iff r^{\underline{t}-1} = \frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\\
&\iff \underline{t} = 1 + \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r},
\end{aligned} \tag{76}$$

where the third equivalence follows from implication (5). Since $\mu \in \big(\frac{\theta_\ell}{\theta_h}, 1\big)$, $\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell} > 1$; moreover, $r > 1$. Therefore, $\underline{t} > 1$, which implies that $t^* = \inf\{t \in \mathcal{T}_0 : t \ge \underline{t}\} \ge 2$.

Step 8. The choice of $T(\delta,\mu)$ is part of the choice of the Lagrange multipliers (in particular, that of $(\gamma_t^*)_{t=2}^\infty$). Thus, $T(\delta,\mu)$ must be chosen to minimize the objective function of program (P2c).

Let $V(\text{P2c};\delta)$ denote the optimal value of program (P2c). Note that

$$\begin{aligned}
V(\text{P2c};\delta) &= \alpha^* + \beta^* - \sum_{t=2}^\infty (\theta_\ell + \varepsilon(\delta))\gamma_t^*\\
&= \alpha^* + \beta^* - \sum_{t=2}^\infty \theta_\ell\gamma_t^* - \sum_{t=2}^\infty \varepsilon(\delta)\gamma_t^*\\
&= \theta_h - \delta^{t^*-1}\Delta\theta\big(\mu + (1-\mu)r^{t^*-1}\big) - \frac{\theta_h(1-\delta)\varepsilon(\delta)}{\Delta\theta}\frac{1-\rho^{T(\delta,\mu)}}{1-\rho},
\end{aligned} \tag{77}$$

where the third equality holds by the parallels with the analysis in Section B.3, the guess that $\gamma_t^* = 0$ for all $t > T(\delta,\mu)$, and equation (70).

Note that $V(\text{P2c};\delta)$ is decreasing in

$$\frac{\theta_h(1-\delta)\varepsilon(\delta)}{\Delta\theta}\frac{1-\rho^{T(\delta,\mu)}}{1-\rho}$$

which, in turn, is increasing in $T(\delta,\mu)$. Thus, $V(\text{P2c};\delta)$ is decreasing in $T(\delta,\mu)$. Since $T(\delta,\mu)$ must be chosen in order to minimize the objective function of program (P2c), we set it as large as possible; that is

$$T(\delta,\mu) = t^*. \tag{78}$$

Step 9. Summing up, a candidate solution

$$\boldsymbol{\xi}^* := (\alpha^*, \beta^*, \zeta^*, (\lambda^*_{\ell,\tau}, \lambda^*_{h,\tau+1}, \gamma^*_{\tau+1})_{t=1}^{\infty}) \in \mathbb{R}^2 \times \mathbb{R}^{3^\infty}$$

to program (P2c) is as follows. For

$$T(\delta,\mu) = \inf\{t \in \mathcal{T}_1 : t \geq \underline{t}\},$$

where

$$\underline{t} = 1 + \frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r},$$

we have:

$$\alpha^* = \delta^{T(\delta,\mu)-1}\Big[(1-\mu)\theta_\ell r^{T(\delta,\mu)-1} - \mu\Delta\theta\Big], \tag{79}$$

$$\beta^* = \mu\theta_h + \mu\theta_\ell\left(\sum_{t=2}^{T(\delta,\mu)} \gamma^*_t\right) \tag{80}$$

$$\zeta^* = \mu \tag{81}$$

$$\lambda^*_{\ell,t} = \begin{cases} 1 & \text{if } t = 1 \\ (1-\mu)\gamma^*_t & \text{otherwise} \end{cases}, \tag{82}$$

$$\lambda^*_{h,t} = \mu\gamma^*_t, \tag{83}$$

$$\gamma^*_t = \begin{cases} \frac{\theta_h(1-\delta)}{\Delta\theta}\rho^{t-2} & \text{if } t \in \{2,\ldots,T(\delta,\mu)\} \\ 0 & \text{if } t > T(\delta,\mu) \end{cases}. \tag{84}$$

Except for α^*, all elements of $\boldsymbol{\xi}^*$ are clearly non-negative. To see that α^* is also non-negative, note that

$$\alpha^* \geq \delta^{\underline{t}-1}\Big[(1-\mu)\theta_\ell r^{\underline{t}-1} - \mu\Delta\theta\Big] = 0$$

where the inequality holds by definition of $\underline{t}$ and the equality holds by equation (25).

Hereafter, for simplicity, we take

$$T(\delta,\mu)=\underline{t}=1+\frac{\log\left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)}{\log r} \tag{85}$$

or, equivalently,

$$r^{T(\delta,\mu)-1}=\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}. \tag{86}$$

From equations (77), (78), (85), and (86), and the parallels with the analysis in Section B.3, we obtain

$$V(\text{P2c})=\theta_h\left(1-\delta^{T(\delta,\mu)-1}\frac{\mu\Delta\theta}{\theta_\ell}\right)-\frac{\theta_h(1-\delta)\varepsilon(\delta)}{\Delta\theta}\frac{1-\rho^{T(\delta,\mu)}}{1-\rho}. \tag{87}$$

D.4 A Solution to Programs (P2a) and (P2)

Step 11. In this step, we recover a candidate solution

$$(\boldsymbol{x}^*(\delta),\boldsymbol{p}^*(\delta)):=(x^*_{\ell,t}(\delta),x^*_{h,t}(\delta),p^*_{\ell,t}(\delta),p^*_{h,t}(\delta))_{t=1}^\infty$$

to program (P2a) and show that $x^*_{h,1}(\delta)>0$ as $\delta\to 1$.

Since $\lambda^*_{\ell,t}>0$ for all $t\in\{1,\dots,t^*\}$, (IRℓ) is binding for all such t. Thus,

$$p^*_{\ell,t}(\delta)=\theta_\ell x^*_{\ell,t}(\delta)\qquad\text{for all } t\in\{1,\dots,t^*\}. \tag{88}$$

Similarly, since $\lambda^*_{h,t}>0$ for all $t\in\{2,\dots,t^*\}$, (IRh) is binding for all such t. Thus,

$$p^*_{h,t}(\delta)=\theta_h x^*_{h,t}(\delta)\qquad\text{for all } t\in\{2,\dots,t^*\}. \tag{89}$$

Moreover, we conjecture that the solution to program (P2a) is as follows:

$$x^*_{\ell,t}(\delta)=\begin{cases}1 & \text{if } t=t^*\\ 0 & \text{otherwise}\end{cases}. \tag{90}$$

and

$$x^*_{h,t}(\delta)=0\qquad\text{for all } t\geq t^*. \tag{91}$$

From (88) and (90), we have

$$p^*_{\ell,t}(\delta)=\begin{cases}\theta_\ell & \text{if } t=t^*\\ 0 & \text{otherwise}\end{cases}. \tag{92}$$

From (89) and (91), we have

$$p^*_{h,t}(\delta)=0\qquad\text{for all } t\geq t^*.$$

Since $\lambda^*_{h,1}>0$, (ICh) is binding. This, together with (89), (90), and (92), implies that

$$\theta_h x^*_{h,1}(\delta)-p^*_{h,1}(\delta)=\delta^{t^*-1}\Delta\theta$$

or, equivalently,

$$p^*_{h,1}(\delta) = \theta_h x^*_{h,1}(\delta) - \delta^{t^*-1}\Delta\theta,$$

where $x^*_{h,1}(\delta)$ is yet to be determined.

To find $x^*_{h,t}(\delta)$ for all $t \in \{2, \dots, t^* - 1\}$, we use that constraint (O′) is binding for all $t \in \{2, \dots, t^* - 1\}$ (as $\gamma^*_t > 0$ for all such t). From constraint (O′) binding at $\tau = t^* - 1$, we obtain

$$\mu[\theta_\ell(1 - x^*_{h,t^*-1}) + \theta_h x^*_{h,t^*-1}] + (1-\mu)\delta\theta_\ell = \theta_\ell + \varepsilon(\delta),$$

or, equivalently

$$x^*_{h,t^*-1}(\delta) = \frac{\theta_\ell}{\mu\Delta\theta}(1-\delta)(1-\mu) + \frac{\varepsilon(\delta)}{\mu\Delta\theta}.$$

Similarly, for all $t \in \{2, \dots, t^* - 2\}$, we obtain

$$\begin{aligned} x^*_{h,t}(\delta) &= \tfrac{\theta_\ell}{\mu\Delta\theta}(1-\delta)(1-\mu)\left(\tfrac{\theta_h - \delta\theta_\ell}{\Delta\theta}\right)^{t^*-t-1} \\ &= \tfrac{\theta_\ell}{\mu\Delta\theta}(1-\delta)(1-\mu)(\delta r)^{t^*-t-1}. \end{aligned}$$

Note that, for all $t \in \{2, \dots, t^* - 2\}$, $x^*_{h,t}(\delta)$ does not depend $\varepsilon(\delta)$. Finally, we have

$$\begin{aligned} x^*_{h,1}(\delta) &= 1 - \sum_{t=2}^{t^*-1} x^*_{h,t} \\ &= \frac{1}{\mu}\left(1 - \frac{\Delta\theta}{\theta_\ell}\frac{\Delta\theta}{\theta_h - \delta\theta_\ell}\mu\delta^{t^*-1}\right) - \frac{\varepsilon(\delta)}{\mu\Delta\theta}. \end{aligned} \tag{93}$$

Let $(\boldsymbol{x}^*, \boldsymbol{p}^*) := (x^*_{\ell,t}, x^*_{h,t}, p^*_{\ell,t}, p^*_{h,t})_{t=1}^{\infty}$ denote the solution to program (P1) we characterize in Appendix B. Note that

$$x^*_{h,1}(\delta) = x^*_{h,1} - \frac{\varepsilon(\delta)}{\mu\Delta\theta}.$$

Since $x^*_{h,1} > 0$ and $\varepsilon(\delta) \to 0$ as $\delta \to 1$, $x^*_{h,1}(\delta)$ is bounded away from 0 as $\delta \to 1$.

Step 11. Since for all $t \in \mathcal{T}_0$ we have, as $\delta \to 1$,

$$x^*_{\ell,t}(\delta) \to x^*_{\ell,t}, \qquad x^*_{h,t}(\delta) \to x^*_{h,t}, \qquad p^*_{\ell,t}(\delta) \to p^*_{\ell,t}, \qquad p^*_{h,t}(\delta) \to p^*_{h,t},$$

it follows that the candidate solution to program (P2a) satisfies all constraints in program (P2) as $\delta \to 1$.

Step 12. Summing up, we have that, as $\delta \to 1$,

$$V(\text{P2c}; \delta) \to V(\text{P1c}) \qquad \text{and} \qquad V(\text{P2}; \delta) \to V(\text{P1})$$

Since $V(\text{P1c}) = V(\text{P1})$, we have that, as $\delta \to 1$, $V(\text{P2c}; \delta) = V(\text{P2}; \delta)$. Thus, $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ is an approximate solution to program (P2) as $\delta \to 1$.

Finally, since $V(\text{P2};\delta) \leq U_P^* \leq V(\text{P1})$ and $V(\text{P2};\delta) \to V(\text{P1})$, as $\delta \to 1$, it follows that $(\boldsymbol{x}^*(\delta), \boldsymbol{p}^*(\delta))$ approximates a seller-optimal wPBE outcome of $\mathcal{G}$, establishing part (a) of Theorem (5).

Step 13. We find the buyer's payoff and the total surplus. For type θ_ℓ, obviously her payoff is 0. For type θ_h, her information rent is positive, but only comes from the initial period. From the second period on, her continuation payoff is 0. By the truth-telling constraint (binding), the payoff of the buyer of type θ_h is $\theta_\ell \delta^{t^*-1}$. Thus, the payoff of the buyer is

$$\mu\theta_\ell\delta^{t^*-1}.$$

Therefore, the total surplus, denoted by $S(\mu, \delta)$ is

$$S(\mu,\delta) = V(\text{P1}) + \mu\theta_\ell\delta^{t^*-1} = \theta_h - \mu\delta^{t^*-1}\left(\frac{\Delta\theta}{\theta_\ell} + \theta_\ell\right)\Big).$$

Step 14. By taking $\delta \to 1$, we obtain

$$\delta^{t^*-1} \approx \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}},$$

and so

$$V(\text{P1}) \approx \theta_h\left(1 - \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \mu\left(\frac{\Delta\theta}{\theta_\ell}\right)\right) > \theta_\ell,$$

establishing part (c) of Theorem (5). Similarly, $\lim_{\delta\to 1} S(\mu,\delta) < \mu\theta_h + (1-\mu)\theta_\ell$, establishing part (b) of Theorem (5).

Step 15. Let $\theta_\ell = \frac{\theta_h}{k}$ for some $k \geq 1$. Then, $\Delta\theta = \theta_h\frac{k-1}{k}$. As $k \to \infty$, we have:

$$\begin{aligned}
v(\text{P1}) = \approx \theta_h &\left(1 - \left(\frac{\mu\Delta\theta}{(1-\mu)\theta_\ell}\right)^{-\frac{\Delta\theta}{\theta_h}} \mu\left(\frac{\Delta\theta}{\theta_\ell}\right)\right) \\
= \theta_h &\left(1 - \left(\frac{\mu\theta_h\frac{k-1}{k}}{(1-\mu)\theta_h\frac{1}{k}}\right)^{-\frac{k-1}{k}} \mu(k-1)\right) \\
= \theta_h &\left(1 - \left(\frac{\mu(k-1)}{(1-\mu)}\right)^{-\frac{k-1}{k}} \mu(k-1)\right) \\
= \theta_h &\left(1 - \left(\frac{\mu(k-1)}{(1-\mu)}\right)^{\frac{1}{k}} (1-\mu)\right),
\end{aligned}$$

where

$$\left(\frac{\mu(k-1)}{(1-\mu)}\right)^{\frac{1}{k}} = \exp\left(\frac{1}{k}\log\frac{\mu(k-1)}{(1-\mu)}\right) \approx \exp(0) = 1,$$

and thus

$$v(\text{P1}) = \approx \theta_h\left(1 - \left(\frac{\mu(k-1)}{(1-\mu)}\right)^{\frac{1}{k}}(1-\mu)\right) \approx \mu\theta_h,$$

which is the commitment payoff, establishing part (d) of Theorem (5). ■

E Calculations for Section 3.1

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