Monotone Additive Statistics

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Talk Overview

- Definition of **monotone additive statistics**.
- Characterization.
- Applications.
  - Posted prices for **sacks of potatoes**.
  - Fishburn-Rubinstein **time preferences**.
  - Rabin-Weizsäcker preferences over **gambles**.
- Monotone additive costs of **Blackwell experiments**
  - Different paper: "From Blackwell Dominance in Large Samples to Rényi Divergences and Back Again."
  - Same authors.
  - Related ideas.
- **Work in progress.**

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- A **statistic** is a way of capturing distributions by a single number.
  - Expectation.
  - Median.
  - Value at risk.
  - Certainty equivalent.
- Let L^∞ be the set of all bounded random variables.
- A **statistic** is a map $\Phi\colon L^\infty \to \mathbb{R}$ such that
  1. $\Phi(c) = c$.
  2. If X and Y have the same distribution then $\Phi(X) = \Phi(Y)$.
- It is **monotone** if $X \geq_1 Y$ implies $\Phi(X) \geq \Phi(Y)$.
- Equivalently: it is monotone if $X \geq Y$ implies $\Phi(X) \geq \Phi(Y)$.
  - Because $X \geq_1 Y$ iff $\exists \tilde{X} \sim X, \tilde{Y} \sim Y$ s.t. $\tilde{X} \geq \tilde{Y}$ a.s.
- A statistic is **additive** if $\Phi(X+Y) = \Phi(X) + \Phi(Y)$ whenever X and Y are **independent**.

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Monotone Additive Statistics

- **Question:** What are the additive monotone statistics?

Examples of Monotone Additive Statistics

- $\mathbb{E}[X]$.
- $\max[X] = \sup\{c \in \mathbb{R} : \mathbb{P}[X \geq c] > 0\}$.
- $\min[X]$.
- For $a \neq 0$,

$$S_a(X) = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aX}\right].$$

- By continuity
  - $S_0(X) = \mathbb{E}[X]$,
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Characterization

- Is there anything beside the S_a's?
- Main result: this is it.
- Well... we can also take weighted averages.

Theorem

Let Φ be a monotone additive statistic. Then there is a probability measure m on $\mathbb{R} \cup \{+\infty, -\infty\}$ such that

$$\Phi(X) = \int S_a(X)\,\mathrm{d}m(a).$$

- $\{S_a\}$ are the extreme points of the set of additive monotone statistics.

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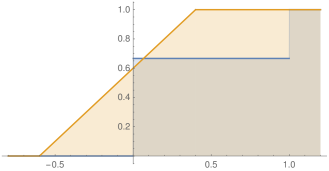
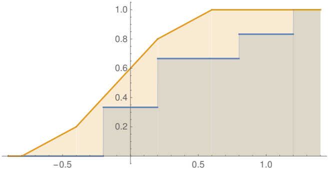
Proof ideas

- Take X, Y that are not ranked under FOSD.
- Is it possible that there is a independent R such that $X + R \geq_1 Y + R$?
- Example: $X \sim B(1/3)$, $Y \sim U([-3/5, 2/5])$.



- Works for $\mathbb{P}[R = \pm 1/5] = 1/2$.



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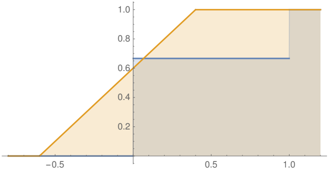

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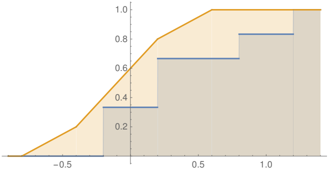

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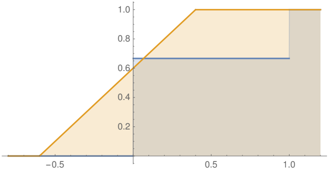

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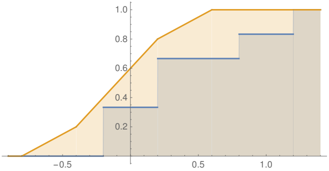
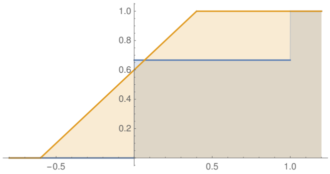


Proof ideas

- Take X, Y that are not ranked under FOSD.
- Is it possible that there is a independent R such that $X + R \geq_1 Y + R$?
- Example: $X \sim B(1/3)$, $Y \sim U([-3/5, 2/5])$.



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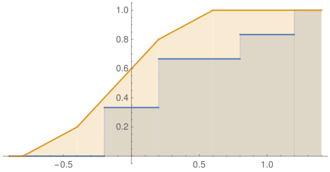

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- Pomatto, Strack, Tamuz (2019): If $\mathbb{E}[X] > \mathbb{E}[Y]$ then $X + R \geq_1 Y + R$ for some independent R.
- Under what conditions on X, Y is there a **bounded** independent r.v. R such that $X + R \geq_1 Y + R$?
- If $S_a(X) < S_a(Y)$ for some a this is impossible, since

$$S_a(X + R) = S_a(X) + S_a(R) < S_a(Y) + S_a(R) = S_a(Y + R).$$

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For $X, Y \in L^\infty$, if $S_a(X) > S_a(Y)$ for all a, then there exists an $R \in L^\infty$ such that $X + R \geq_1 Y + R$.

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$$\Phi(X) + \Phi(R) = \Phi(X + R) \geq \Phi(Y + R) = \Phi(Y) + \Phi(R)$$

- Rest of the proof: exercise in analysis.

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$$[(G - F) * h](y) = \int_{-N}^{N} [G(x) - F(x)] \cdot h(y - x)\,\mathrm{d}x$$
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  1. $\mathrm{e}^{-\frac{x^2}{2V}} \approx 1$ for $x \in [-N, N]$ and large V.
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Application: Posted Prices for Sacks of Potatoes

- Consider a buyer who posts her prices for **potatoes**.
- Farmers come and sell her their crops.

Potatoes	Price
1	\$1
2	\$2
3	\$3.10
4	\$4
5	\$5
6	\$6
7	\$5

- Price $P\colon \mathbb{R}_+ \to \mathbb{R}_+$.
- **Free disposal**: $x \geq y$ implies $P(x) \geq P(y)$.
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- Example: $f(x,T) = \dfrac{u(x)}{\mathbb{E}\left[\mathrm{e}^{rT}\right]}$.
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- Let L^∞ be the set of **bounded gambles**.
- Consider an expected utility agent with an increasing utility function u for money.
- Write $X \succ Y$ if $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$.

Axiom

Suppose X_1, X_2 are independent. Y_1, Y_2 are independent. If $X_1 \succ Y_1$ and $X_2 \succ Y_2$ then $Y_1 + Y_2$ does not stochastically dominate $X_1 + X_2$.

- What does this tell us about u?

Theorem (Rabin-Weizsäcker)

The axiom implies that either $u(x) = ae^{ax}$ for some $a \neq 0$, or $u(x) = x$ (up to affine transformations).

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2. $X + \varepsilon \succ X$.
3. *for all* X *there is a* $c \in \mathbb{R}$ *such that* $X \sim c$.

- Such preferences can be represented by a monotone additive statistic.

Proposition

The axioms imply that $\succ$ *is represented by some monotone additive statistic.*

- Φ is the average of CARA certainty equivalents $S_a(X) = \frac{1}{a} \log \mathbb{E}\left[e^{aX}\right]$.

Application: Rabin-Weizsäcker Preferences

- What about general (non-expected utility) preferences?
- Write $X \succ Y$ if the agent strictly prefers X to Y.

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Monotone Additive Costs of Blackwell Experiments

- Binary state of the world $\theta \in \{0, 1\}$.
- A **Blackwell Experiment** is a pair $\mu = (\mu_0, \mu_1)$ of probability measures on some measurable space Ω.
- We say that it is **bounded** if $\log \frac{d\mu_0}{d\mu_1}$ is bounded.
- The collection of bounded experiments is $\mathcal{B}$.
- The Blackwell order captures a strong sense of when one experiment is more informative than another.
- The product experiment $\mu \otimes \nu$ is given by $(\mu_0 \times \nu_0, \mu_1 \times \nu_1)$.

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