Money Mining and Price Dynamics*

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May 4, 2020

Abstract

We develop a random-matching model to study the price dynamics of monies produced privately according to a time-consuming mining technology. For our leading example, there exists a unique equilibrium where the value of money increases over time and reaches a steady state. There is also a continuum of perfect-foresight equilibria where the price of money inflates and bursts gradually over time. Initially, money is held for a speculative motive but it acquires a transactional role as it becomes sufficiently abundant. We study fiat, commodity, and crypto monies, endogenous acceptability, and adopt implementation and equilibrium approaches.

Keywords: Money, Search, Private Monies, Mining
JEL codes: E40, E50

*We thank the editor, Simon Gilchrist, and two anonymous referees for their comments. We also thank Johnathan Chiu, Paul Jackson, Lucie Lebeau, Fan Liang, Sebastien Lotz, Diana Xiuyao Yang, Cathy Zhang, and seminar participants at UC Riverside, UC Irvine, 2018 West Coast Search-and-Matching Workshop at UC Irvine, 1st DC Search and Matching Workshop at Federal Reserve Board, University of Saskatchewan, 9th European Search-and-Matching Workshop at Oslo, Midwest Macro at University of Georgia Athens, National University of Singapore, Singapore Management University and participants to the 2020 AEA session on "Models of Cryptocurrencies: Pricing and design". Usual disclaimers apply.

1 Introduction

The most successful currencies in Medieval Europe (e.g., the Florentine florin) were coins made of gold or silver obtained through mining activities. The first crypto-currency, Bitcoin, was designed to mimic gold-based coinage: the supply of money increases gradually over time and becomes constant in the long run.[1] Following Bitcoin, hundreds of new crypto-currencies have been introduced over the last decade, bringing some foundational questions of monetary theory to the forefront. Can privately-produced, intrinsically useless objects be traded at a positive price? How is the initial value of a new money determined and how does its price evolve over time? Is a boom and burst of crypto-currency prices consistent with rational expectations? Is the private production of money socially efficient?

The goal of this paper is to revisit these questions by studying the dynamics of an economy where money is privately produced at some cost, possibly endogenous, through mining — a time-consuming activity. Our theory applies to the mining of commodity monies, e.g., gold and silver, as well as the production of fiat currencies, e.g., Bitcoins. (Throughout the paper, we use Wallace's (1980) definition of a fiat money as an object that is inconvertible and intrinsically useless.)[2] Because the determination of currency prices is better understood in models where there is an essential role for a medium of exchange, we adopt the search-theoretic model of monetary exchange of Shi (1995) and Trejos and Wright (1995). In this environment, trades take place within pairwise meetings that are formed randomly. Heterogeneity in preferences and specialization in production generate a lack-of-double-coincidence-of-wants problem and rule out barter trades. In addition, agents, who are anonymous, cannot finance random consumption opportunities by issuing private debts, hence a role for money (Kocherlakota, 1998). Money is indivisible and there is a unit upper bound on individual money holdings. While this assumption was originally made for tractability, it captures the notion that the quantity of liquid assets is scarce and affects the measure of transactions.[3]

We add two components to the Shi-Trejos-Wright model. First, we introduce a mining technology. We distinguish technologies that correspond to the mining of tangible objects (e.g., gold) from mining technologies for crypto-currencies. Second, we add a cost to mining that can take various forms. It can be

[1] In Nakamoto (2008), the creator of Bitcoin, Satoshi Nakamoto, wrote: "The steady addition of a constant amount of new coins is analogous to gold miners expending resources to add gold to circulation. In our case, it is CPU time and electricity that is expended."

[2] Goldberg (2005) discusses the notion of fiat money in monetary economics and disputes the common wisdom that fiat monies defined as inconvertible and intrinsically useless media of exchange ever existed. In that regards, crypto-currencies might be the first creation of fiat monies as defined by monetary theorists.

[3] In the Appendix E we study a version of the model where money is perfectly divisible and agents adjust their unrestricted asset holdings in competitive exchanges. The results are qualitatively similar.

an exogenous cost associated with the use of input factors, such as computers and electricity, or an endogenous opportunity cost due to occupation choice. We will characterize for different mining technologies the set of all deterministic equilibria under perfect foresight starting from the initial time where money is introduced.

1.1 Preview of our results

In accordance with monetary folk-theorems, a privately-produced fiat money can be valued if agents are sufficiently patient and trading frictions are not too severe. The assumption that monies are privately produced makes the condition for the existence of a monetary equilibrium more stringent. The threshold for the rate of time preference below which money is valued decreases with the maximum amount of money that can be mined and the speed of mining.

Our leading example assumes the speed of mining decreases with the amount of money already mined. We show that the initial price of money is indeterminate within a nonempty interval. The largest value in this interval corresponds to the unique equilibrium leading to a positive value of the currency in the long run. For all lower but positive initial values, the equilibrium path for the value of money is first increasing and then decreasing, and it vanishes asymptotically. So unless agents can coordinate on the highest equilibrium – one equilibrium among a continuum of perfect-foresight equilibria — the life cycle of a privately-produced currency is composed of a boom, where agents mine money, and a bust where agents trade a depreciating money. Across equilibria, the peak for the value of money is positively correlated with its initial value. This result shows that the starting value of a new currency is crucial for its long-run viability.

Increases in the amount of money that can be privately mined, e.g., through discoveries of new mines or through an increase in the supply of crypto-currencies, generates price waves. The value of money falls initially and then increases gradually over time. The overall trend for the price of money is downward slopping. The correlation between the quantity of money and its price can change sign in the short and long run: the correlation is positive along the transitional path but negative across steady states.

A critical component to the fundamental value of a new currency is the extent of its transaction role (Tirole, 1985). A new money can have a transactional role in the long run even if does not serve as a means of payment in the short run. In all equilibria of our model, the new money does not circulate initially and this outcome is shown to be constrained-efficient. To an outside observer, the new currency looks like a speculative bubble since it is only held for its capital gains. It is only when money is sufficiently abundant that agents stop hoarding it. We obtain similar outcomes when we endogenize the acceptability of a new currency through a costly ex ante investment.

Dynamics of currency prices depend on the mining technology. If miners compete for the revenue of money creation and the cost of mining is exogenous and constant, then the price of the currency falls over time in all equilibrium trajectories. This result is overturned if acceptability of money is endogenous, as in Lester et al. (2012), or if the cost of mining is an endogenous opportunity cost. Moreover, the model with costly acceptability generates sunspot equilibria according to which the acceptability of the currency varies with a sunspot state that is independent of fundamentals, and the value of money is positively correlated with acceptability. In a version of the model with endogenous opportunity cost, the currency issuer can stabilize prices by choosing a money growth rate that is proportional to the fraction of the money supply that is yet to mine, where the coefficient of proportionality depends on market structure and preferences. This formula resembles the Bitcoin growth rate.

1.2 Empirical evidence.

We present motivating facts regarding the production and pricing of gold and crypto-currencies.

Gold mining and prices. In the following, we describe two historical episodes that illustrate the joint dynamics of the supplies and prices of gold and silver. The left panel of Figure 1 plots the price level in England and the inflow of silver and gold into Europe in 1300-1700.[4] The inflow of precious metals from America started to increase at the beginning of the 16th century. At the same time, Europe experienced the so-called Price Revolution – a sustained increase in the price level. The positive correlation between the price level and the quantity of gold and silver over that period of time is consistent with the quantity theory. It is also consistent with the long-run comparative statics of our model when the changes in the quantity of money are due to exogenous changes in the potential supply of gold or silver, e.g., because of mine discoveries. Note that the late 16th century and early 17th century exhibit multiple price waves that are also consistent with the short-run dynamics of our model following discoveries of new gold mines or progress in mining technologies.

The second period we consider is the end of the 19th century and first half of the 20th century. The right

[4] The data is from Edo and Jacques (2019) on the cause of inflation in Europe in the period 1500-1700. The inflow of precious metals includes the mine production in Europe as well as the import from America. The inflows is measured in tones of silver and the inflow of gold is converted into silver-equivalent tones. The price level is computed by dividing the nominal GDP (in pounds) by a real output index. We HP filtered the price data.

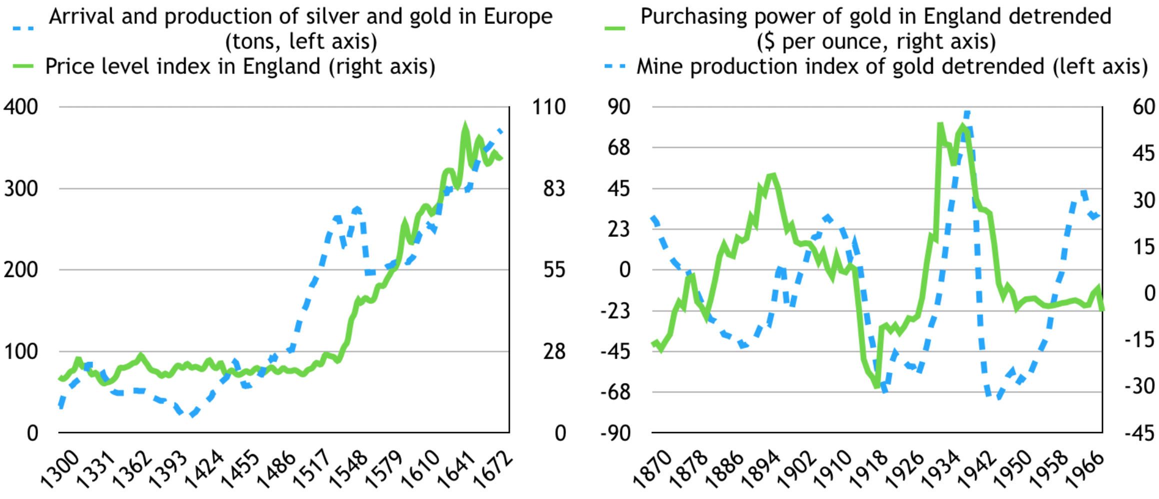

Figure 1: (Left) Price level and inflow of precious metal in Europe during 14th to 17th centuries. (Right) Purchasing power of gold in England and world mine production during 19th and 20th centuries.

panel of Figure 1 plots the deviations from trend of the purchasing power of gold and its production.[5] The two are positively correlated. Moreover, a two-variable vector autoregression model finds that the purchasing power of gold Granger-causes its production in the same time period at a 5-percent significance level. (See our Online Appendix A.) While this result seems to contradict the quantity theory, we will show that the long-run correlations can differ from the short-run ones as the mining of gold in the short-run responds positively to its price.[6]

Mining and pricing of Bitcoin. We take Bitcoin as our leading example for cryptocurrencies. The left panel of Figure 2 shows the daily closing price of Bitcoin and its trend component as estimated by the HP filter.[7] Since 2017 there have been two large boom-and-bust cycles, the first one being larger than the second one.[8] From December 2016 to December 2017, the price of Bitcoin increased from \$1130 to \$19000 — a 17 times increase, then dropped by 5.5 times to around \$3500 in December 2018. Our model will establish conditions to generate boom-and-bust fluctuations under perfect foresight and will show how such fluctuations can repeat themselves. Another feature of the data is the high volatility of the Bitcoin price. According to Klein et al. (2018), the standard deviations of daily return for Bitcoin, gold and S&P500 are

[5]The data on purchasing power and production are from Jastram (2009). The purchasing power of gold is an index of the nominal price of gold in England deflated by an index of commodity prices in England.

[6]Bordo (1981) uses a similar idea to show a rising purchasing power of gold induces an increase in the monetary gold stock.

[7]We use a smoothing parameter of 80000 which is in between the value used by Hodrick and Prescott's original paper and the one recommended by the Ravn-Uhlig rule.

[8]In 2017 and 2018 there are around 20 Bitcoin hard forks. Although the crypto-currencies created by these hard forks are not necessarily perfect substitutes of Bitcoin, one can view them as increases in the potential total supply of crypto-currencies.

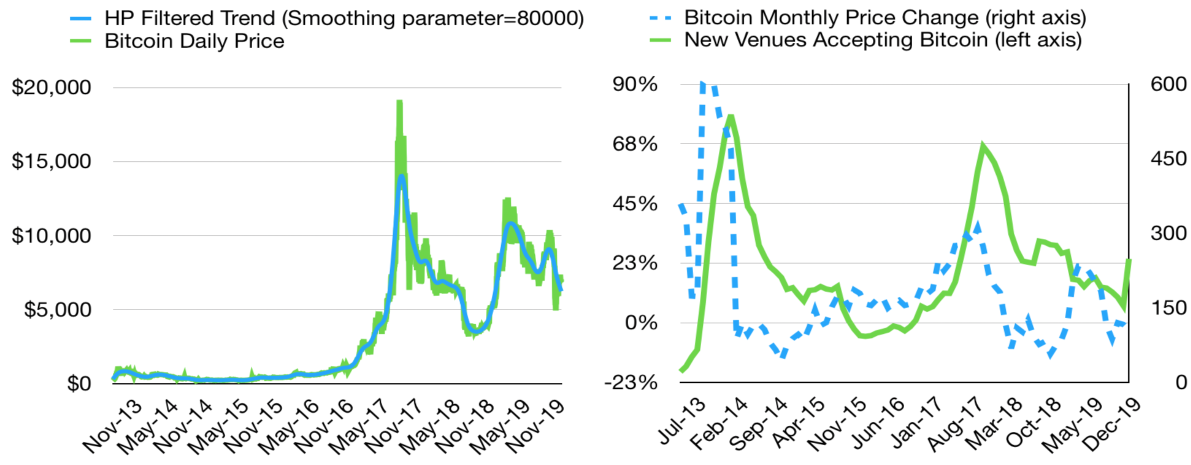

Figure 2: (Left) Daily Bitcoin price. (Right) Bitcoin price change and acceptability.

5.76, 1.05, and 0.89, respectively, from July 2011 to December 2017. A version of our model with endogenous acceptability generates sunspot equilibria that can help explain the high volatility of currency prices.

Bitcoin has been designed so that its supply is predictable. The total supply of Bitcoins is controlled by varying the difficulty level of the mathematical puzzles that miners have to solve. If there is a sudden rise in the number of miners, then the difficulty level increases to keep the money supply along a pre-determined path. As a result, one can infer the intensity of the mining activities (e.g., the number of miners and the CPU time they invest into mining) by looking at the difficulty level of the puzzles. We show in Online Appendix A that the growth rate of Bitcoin prices Granger-causes the growth rate of the mining difficulty level at a 1-percent significance level. This finding is consistent with our assumption that the intensity of Bitcoin mining is driven by the real value of Bitcoins. Relatedly, Prat and Walter (2018) use the Bitcoin-to-US dollar exchange rate to predict the computing power of Bitcoin's network.

In our model, the decision of an agent to accept money depends on its anticipated value. We test this mechanism by comparing the number of new venues accepting Bitcoin each month and the growth rate of Bitcoin prices at a monthly frequency.[9] The right panel of Figure 2 shows that the growth rate of Bitcoin prices leads the number of new venues accepting it. Statistically, the growth rate of prices Granger-causes the number of new venues at a 1-percent significance level, and the correlation between the two series is 0.17.

[9] The data on Bitcoin's acceptability is from CoinMap. It documents venues accepting Bitcoin as a means of payment since 2013. These venues include retailers, restaurants, ATMs, lodging, attractions etc. The daily data of Bitcoin prices is from CoinMarketCap.com. For both data series, we plot the 6-month moving average.

1.3 Literature review

Our model builds on the search-theoretic models of monetary exchange of Shi (1995) and Trejos and Wright (1995) by adding a time-consuming mining activity and, in one version of the model, an occupation choice with an endogenous opportunity cost of money production.[10] Related papers include Burdett, Trejos and Wright (2001) where the quantity of commodity money (cigarettes) is endogenous, Cavalcanti and Wallace (1999) and Williamson (1999) where banks issue inside money, Lotz and Rocheteau (2002) and Lotz (2004) who study the launching and adoption of a new fiat money, Cavalcanti and Nosal (2011) who interpret the production of counterfeited notes as the issuance of a private money that is difficult to monitor, Hendrickson and Luther (2017) who study the coexistence of Bitcoin and a regular currency under endogenous matching. A thorough review of this class of models is provided by Lagos et al. (2017).

Fernandez-Villaverde and Sanches (2018) study currency competition in the Lagos-Wright model extended to have a unit measure of entrepreneurs who can issue distinguishable tokens at an exogenous cost. Complementing their approach, in our model the measure of miners is endogenous. We study the case of an exogenous cost of mining and the case of an endogenous opportunity cost and compare price dynamics. Our description of the mining technology differs as we model its time dimension explicitly. Our focus is also different as we emphasize price dynamics starting from the creation of a new currency up to its disappearance. We use the Shi-Trejos-Wright model with indivisible money instead of the Lagos-Wright model with divisible money as it is simpler to illustrate price dynamics in continuous time. Also, in the Shi-Trejos-Wright model, there is an optimum stock of money, so mining is a meaningful activity, i.e., it is part of the planner's problem. In Appendix E, we present a version of the Lagos-Wright model in continuous time (as in Choi and Rocheteau, 2019b) with mining and show that the dynamics are qualitatively equivalent.

We adopt an implementation approach to study the constrained-efficient production of money and price stabilization. Chiu and Koeppl (2017) study the optimal design of crypto-currencies to overcome double-spending and show that the Bitcoin scheme creates a large welfare loss. Chiu and Koeppl (2018) provide necessary conditions for blockchain-based settlement to be feasible. Biais et al. (2019) formalize the proof-of-work blockchain protocol as a stochastic game and show it has multiple equilibria, including ones with forks and orphaned blocks. They also identify negative externalities that lead to excessive investment in

[10] While we adopt the search-theoretic approach to obtain an essential role for media of exchange, there is a related literature on rational bubbles in the context of OLG models, e.g., Wallace (1980) and Tirole (1985), among many others. An application to crypto-currencies is provided by Garratt and Wallace (2018).

computing capacity. Pagnotta (2018) adopts a version of Rocheteau and Wright (2005) and assumes miners contribute resources that enhance network security and compete for mining rewards in the form of Bitcoins. The equilibrium level of network security and the price of Bitcoins are jointly determined and, among many insights, the price of Bitcoins can vary non-monotonically with the growth rate of Bitcoin supply.

2 The model

2.1 Environment

Time, agents, and goods Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is composed of a unit measure of ex ante identical, infinitely-lived agents indexed on $[0, 1]$, and a perishable good that comes in $J \geq 3$ distinct varieties. In order to create a need for trade, agents are divided evenly across J types corresponding to their specialization in consumption and production. Agent of type $j \in \{1, ..., J\}$ can produce variety j but she only consumes variety $j + 1$ (modulo J). The type-j's utility from consuming $q \in \mathbb{R}_+$ units of good $j+1$ is $u(q)$ with $u(0)=0$, $u'>0$, $u'(0)=+\infty$, and $u''<0$. The type-j's disutility from producing q units of good j is q. There exists a $q^* > 0$ such that $u'(q^*) = 1$ and a $\bar{q}<+\infty$ such that $u(\bar{q})=\bar{q}$. Agents discount future utility at rate $r>0$. Agents' preferences are represented in the left panel of Figure 3.

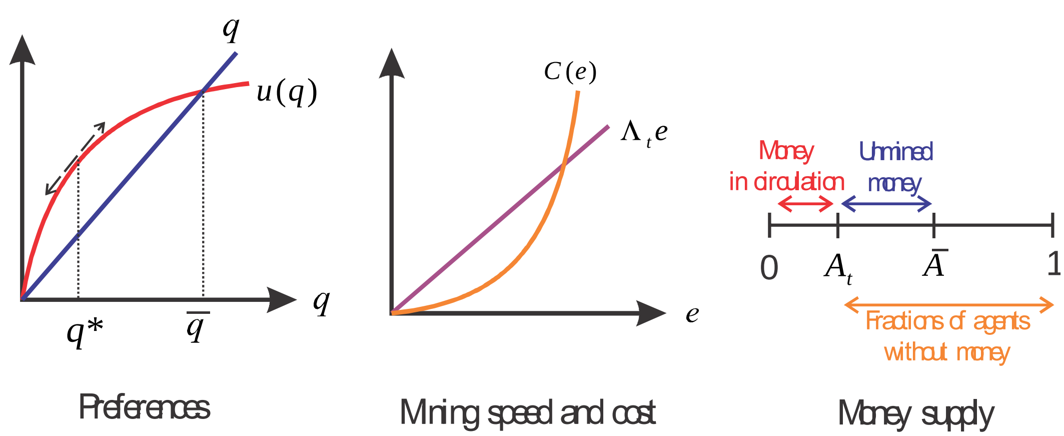

Figure 3: Description of the environment.

Random pairwise matching Agents meet bilaterally and randomly according to a Poisson process with arrival rate $\alpha>0$. Conditional on a meeting, the probability that an agent is matched with a type-j partner is $\sigma \equiv 1/J \in [0, 1]$, where σ is the probability of single coincidence of wants. The specialization in preferences and technologies described earlier rules out double-coincidence-of-wants matches. The terms of trade in pairwise meetings are determined through bargaining.

Frictions and money Agents are anonymous (i.e., there is no public record of trading histories), they lack commitment, and there is no technology to enforce private debt contracts. These frictions create a need for a medium of exchange (Kocherlakota, 1998). There is an intrinsically useless object, called money, that is perfectly storable and durable. It is indivisible, and individual holdings of money, a_i, are restricted to $\{0,1\}$.[11] We denote $A_t \equiv \int_0^1 a_{t,i} di$ which is both the measure of agents with one unit of money at time t and the aggregate money supply. The flow of meetings between money holders and agents without money is $\alpha A_t(1-A_t)$. Among those meetings, only a fraction will generate a trade because the buyer must like the seller's output, the potential seller must choose to produce instead of mining (in a version of the model with endogenous occupation choices) and he must have the technology or expertise to accept money (in another version with endogenous acceptability).

Money mining Money is produced privately according to a time consuming activity called mining and the initial stock of money A_0 is given. The individual effort devoted to mining by agent $i \in [0,1]$ is denoted $e_i \in \mathcal{E}$, where $\mathcal{E}$ is the set of feasible mining intensities from which the agent can choose. If $\mathcal{E} = \mathbb{R}_+$, then mining effort is a continuous variable, e.g., e_i is a variable input such as CPU time and electricity. If $\mathcal{E} = \{0,1\}$, then mining is a discrete choice, e.g., mining is an indivisible occupation choice. The aggregate mining effort across all agents is

$$m_t = \int_0^1 e_{i,t} di. \tag{1}$$

If $\mathcal{E} = \{0,1\}$, m_t is simply the measure of miners. Given the effort e, an agent mines a unit of money according to a Poisson process with time-varying intensity $\Lambda(A_t, m_t)e$. The factor, $\Lambda(A_t, m_t)$, captures the efficiency of mining. It is a function of the amount of money already mined, A_t, and the aggregate mining effort, m_t. We provide two examples of functional forms for Λ.

Example 1: Gold mining. A leading example of a mining technology is

$$\Lambda(A_t, m_t) = \lambda(\bar{A} - A_t), \tag{2}$$

[11] We relax the indivisible money assumption in a working paper, Choi and Rocheteau (2019a), and show the results are robust. We also consider variants of the model where different competing monies, with different physical properties or acceptabilities, coexist.

where $\bar{A} \in (0,1)$ is the overall fixed quantity of money and $\bar{A} - A_t$ is the amount of money that has yet to be mined. We interpret this technology as miners being randomly allocated at locations where units of money can potentially be found. With that specification, the individual mining rate declines as the quantity of money that has been mined, A_t, increases. The individual mining rate, however, is unaffected by the aggregate mining intensity, i.e., the congestion effect from other miners is only indirect through A_t. One can also interpret this mining technology as the creation of many distinct crypto-currencies, where the total number of potential crypto-currencies is $\bar{A}$.

Example 2: Crypto mining. Our second example captures the virtual mining of a crypto-currency:

$$\Lambda(A_t, m_t) = \frac{\pi(A_t)A_t}{m_t}, \tag{3}$$

where $\pi(A_t)$ is the exogenous money growth rate of the currency expressed as a function of A_t and set by the designer of the currency at the time of its creation. The total money creation at time t is $\pi(A_t)A_t$. It is allocated to miners randomly with probabilities proportional to their mining effort: if there is a small measure di of agents mining with intensity e_i, their probability to be allocated a unit of money newly created is $e_i di/m$. By construction, the aggregate quantity of money mined is $\int_0^1 \Lambda(A_t, m_t)e_i di = \Lambda(A_t, m_t)m_t = \pi(A_t)A_t$. The money growth rate of Bitcoin can be approximated by $\pi(A) = \lambda(\bar{A} - A)/A$.[12] From (3),

$$\Lambda^{\text{Bitcoin}}(A_t, m_t) = \frac{\lambda(\bar{A} - A_t)}{m_t}. \tag{4}$$

The only difference between the two technologies, (2) and (4), is the congestion factor, $1/m_t$.

Cost of mining The flow cost of mining is $C(e)$ where $C(0)=0$, $C'>0$, and $C''\geq 0$. A simple specification is $C(e)=ek$ where $k>0$ is a constant representing the unit cost of the variable input going into mining. We represent graphically the mining intensity and cost of mining in the middle panel of Figure 3. In one version of the model, $e \in \{0,1\}$ and $C(1)$ is the endogenous opportunity cost from mining instead of producing consumption goods. According to this version, mining is an occupation choice and agents who choose to mine cannot take advantage of production opportunities in pairwise meetings.

In order to take into account how occupation choices affect buyers' trading probabilities, we denote χ_t the fraction of agents without money who are active producers, e.g., they choose not to mine money when mining

[12] In July 2016 the reward for mining a block is 12.5 bitcoins, plus any transaction fees from payments. The reward for adding a block will be halved every 210,000 blocks (approximately 4 years). The reward will eventually vanish and the limit of 21 million bitcoins will be reached in 2140. Given this description the supply of Bitcoin can be approximated by $A_t = \bar{A}[1 - 2^{-t/4}]$ where t is the number of years since Bitcoin is introduced. Hence the growth rate of Bitcoin is $\pi(A) = \dot{A}/A = (1/A - 1/\bar{A})Log(2)\bar{A}/4$.

and producing are mutually exclusive occupations. In the version of the model with costly acceptability of money, χ_t will denote the fraction of producers who have the technology or expertise to accept money. Altogether the unit measure of agents is divided between buyers, active producers, and inactive producers as shown in Figure 4. There is a measure A_t of buyers, all agents with one unit of money. The remaining $1 - A_t$ agents are potential producers. A fraction χ_t of those potential producers are active, either because they choose not to mine, if mining is an occupation choice, or because they invest in a costly technology to accept money, depending on the version of the model. The remaining $(1 - A_t)(1 - \chi_t)$ agents are inactive as they either decide not to produce or have not made the required investment to accept money.

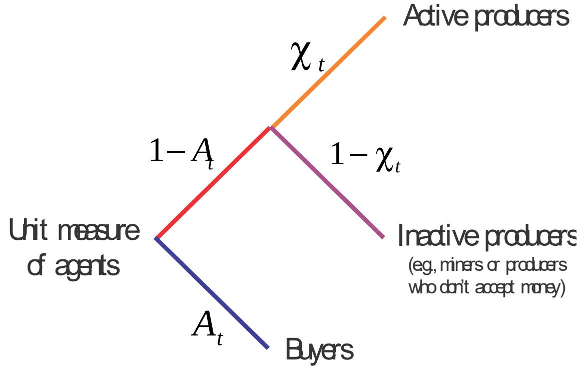

Figure 4: Distribution of agents' roles.

2.2 Definition of equilibria

We define an equilibrium as a list of Bellman equations, bargaining outcomes, optimal mining choices, and a law of motion for the money supply.

Bellman equations Let $V_{a,t}$ be the value of an agent holding $a \in \{0, 1\}$ unit of money at time t. The lifetime expected discounted utility of a money holder solves the Hamilton-Jacobi-Bellman (HJB) equation

$$rV_{1,t} = \alpha\sigma (1 - A_t) \chi_t [u(q_t) + V_{0,t} - V_{1,t}] + \dot{V}_{1,t}, \tag{5}$$

where a dot over a variable represents a time derivative. According to the right side of (5) the agent meets another agent at Poisson rate α; this potential trading partner drawn at random from the whole population holds no money with probability $1 - A_t$; her production type corresponds to the good that the agent likes with probability σ; and she is an active producer with probability χ_t. In versions of the model where agents can produce and mine simultaneously, i.e., these activities are not mutually exclusive, $\chi_t = 1$. In contrast, if mining is a full time occupation, then $\chi_t = (1 - A_t - m_t)/(1 - A_t)$ is the fraction of agents with $a = 0$ who choose $e = 0$ instead of $e = 1$. When a trade takes place, one unit of money buys q_t units of output, where q_t is determined through bargaining. The last term on the right side is the change in the value function over time. Note that money holders choose not to exert any mining effort, $e = 0$, since they have reached the upper bound on money holdings and cannot accumulate an additional unit.

The value function of an agent with $a = 0$ solves the following HJB equation:

$$rV_{0,t} = \alpha\sigma A_t \left(-q_t + V_{1,t} - V_{0,t}\right) + \max_{e \in \mathcal{E}} \left\{\Lambda(A_t, m_t)e(V_{1,t} - V_{0,t}) - C(e)\right\} + \dot{V}_{0,t}. \tag{6}$$

By the first term in the right side, a non-holder of money meets a money holder who likes her good at Poisson arrival rate $\alpha\sigma A_t$ in which case she produces q_t at a linear cost in exchange for one unit of money. By the second term, the agent chooses the mining effort to maximize the expected gain from mining net of the flow cost, $C(e)$. At Poisson rate $\Lambda_t e$ she is rewarded a unit of money and enjoys the capital gain, $V_{1,t} - V_{0,t}$. In the version of the model where mining corresponds to an occupation choice, $C(1)$ is the opportunity cost from not producing, which is equal to the first term on the right side.

Bargaining over output The quantity q produced in a bilateral match is determined according to the Kalai (1977) bargaining solution that gives a constant fraction of the match surplus to the buyer.[13] Formally,

$$u(q_t) + V_{0,t} - V_{1,t} = \theta \left[u(q_t) - q_t\right], \tag{7}$$

where $\theta \in [0, 1]$ is the buyer's share. Solving for the value of money, $V_{1,t} - V_{0,t}$, we obtain:

$$V_{1,t} - V_{0,t} = \omega(q_t) \equiv (1 - \theta)u(q_t) + \theta q_t. \tag{8}$$

Substituting $V_{1,t} - V_{0,t}$ by its expression given by (8) into (5) and (6) leads to:

$$rV_{1,t} = \alpha\sigma \left(1 - A_t\right) \chi_t \theta \left[u(q_t) - q_t\right] + \dot{V}_{1,t} \tag{9}$$

$$rV_{0,t} = \alpha\sigma A_t(1 - \theta) \left[u(q_t) - q_t\right] + \max_{e \in \mathcal{E}} \left\{\Lambda(A_t, m_t)e\omega(q_t) - C(e)\right\} + \dot{V}_{0,t}. \tag{10}$$

Upon trading, the surplus of the buyer is $\theta \left[u(q) - q\right]$ while the surplus of the seller is $(1 - \theta) \left[u(q) - q\right]$. If rewarded a unit of money, the gain of the miner is $\omega(q)$. Subtracting (10) from (9) and making use of (8)

[13] We use Kalai bargaining instead of Nash because of its simplicity. For the merits of this solution, see Aruoba et al. (2007).

we obtain that q is the solution to the following ODE:

$$r\omega(q_t) = \alpha\sigma[(1-A)\,\chi_t\theta - A_t(1-\theta)]\,[u(q_t) - q_t] - \max_{e\in\mathcal{E}}\{\Lambda(A_t, m_t)e\omega(q_t) - C(e)\} + \omega'(q_t)\dot{q}_t. \tag{11}$$

Optimal mining choice From (11) an optimal mining intensity is

$$e^* \in \mathbb{E}^*_t \equiv \arg\max_{e\in\mathcal{E}}\{\Lambda(A_t, m_t)e\omega(q) - C(e)\}. \tag{12}$$

Let $\underline{e}^*$ ($\overline{e}^*$) be the lowest (highest) element in $\mathbb{E}^*$. Then, allowing for asymmetric choices, aggregate mining intensity is

$$m_t \in [(1-A_t)\underline{e}^*_t, (1-A_t)\overline{e}^*_t]. \tag{13}$$

There is a measure $1-A_t$ of agents without money who choose their mining effort in $\mathbb{E}^*$.

The law of motion for the supply of money in circulation in the economy is:

$$\dot{A} = m\Lambda(A, m). \tag{14}$$

Given the aggregate mining intensity, m, money creation is $m\Lambda(A, m)$. We now define an equilibrium.

Definition 1 *An equilibrium is a pair of value functions, $V_{0,t}$ and $V_{1,t}$, the quantity traded in each match, q_t, the aggregate mining intensity, m_t, and the quantity of money in circulation, A_t, that solve: (9), (10), (11), (13), (14), and the initial condition A_0.*

Below we characterize the set of equilibria for different classes of mining technologies and cost functions.

3 Gold mining

We first adopt the mining technology in (2), $\Lambda(A, q) = \lambda(\bar{A} - A)$. This technology has the key feature that the congestion from mining occurs only indirectly through A. In addition, mining is an occupation choice, $e \in \{0, 1\}$, and the mining cost is an opportunity cost equal to $C(1) = \alpha\sigma A(-q + V_1 - V_0)$.[14] An agent who mines gives up the opportunities to produce, but agents can move freely between the production and mining sectors. The chance that an agent without money chosen at random in the population is able to produce, i.e., she is a producer rather than a miner, is

$$\chi = \frac{1 - A - m}{1 - A}.$$

[14] There is plenty of evidence to justify that money mining has an endogenous opportunity cost by diverting input factors from alternative productive uses. The California Gold Rush (1848–1855) is a case in point. The Gold Rush tripled the population in California by bringing approximately 300,000 people from the rest of the world (see Britannica). South Africa offers another example where gold mining had a large impact on the allocation of workers across sectors of the economy (Gilbert, 1933).

Occupation choice

The net instantaneous gain from being a miner rather than a producer is $\Delta(q,A) \equiv \lambda\left(\bar{A}-A\right)\omega(q)-C(1)$, i.e.,

$$\Delta(q,A) \equiv \lambda\left(\bar{A}-A\right)\omega(q)-\alpha\sigma A(1-\theta)\left[u(q)-q\right]. \tag{15}$$

From (10) or (11) the measure of miners is given by:

$$m \begin{array}{l} =1-A \\ \in[0,1-A] \\ =0 \end{array} \quad \text{if} \quad \Delta(q,A) \begin{array}{c} > \\ = \\ < \end{array} 0. \tag{16}$$

By (15) the indifference condition, $\Delta(q,A)=0$, can be rewritten as:

$$A=\mu(q) \equiv \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1-\theta)\left[u(q)-q\right]+\lambda\omega(q)}. \tag{17}$$

Since $\omega(q)/\left[u(q)-q\right]$ increases in q by the concavity of $u(q)$, so does $\mu(q)$. Therefore, as A increases, so must q for agents to be indifferent across occupations.

3.1 Steady states

We first describe steady-state equilibria where q and A are constant over time and $m=0$. We focus on the steady state with the lowest A as it is the one that will be reached from the initial condition $A_0=0$. By (11):

$$r\omega(q)=\alpha\sigma\left(\theta-A\right)\left[u(q)-q\right]. \tag{18}$$

Substituting $\omega(q)$ by its expression given by (8) and rearranging,

$$rq=\left\{\alpha\sigma(\theta-A)-r(1-\theta)\right\}\left[u(q)-q\right]. \tag{19}$$

There is a unique $q>0$ solution to (19) provided that $r<\alpha\sigma(\theta-A)/(1-\theta)$. Hence, a necessary (but not sufficient) condition for a monetary equilibrium to exist is $\theta>A$. Moreover, $\partial q/\partial A<0$, i.e., an increase in the money supply reduces the purchasing power of money.

The condition for $m=0$, $\Delta(q,A)\leq 0$, holds if $A\geq\mu(q)$, which from (17) and (18) can be reexpressed as

$$rA(1-\theta)\geq\lambda\left(\bar{A}-A\right)\left(\theta-A\right). \tag{20}$$

We represent inequality (20) in Figure 5. The left side is linear in A while the right side is quadratic with two roots, $A=\bar{A}$ and $A=\theta$. They intersect for two values, $A_1<\min\{\bar{A},\theta\}$ and $A_2>\max\{\bar{A},\theta\}$. The left side is located above the right side for all $A\in(A_1,A_2)$. Since A cannot be greater than θ for a monetary

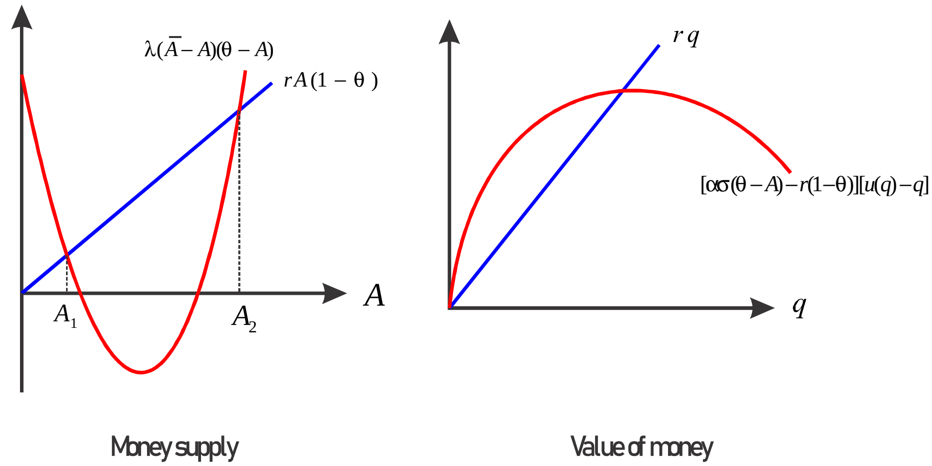

Figure 5: Steady states.

equilibrium to exist, we must have $A < \min\{\bar{A}, \theta\}$. So a steady-state monetary equilibrium exists for all A in the half-closed interval $\big[A_1, \min\{\bar{A}, \theta\}\big)$. In the following we focus on the steady state $A^s = A_1$.

The steady-state equilibrium is determined recursively. First, A^s is obtained as the smallest solution to (20). Given $A = A^s$, q^s exists if and only if $r < \alpha\sigma(\theta - A^s)/(1-\theta)$ by (19) or, equivalently,

$$A^s < \theta - \frac{r(1-\theta)}{\alpha\sigma}. \tag{21}$$

Figure 5 provides a graphical representation of the determination of the steady-state equilibrium.

Proposition 1 ***(Steady-state monetary equilibria)*** *There exists a unique steady-state monetary equilibrium (where* $\Delta(q, A) = 0$*) if and only if*

$$r < \frac{\alpha\sigma\theta}{1-\theta}\left[1 - \frac{\lambda\bar{A}}{\theta\left(\alpha\sigma + \lambda\right)}\right] \tag{22}$$

where the steady-state money supply is

$$A^s = \frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda} - \sqrt{\left(\frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda}\right)^2 - \bar{A}\theta}. \tag{23}$$

Comparative statics are summarized in the following table:

	$\partial\lambda$	$\partial\bar{A}$	$\partial\theta$	∂r	$\partial(\alpha\sigma)$
$\partial A^s/$	$+$	$+$	$+$	$-$	0
$\partial q^s/$	$-$	$-$	$\pm$	$-$	$+$
$corr(q^s, A^s)$	$-$	$-$	$\pm$	$+$	0

Table 1: Comparative statics.

Under which conditions can fiat money be privately produced and maintain a positive value in the long run? According to (22), the existence of a monetary equilibrium requires agents to be sufficiently patient — a standard "folk theorem" in monetary theory. The first term on the right side of (22), $\alpha\sigma\theta/(1-\theta)$, is the threshold for r below which money is valued in the Shi-Trejos-Wright model with a fixed supply of money. The second term between squared brackets is new and captures the effect of the private production of money on the existence of a monetary equilibrium. Because this term is less than one, the private production of money makes it harder to sustain an equilibrium with a valued fiat money. Higher $\bar{A}$ or λ are associated with higher incentives to mine, and hence a higher money supply at the steady state. A higher A^s reduces buyers' trading opportunities, thereby making it harder for money to be valued.

Comparative statics in Table 1 have implications for the correlation between the endogenous money supply and price level. The sign of this correlation (the bottom row in Table 1) depends on which fundamentals drive the movements of A^s and q^s. If λ or $\bar{A}$ rises, then A^s rises and q^s falls. Then there is a positive correlation between money supply and price level $(1/q^s)$, which is consistent with the quantity theory. However, if r rises, then both A^s and q^s fall. There is now a negative correlation between the money supply and the price level. A change in θ generates a non-monotone relation between A^s and q^s: numerical examples show that for low θ, there is a positive correlation between A^s and q^s while for high θ there is a negative correlation. Finally, an increase in α or σ does not affect incentives to mine and the money supply but it raises q^s.

3.2 Dynamics

We now turn to transitional dynamics from an arbitrary initial condition, A_0. Without loss of generality, we set $A_0 = 0$ because the equilibrium is time consistent, i.e., if the equilibrium reaches A_t at time t from $A_0 = 0$, then the path onward is the same as the one obtained from the initial condition A_t.

From (11), $C(1) = \alpha\sigma A(-q+V_1-V_0)$, and (14), (q, A, m) solve the following system of differential equations:

$$\omega'(q)\dot{q} = \left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] \quad (24)$$

$$\dot{A} = m\lambda\left(\bar{A} - A\right) \quad (25)$$

$$m \leq 1 - A \quad \text{“}=\text{” if } \Delta(q, A) > 0. \quad (26)$$

Equation (24) is an asset pricing equation for the value of money. The first term on the right side is the appreciation of the value of money over time if it does not provide transactional services: ω grows at rate $r + \lambda\left(\bar{A} - A\right)$, which compensates the buyer for her rate of time preference and the foregone opportunities of mining. The second term on the right side corresponds to the liquidity services that money provides to a buyer as measured by the expected surplus from a trade. These liquidity services constitute a non-pecuniary return that reduces the appreciation rate of money. Equation (25) is the law of motion of the stock of money. Equation (26) is the optimality condition for the occupation choice between producing or mining.

We distinguish two regimes. In the first regime, all agents without money (the potential producers) engage in mining, namely $m = 1 - A$. Then by (24) and (25):

$$\dot{q} = \left[r + \lambda\left(\bar{A} - A\right)\right]\frac{\omega(q)}{\omega'(q)} \tag{27}$$

$$\dot{A} = (1-A)\lambda\left(\bar{A} - A\right). \tag{28}$$

By (27) the value of money, ω, grows at a rate larger than r because it provides no liquidity services yet. Along the equilibrium path, by (27) and (28), the relation between q and A is given by

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} = \frac{\omega(q)[r + \lambda\left(\bar{A} - A\right)]}{\omega'(q)(1-A)\lambda\left(\bar{A} - A\right)}. \tag{29}$$

Since the right side is positive, the path is upward sloping in the (A, q) space.

Consider next the regime where miners and producers coexist, $m \in (0, 1-A)$. In that case $A = \mu(q)$ and by (25) the measure of miners is

$$m = \frac{\mu'(q)}{\lambda\left[\bar{A} - \mu(q)\right]}\dot{q}. \tag{30}$$

The measure of miners increases with the capital gain $\dot{q}$. The next proposition characterizes the unique dynamic equilibrium that converges to (q^s, A^s).

Proposition 2 ***(Transitional Dynamics to Steady-State Monetary Equilibrium)*** *Suppose (22) holds and $A_0 = 0$. There exists a unique monetary equilibrium such that (q_t, A_t) converges to $(q^s, A^s) > 0$. Along the equilibrium path q_t and A_t increase over time. Moreover:*

1. *There exists $t_0 > 0$, such that for all $t < t_0$, $m_t = 1 - A_t$, and*

$$A_t = \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \tag{31}$$

$$\omega(q_t) = e^{rt}\omega_0\left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1-\bar{A}}\right]. \tag{32}$$

2. *If*

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}, \tag{33}$$

then $m_t < 1 - A_t$ in the neighborhood of the steady state and convergence to (q^s, A^s) is asymptotic. Otherwise, $m_t = 1 - A_t$ until the steady state is reached in finite time.

Proposition 2 proves the existence and uniqueness of a dynamic equilibrium leading to (q^s, A^s) starting from an initial condition $A_0 = 0$. It allows us to study how the supply of privately-produced money and its price covary over time. The equilibrium features monotone trajectories for q_t and A_t. As the money supply increases, the price level falls, and quantities traded in pairwise meetings increase.[15] This result seems in contradiction with the quantity theory according to which the price level increases with the money supply and the long-run comparative statics in Table 1 where an increase in $\bar{A}$ reduces q. Intuitively, the value of money must appreciate over time in order to induce agents to mine because as A increases the mining speed $\lambda(\bar{A} - A)$ falls but the frequency $\alpha\sigma A$ of trading opportunities in the production sector rises.

Proposition 2 also answers the question: can money be valued if it does not serve as a medium of exchange? Early on, when A is close to 0, all agents without money choose to be miners and all agents with money hoard it because they have no opportunity to use it as a medium of exchange. From the viewpoint of an outside observer, money resembles a pure speculative bubble: it does not play any role in exchange, and hence it should not have any liquidity premium, but its value grows at a rate larger than r. This path for the value of money is sustainable because in finite time money starts being used as a medium of exchange.

Can a government prevent the emergence of a private money? The government can discourage money mining by supplying $A_0 > A^s$. If A_0 is sufficiently large, then the gains from being a producer exceeds that from mining. As shown in Table 1, the larger λ and $\bar{A}$, the larger A_0 has to be to prevent money mining.

Finally, we showcase the tractability of the model by solving the equilibrium path in closed form in (31)-(32). This result follows from the observation that the law of motion for A, (25), when $m = 1 - A$, is a Riccati equation that admits an analytical solution (see Section 2.15 in Ince (1956) for details).

The equilibrium in the neighborhood of the steady state can take two forms as illustrated in Figure 6. There are equilibria where miners and producers coexist. In this case, the steady state is only reached asymptotically. There is another type of equilibrium where all agents without money strictly prefer mining until the steady-state money supply is reached, which occurs in finite time. These regimes have implications for the transaction velocity of money measured by $\mathcal{V}_t \equiv \alpha\sigma(1 - A_t - m_t)$. Early on $\mathcal{V}_t = 0$ since all potential

[15] As shown in Online Appendix C, we can obtain less dramatic results with alternative matching functions, i.e., agents trade at all dates, but the insight that market tightness measured by the ratio of producers to buyers increases over time is robust.

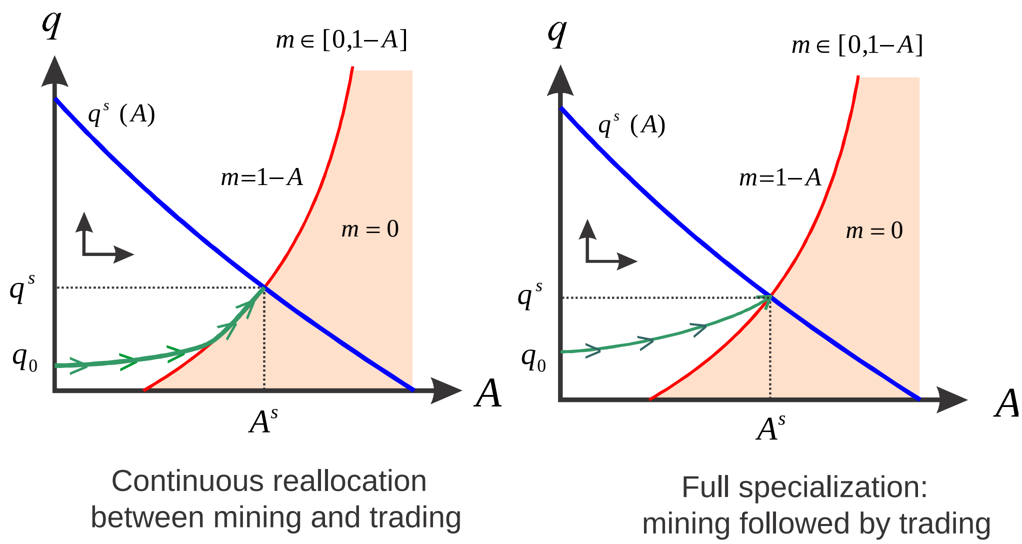

Figure 6: Dynamic equilibria with mining.

producers prefer to mine. If (33) holds, then $\mathcal{V}_t > 0$ for some $t>0$ and it rises as m_t falls toward its steady-state value. If (33) does not hold, then $\mathcal{V}_t = 0$ until the steady state is reached, at which point $\mathcal{V} = \alpha\sigma\,(1 - A^s)$. Along the equilibrium path, the velocity, price and supply of mone are positively correlated. The next lemma provides conditions for mining and production to coexist along the equilibrium path.

Lemma 1 ***(Coexistence of trades and mining)***

1. *If $\epsilon(q) \equiv u'(q)q/u(q)$ is non-increasing in q, then there exists $\lambda^* < +\infty$ and $\kappa^* \in (0, +\infty)$ such that $m_t < 1 - A_t$ in the neighborhood of (A^s, q^s) if and only if $\lambda < \lambda^*$ or $\sigma\alpha \geq \kappa^*$. Moreover, if $\alpha\sigma > \lambda\theta/(1-\theta)$, then the equilibrium features at most one regime switch.*

2. *If $\theta \leq 1/2$ and λ is sufficiently large, then $m_t = 1 - A_t$ for all t such that $A_t < A^s$.*

The condition on the elasticity of $u(q)$ in Lemma 1 is satisfied by $u(q) = q^{1-a}$ or $u(q) = 1 - e^{-aq}$. Part 1 of Lemma 1 establishes that if the efficiency of mining is low and the matching rate is high, then mining and trades coexist near the steady state. Part 2 provides a global characterization of the occupation choice. If the mining technology is sufficiently efficient and producers have more bargaining power than buyers, then no trade takes place until the supply of money reaches its steady-state level.

The next proposition addresses the question of the determination of the initial value of money by characterizing the set of all initial values of a new currency that are consistent with a monetary equilibrium.

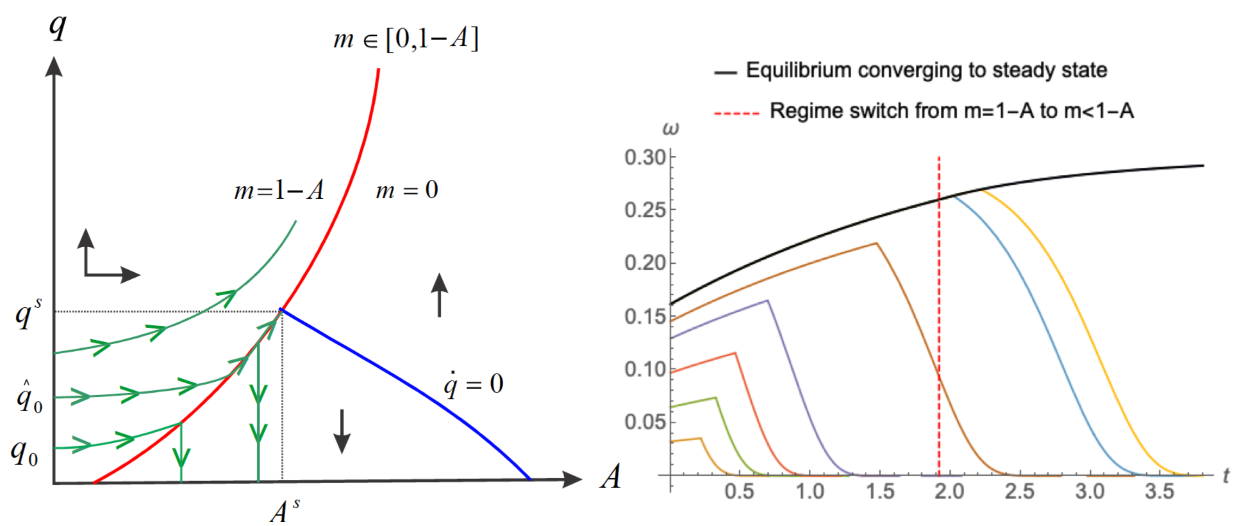

Figure 7: (Left) Phase diagram of different equilibria (Right) The value of money under different equilibria for the same parameters.

Money can be valued and privately produced even it is anticipated that it will be worthless in the long run. Below we use $(\hat{A}_t, \hat{q}_t)$ to denote the unique equilibrium that leads to the monetary steady state (A^s, q^s).

Proposition 3 ***(Boom/Bust equilibria)***

1. *For all $q_0 \in (0, \hat{q}_0)$, there exist $0 < T_0 \le T_1 < +\infty$ such that a monetary equilibrium exists with the following properties:*

   (a) Boom phase: For all $t \le T_0$, $m_t = 1 - A_t$ and $\dot{\omega}/\omega = r + \lambda \left(\bar{A} - A\right) > 0$.

   (b) Bust phase: For $t > T_1$, $m_t = 0$, $\dot{\omega}_t = r\omega_t - \alpha\sigma \left(\theta - A_{T_1}\right) \left[u(q_t) - q_t\right] < 0$, and $\lim_{t \to +\infty} \omega_t = 0$.

2. *If (33) holds, then there is a continuum of monetary equilibria indexed by $T \in \{t \in \mathbb{R}_+ : \hat{A}_t = \mu(\hat{q}_t)\}$ such that $q_0 = \hat{q}_0$ and:*

   (a) Boom phase: For all $t \le T$, $(A_t, q_t) = (\hat{A}_t, \hat{q}_t)$.

   (b) Bust phase: For $t > T$, $m_t = 0$, $A_t = \hat{A}_T$, $\dot{\omega}_t = r\omega_t - \alpha\sigma \left(\theta - \hat{A}_T\right) \left[u(q_t) - q_t\right] < 0$, and $\lim_{t \to +\infty} \omega_t = 0$.

There is a continuum of monetary equilibria featuring a boom and a bust of the currency price. Those equilibria are indexed by the initial value of money in the interval $(0, \hat{q}_0)$. If the initial beliefs are not optimistic enough to bootstrap the value of money to $\hat{q}_0$, then a boom/bust equilibrium exists.[16] Along the

[16] Such equilibria capture the idea that new currencies might be likely to fail in the absence of coordination mechanisms. See Selgin (1994) for historical examples.

equilibrium path the value of money first increases at a rate larger than r. It reaches a maximum at which point agents stop mining. Even though the money supply remains constant afterwards, the value of money declines and converges to 0 asymptotically. In the phase diagram of Figure 7, the equilibrium path is upward sloping until it reaches the locus $A = \mu(q)$. At that point it becomes vertical since the money supply remains constant with arrows of motion oriented toward the horizontal axis as money loses its value over time.

There can also be boom/bust equilibria where $q_0 = \hat{q}_0$. Such equilibria occur when agents are indifferent between mining or producing in the neighborhood of the steady state, i.e., (33) holds, and they are indexed by the time T at which the value of money starts falling. Such an equilibrium path is represented in the left panel of Figure 7 by a trajectory starting at $q_0 = \hat{q}_0$. The trajectory is upward sloping and follows the $A = \mu(q)$ locus for a while until it becomes vertical and falls toward the horizontal axis. From the viewpoint of an outside observer, it would be impossible to tell whether the currency will be successful until the time T at which the value of currency starts declining.

Models with a fixed supply of fiat money also feature a continuum of deterministic monetary equilibria, see, for example, Trejos and Wright (1995), Coles and Wright (1998), and more recently He and Wright (2018). There is typically a unique steady-state monetary equilibrium (there can be multiple steady states in some versions with barter trades) and a continuum of equilibria where the value of money declines over time and vanishes asymptotically. Those equilibria generate outcomes that are analogous to the bust phase of our equilibria with privately-produced monies. The boom phase is new and illustrates how the dynamics of the money supply and its price are intertwined.

Given the existence of a continuum of equilibria, is it possible to refine the equilibrium set and focus on a single one? In order to answer this question, we consider the equilibrium set of a similar economy where money is endowed with a commodity value, $d > 0$, e.g., a utility flow from a commodity money or a real interest payment, and we take the limit as d goes to 0. (The value functions and equilibrium conditions are detailed in Online Appendix D.) This selection method is sometimes referred to as the commodity-money refinement (e.g., Wallace and Zhu, 2004, or Garratt and Wallace, 2018). If money creates a flow dividend $d > 0$, then in any equilibrium the value of money ω_t is bounded below by the discounted sum of dividends, d/r, at all t. This rules out the continuum of boom-and-bust equilibria where the value of money vanishes asymptotically. Hence, if money pays an arbitrarily small interest $d > 0$, there exists a unique equilibrium and it is such that $(A_t, q_t) \to (A^s, q^s)$ as $t \to \infty$.

3.3 Price waves

Historically, the world supplies of silver and gold have increased through sequential discoveries of new mining sites, e.g., South America during the 16-17th centuries, South Africa, and Australia during the 19-20th centuries. In the context of crypto-currencies, one can interpret mine discoveries as an unexpected increase in the potential supply of a currency or the introduction of new currencies.[17]

To capture mine discoveries and their impact on price dynamics, we describe a sequence of unanticipated shocks on $\bar{A}$ starting from a steady state. Initially, $\bar{A}=\bar{A}_0$ and the economy is at a stationary equilibrium (q_0^s, A_0^s). At time 0, the maximum amount of money agents can mine, $\bar{A}$, rises from $\bar{A}_0$ to $\bar{A}_1$. This could correspond to a new estimate of the gold resources of the planet. After the economy reaches a new steady state, (q_1^s, A_1^s), another discovery happens that raises the potential money supply from $\bar{A}_1$ to $\bar{A}_2$. And so on.

In the phase diagram of the left panel of Figure 8, the locus $A = \mu(q)$ shifts to the right. The new steady state is such that the money supply increases, $A_1^s > A_0^s$, and money loses some value, $q_1^s < q_0^s$. At time 0^+, q falls below q_1^s so that the value of money overshoots its steady state. Along the transition to the new steady state the value of money increases. The sequence of unanticipated increases in $\bar{A}$ generates fluctuations in the value of money around a downward trend. The impact of an unanticipated increase in the mining intensity λ is similar to that of an increase in $\bar{A}$: the value of money falls on impact and then rises to reach a new steady state with a lower q and higher A.

We now compare the dynamics of our model where the role of money is endogenous to the dynamics of a commodity price (e.g., minerals) if the commodity is durable and produced slowly through time but it does not serve as a medium of exchange. Suppose the commodity generates a flow of marginal utility, $\vartheta(A)$, to its holder, where A is the supply of the commodity and $\vartheta'(A) < 0$. The value of this commodity, ω, obeys the following HJB equation:

$$r\omega = \vartheta(A) + \dot{\omega}. \tag{34}$$

The flow value from holding the commodity is composed of its marginal utility and the capital gain (or loss) as the value of the commodity varies over time. We are agnostic as to the exact functional form of the mining technology and simply assume $\dot{A} > 0$ for all $A < \bar{A}$ and $\dot{A} = 0$ otherwise. The supply of the commodity grows continuously until it reaches a maximum potential supply, $\bar{A}$. The steady-state value of

[17] Another interpretation is the "forking" of an existing crypto-currency into an old and new one, e.g., the fork between Bitcoin and Bitcoin Gold. But the old and new currency are often imperfect substitutes, as their prices might not comove.

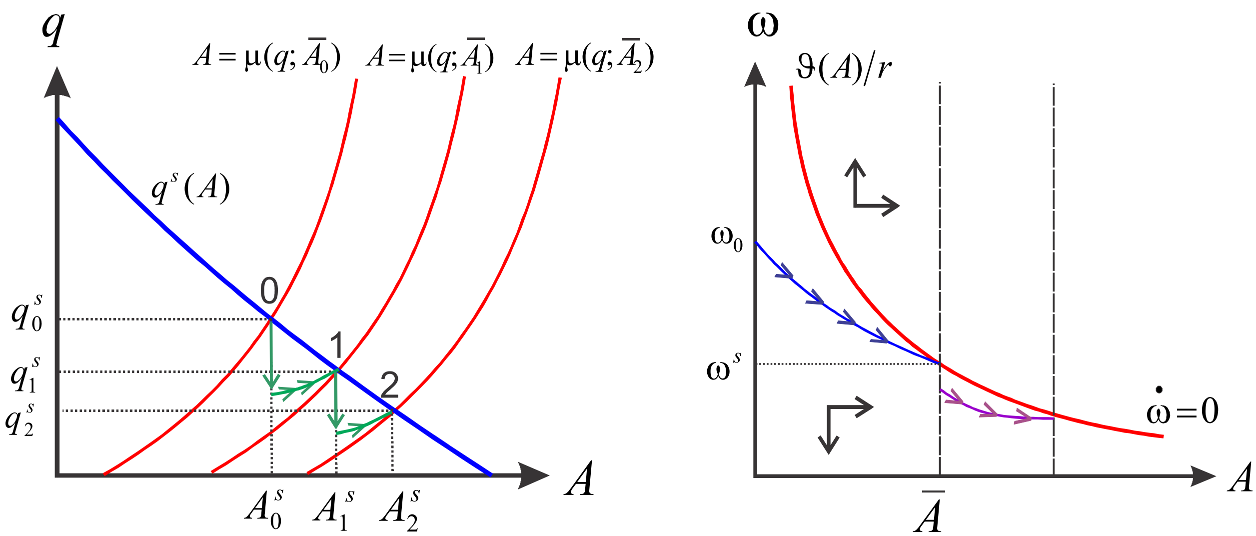

Figure 8: (Left) Mine discoveries: Unanticipated increases of $\bar{A}$. (Right) Dynamics of commodity prices

the commodity is $\omega^s = \vartheta(\bar{A})/r$. Starting from some initial condition $A_0 = 0$, the value of the commodity decreases over time, $\dot{\omega} < 0$, as illustrated in the phase diagram in the right panel of Figure 8. Now suppose that the economy is at a steady state and $\bar{A}$ increases. Graphically, the vertical dashed line moves to the right. In that case, the price of the commodity jumps downward and keeps falling afterwards until it reaches its new steady state.

So why are the dynamics of a commodity price different from the dynamics of the price of money? The answer has to do with the endogenous role of money as a medium of exchange. One might think that the non-pecuniary services that money provides to its holder decrease with its stock, so that there exists an indirect utility function of the form $\vartheta(A)$, which would make the dynamics of the value of money analogous to (34). The analogy fails in this version of the model for two reasons. First, the surplus that the money holder obtains in a trade match depends on the real value of money and not its nominal stock. In other words, q depends on $\omega = V_1 - V_0$ but not A. A higher A reduces the matching probability of a buyer, so that for given q the expected surplus of a money holder decreases with A. But this congestion effect alone does not dictate the dynamics of the value of money. Second, and importantly, the use of money as a medium of exchange and its velocity, $\alpha\sigma\chi_t A_t(1 - A_t)$, are endogenous and depend on χ_t. As A_t increases over time, money becomes more widely held and, as a result, potential producers are more likely to meet buyers with a positive payment capacity. Hence, potential producers have more incentives to participate in the market for goods and services, i.e., χ_t is weakly higher. Since χ_t rises, buyers have more opportunities to spend money and thus the value of money can rise over time.

3.4 Efficient gold mining

We now ask whether the decentralized private production of money can generate a socially efficient outcome. We describe the problem of a social planner who is subject to both the mining technology and the matching technology between money holders and producers. Implicit in the latter constraint is the requirement that all trades take the form of one unit of money for some q, i.e., trades are quid pro quo. The planner chooses agents' occupation and output in pairwise meetings to maximize the discounted sum of all agents' utilities (The planner's problem is defined explicitly in Lemma 2 in the Appendix). We then provide an incentive-feasible mechanism to implement such constrained-efficient allocations. In the following recall that q^* is the solution to $u'(q^*) = 1$ and it is the efficient level of production in a trade meeting.

Proposition 4 ***(Constrained-efficient allocation)*** *Assume* $A_0 = 0$.

1 Efficient allocations are such that $q_t = q^*$ *for all* t *and* $m_t = 1 - A_t$ *for all* $t < T^*$ *where* $T^* > 0$ *is the time it takes to mine* A^* *where*

$$A^* = \frac{1}{4}\left[\left(2\bar{A} + 1 + \frac{r}{\lambda}\right) - \sqrt{\left(2\bar{A} + 1 + \frac{r}{\lambda}\right)^2 - 8\bar{A}}\right]. \tag{35}$$

For all $t \geq T^*$, $m_t = 0$ *and* $A_t = A^*$.

2 Implementation. If

$$r \leq \frac{\alpha\sigma(1 - A^*)\left[u(q^*) - q^*\right]}{q^*} \text{ and} \tag{36}$$

$$\lambda\left(\bar{A} - A^*\right) \leq \frac{\alpha\sigma A^*(1 - A^*)\left[u(q^*) - q^*\right] - rq^*}{(1 - A^*)u(q^*) + A^* q^*}, \tag{37}$$

then the constrained-efficient allocation is implementable with

$$\begin{aligned} \theta_t &= 1 \text{ if } t < T^* \\ &= \theta^* \equiv \frac{rq^* + (\alpha\sigma A^* + r)\left[u(q^*) - q^*\right]}{\left[u(q^*) - q^*\right](\alpha\sigma + r)} \text{ if } t \geq T^*. \end{aligned} \tag{38}$$

The planner chooses q^* in all trade matches and it assigns all non-asset holders to mining until the efficient quantity of money A^* has been produced. Intuitively, it is more efficient to assign agents to the production sector when the chance of forming trade matches is higher. Since the chance that a non-asset holder matches with a trading partner rises in A, the planner assigns agents to the production sector only after A^* is reached. We show in the proof of Proposition 4 that along the optimal path the shadow value of money ξ (i.e. the co-state variable associated with A_t) satisfies

$$\frac{\dot{\xi}}{\xi} = r + \lambda(1 + \bar{A} - 2A).$$

If we compare with the equilibrium ODE, (24), when $m = 1 - A$,

$$\frac{\dot{\omega}}{\omega} = r + \lambda\left(\bar{A} - A\right),$$

we see that the rate of growth of ξ is larger than the rate of growth of ω by a term equal to $\lambda(1 - A)$. According to this additional term, the planner internalizes the fact that as more money is taken out of the ground, it becomes harder for future miners to find new units of money. The optimal quantity of money, A^*, is less than $1/2$, which is the quantity that would maximize the measure of trades. As agents become infinitely patient, $\lim_{r \to 0} A^* = \min\{1/2, \bar{A}\}$. By comparing (35) and (23) we obtain that $A^s > A^*$ if $\theta > 1/2$ and $A^s < A^*$ if $\theta < 1/2$. There is over-production of money in equilibrium if buyers get more than half of the trade surplus. Even if $\theta = 1/2$ so that $A^s = A^*$, the number of trades is constrained-efficient but the equilibrium output in trade matches might differ from q^*.

In the second part of Proposition 4, we propose an incentive-feasible trading mechanism that implements the constrained-efficient allocation. The mechanism is incentive feasible if it satisfies the individual rationality constraints of the buyer, $u(q) + V_0 - V_1 \geq 0$, and the producer, $-q + V_1 - V_0 \geq 0$, in a pairwise meeting. Any incentive-feasible trading mechanism is described by a sequence of time-varying bargaining shares, θ_t. By (38) an incentive-feasible trading mechanism that implements the constrained-efficient allocation is such that buyers have all the bargaining power until the efficient quantity of money, A^*, has been dug at time T^*. Giving no bargaining power to producers initially guarantees that agents without money choose to be miners rather than producers. Following T^* the buyer's bargaining power is $\theta^* > 0$, which is the value that implements q^* in all pairwise meetings.

Condition (36) is a standard implementation condition of the first best in monetary search models (see, e.g., Wright 1999). It requires the opportunity cost of holding money, as measured by rq^*, to be smaller than the expected surplus from holding money when the buyer has all the bargaining power, $\alpha\sigma(1 - A^*)\left[u(q^*) - q^*\right]$. A key difference from the existing literature is that the money supply here is endogenous and depends on fundamentals. As r vanishes, A^* tends to $\min\{\bar{A}, 1/2\}$. Hence (36) is satisfied for r sufficiently small.

Condition (37) is new and guarantees that agents have no incentive to over-produce money. Assuming that the buyer's bargaining share is θ^*, it requires that the expected gain from mining, $\lambda \left(\bar{A} - A^*\right) \omega^*$ where $\omega^* \equiv (1-\theta^*)u(q^*) + \theta^* q^*$, is smaller than the expected gain from being a producer $\alpha\sigma(1-\theta^*)\left[u(q^*) - q^*\right]$. If $\bar{A} < 1/2$, then this condition holds for r sufficiently close to 0.

4 Crypto mining

A characteristic of the gold mining technology described in Section 3 is that the more miners, the more money is created or discovered. In contrast, some crypto-currencies (e.g., Bitcoin) are designed such that the aggregate rate of money creation does not vary with the measure of miners. The designer of the currency chooses a path for the money supply, $\dot{A}_t = \pi(A_t)A_t$, where $\pi(A_t)$ is the state-contingent rate of money creation, which is independent of the measure of miners. Each unit of newly created money is allocated to a miner with a probability proportional to their mining effort, as described by (3). We will study the implications of this mining technology for currency price dynamics under alternative cost functions and compare those dynamics to the ones obtained in Section 3.

4.1 Variable mining intensity

Suppose that every agent without money chooses a mining intensity, $e \in \mathbb{R}_+$, at cost $C(e) = ek$. In this version of the model, mining and producing are not mutually exclusive, hence $\chi_t = 1$. By the first-order condition of (12) where $\Lambda(A_t, m_t) = \pi(A_t)A_t/m_t$, the aggregate mining intensity is

$$m_t = \frac{\pi(A_t)A_t}{k}\omega_t. \tag{39}$$

It is the real value of money creation divided by the unit cost of mining. We focus on symmetric equilibria where all miners choose the same e. Aggregate money supply evolves according to

$$\dot{A} = \pi(A_t)A_t. \tag{40}$$

From (9)-(10), the value of money solves

$$r\omega_t = \alpha\sigma\left(\theta - A_t\right)S(\omega_t) + \dot{\omega}_t, \tag{41}$$

where $S(\omega) \equiv u[q(\omega)] - q(\omega)$ denotes the match surplus as a function of the value of money, where q is a function of ω through (8). An equilibrium is a list, $\{m_t, A_t, \omega_t\}$, that solves (39)-(41) and A_0 given. It can be solved recursively as follows. Equation (40) together with the initial condition A_0 gives A_t. Given A_t, ω_t can be solved by (41). Given $\{A_t, \omega_t\}$, the time-path for the aggregate mining effort is given by (39).[18]

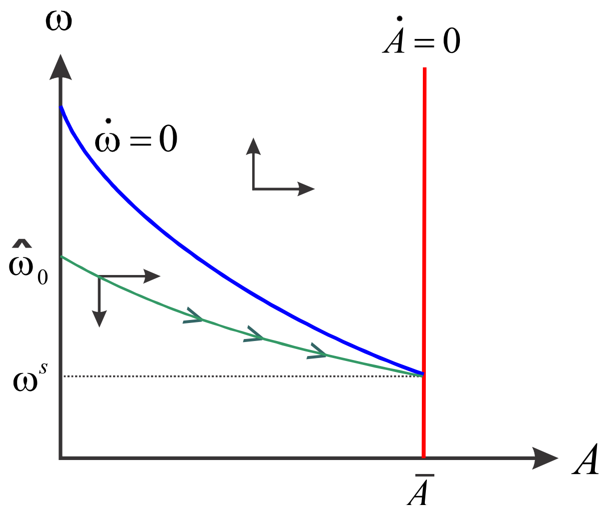

Figure 9: Phase diagram for crypto mining under variable mining intensity.

In Figure 12 we represent the phase diagram associated with (40)-(41) under the assumption that $\pi(A_t) > 0$ for all $A_t < \bar{A}$ and $\pi(A_t) = 0$ for all $A_t \geq \bar{A}$. For instance, such assumptions are satisfied for the Bitcoin money growth rate, $\pi(A) = \lambda(\bar{A} - A)/A$. The ω-isocline is downward-sloping and $\omega = 0$ for all $A > \theta - r(1-\theta)/\alpha\sigma$. It has a strictly positive intercept if $r < \alpha\sigma\theta S'(0) = \alpha\sigma\theta/(1-\theta)$ by (41). By the same logic as in Section 3, we obtain the following proposition:

Proposition 5 ***(Crypto mining with variable intensity)*** *Suppose the mining technology is given by (3) where $\pi(A_t) > 0$ for all $A_t < \bar{A}$, $\pi(A_t) = 0$ for all $A_t \geq \bar{A}$, and $\pi'(\bar{A}) < 0$. Moreover, $C(e) = ek$ for all $e \in \mathbb{R}_+$. There exists a steady-state monetary equilibrium, $\left(\bar{A}, \omega^s\right)$, if and only if*

$$\bar{A} < \frac{\alpha\sigma\theta - r(1-\theta)}{\alpha\sigma}, \tag{42}$$

and, if it exists, it is unique. Given $A_0 = 0$, there is a unique equilibrium leading to $\left(\bar{A}, \omega^s\right)$ where $\omega_0 = \hat{\omega}_0 > 0$ and $\dot{\omega} < 0$. There is also a continuum of equilibria indexed by $\omega_0 \in (0, \hat{\omega}_0)$ such that $\dot{\omega} < 0$ and $\lim_{t\to\infty} \omega_t = 0$.

In contrast to the gold mining model in Section 3, here the value of money declines over time in all monetary equilibria. There are two key differences that explain this result. First, the path for the money

[18] The dichotomy between $\{A_t, \omega_t\}$ and m_t can be broken, e.g., by assuming that m_t facilitates the coordination on a monetary equilibrium (Pagnotta, 2018), or by assuming the cost of mining depends on m (as in Section 4.4).

supply is determined independently from the mining activity. Second, the cost of mining does not depend on the state of the economy, including the money supply and value of money. As a result, as the money supply rises, the liquidity value of money falls, since the buyer's matching probability falls, which reduces the currency price. In the rest of this section, we will show how small changes to this environment can generate dynamics where the currency price rises over time or is non-monotone (rises first and then falls).

4.2 Endogenous acceptability

Following Lester et al. (2012), we assume that in order to accept a new currency a seller must incur a flow cost $\psi > 0$. The variable χ_t now represents the fraction of agents without money (sellers) who incur that cost and accept the new currency. The cost of accepting money has several interpretations: the cost to authenticate a new money, the cost to get informed about the characteristics of this money (supply, security protocols), to acquire the technology to receive it in payment, and so on.

Since in equilibrium mining creates zero expected profit, the HJB equation of an agent without money is:

$$rV_{0,t} = \max\left\{-\psi + \alpha\sigma A_t(1-\theta)S(\omega_t), 0\right\} + \dot{V}_{0,t}. \tag{43}$$

According to the first term on the right side, an agent without money enjoys the gains from trading with money holders by incurring the flow cost ψ to accept money. Hence the fraction of sellers who accept the new currency solves

$$\chi_t \begin{array}{c} =1 \\ \in[0,1] \\ =0 \end{array} \quad \text{if} \quad \psi \begin{array}{c} < \\ = \\ > \end{array} \alpha\sigma A_t(1-\theta)S(\omega_t). \tag{44}$$

Money is universally accepted, $\chi = 1$, if the cost ψ to accept it is no greater than the expected gains from trade of the seller. If ψ is exactly equal to the gains from trade, then money is partially accepted, $\chi \in (0,1)$.

By (5) and (43) the law of motion for ω solves:

$$r\omega_t - \dot{\omega}_t = \left\{ \begin{array}{l} \alpha\sigma\left(\theta - A_t\right)S(\omega_t) + \psi \\ \alpha\sigma(1-A_t)\chi_t\theta S(\omega_t) \end{array} \right. \quad \text{if} \quad \chi_t \begin{array}{c} = \\ \leq \end{array} 1. \tag{45}$$

If money is universally accepted, namely $\chi_t = 1$, then the law of motion for ω is analogous to (41) except for the last term corresponding to the cost of accepting money. If money is only partially accepted, then its flow value is equal to the expected gains from trade of the money holder.

In Figure 10 we show the phase diagram corresponding to (45). The locus of the points where $\chi_t \in (0,1)$ is given by $A_t = \psi/\left[\alpha\sigma(1-\theta)S(\omega_t)\right]$. It is $\subset$-shaped in the (A,ω) space as $S(\omega_t)$ is concave and maximized at $\omega_t = (1-\theta)u(q^*) + \theta q^*$. Assuming $\bar{A} < \theta$, the locus $\dot{\omega} = 0$ conditional on $\chi_t = 1$ is downward sloping for all $A \in [0, \bar{A}]$.

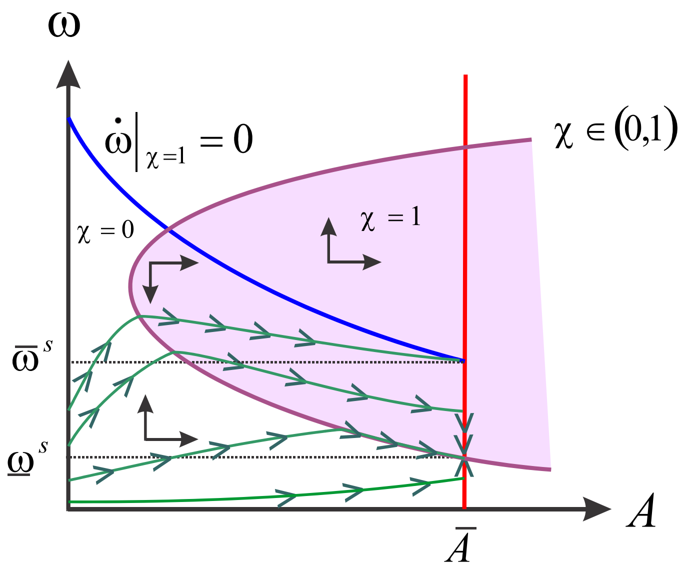

Figure 10: Dynamics with endogenous acceptability.

Proposition 6 ***(Crypto mining with endogenous acceptability)*** *Suppose (42) holds. There is a* $\bar{\psi} > 0$ *such that for all* $\psi < \bar{\psi}$ *there exists:*

1. *A unique steady-state monetary equilibrium,* $(\bar{\omega}^s, \bar{A})$*, with universal acceptability,* $\chi = 1$*. If* $A_0 = 0$*, then there is a unique equilibrium,* (ω_t, A_t)*, leading to* $(\bar{\omega}^s, \bar{A})$*. It is such that* $\omega_t = \hat{\omega}_0 > 0$ *and* $\chi_0 = 0$*. There exists* $(\underline{t}, \bar{t})$ *with* $0 < \underline{t} \leq \bar{t}$ *such that for all* $t < \underline{t}$*,* $\dot{\omega}/\omega = r$ *and* $\chi_t = 0$ *and for all* $t > \bar{t}$*,* $\dot{\omega} < 0$ *and* $\chi_t = 1$*.*

2. *A unique steady-state monetary equilibrium,* $(\underline{\omega}^s, \bar{A})$*, with partial acceptability,* $\chi < 1$*, and* $\underline{\omega}^s < \bar{\omega}^s$ *. If* $A_0 = 0$*, then there is a continuum of equilibria indexed by* $\omega_0 \in (0, \hat{\omega}_0)$ *leading to* $(\underline{\omega}^s, \bar{A})$*.*

The model of crypto mining with an endogenous acceptability decision generates dynamics that are reminiscent to the ones from gold mining, but there are important differences. The price trajectory leading to the high steady state where money is universally accepted is non-monotone. Initially, the supply of money is low and hence sellers do not invest to accept money. Since money is not accepted for transactions, its value must increase at the rate of time preference so that agents are willing to hold it. Once the supply reaches a certain threshold, then sellers start accepting money and its value declines for reasons similar to that behind Proposition 5.

There is no equilibrium where money is valued initially, $\omega_0 > 0$, but its value vanishes asymptotically. If ω_t were close to 0 for some t, then money would not be accepted, in which case $\dot{\omega}_t/\omega_t > r$, which prevents ω_t from converging to 0 from above. But there is a continuum of equilibria leading to a steady state where money is partially accepted and its price is $\underline{\omega}^s > 0$.[19] The time path for ω_t is either hump-shaped or monotone increasing. Hence, if the initial value of money is ω_0, then $\omega_t \geq \min\{\omega_0, \underline{\omega}^s\}$ for all t. For instance, if the initial value of money is very low, then it will keep growing at rate r until it reaches $\underline{\omega}^s$ at the steady state. It is only when the steady state is reached that money becomes partially acceptable and circulates as a medium of exchange. The left panel of Figure 11 provides a numerical example of Proposition 6.[20]

4.3 Sunspot equilibria and volatility of currency prices

Our model with endogenous acceptability can provide an explanation for large changes in crypto-currency prices that are disconnected from fundamentals. Indeed, we can build on the existence of multiple steady states to construct sunspot equilibria where currency price and acceptability depend on some extrinsic uncertainty. We start with the existence of stationary sunspot equilibria when $A_t = \bar{A}$. Suppose there are two sunspot states, ℓ and h, that are unrelated to fundamentals. The economy transitions from state h to ℓ at Poisson rate $\varpi^h > 0$ and from ℓ to h at Poisson rate $\varpi^\ell > 0$. The value of money is ω^h in state h and ω^ℓ in state ℓ. The acceptability of money is one in state h and $\chi < 1$ in state ℓ. At a stationary equilibrium, $(\omega^h, \omega^\ell, \chi)$ is a solution to the following system:

$$r\omega^h = \alpha\sigma(\theta - \bar{A})S(\omega^h) + \psi + \varpi^h\left(\omega^\ell - \omega^h\right) \tag{46}$$

$$r\omega^\ell = \alpha\sigma(1 - \bar{A})\theta\chi S(\omega^\ell) + \varpi^\ell\left(\omega^h - \omega^\ell\right) \tag{47}$$

$$\psi = \alpha\sigma\bar{A}(1 - \theta)S(\omega^\ell). \tag{48}$$

Equation (46) is the HJB equation for the value of money in the state where it is accepted with probability one. The difference with respect to (45) is the last term on the right side according to which the value of

[19] The indeterminacy of the initial value of a new currency was acknowledged by earlier adopters of Bitcoins. Luther (2018) reports the following post on bitcoin-list in January 2009, the month when Bitcoin was first introduced: "One immediate problem with any new currency is how to value it. Even ignoring the practical problem that virtually no one will accept it at first, there is still a difficulty in coming up with a reasonable argument in favor of a particular non-zero value for the coins."

[20] The parameters used in this example are $u(q) = q^B$, $A(t) = \bar{A}(1 - 2^{-\lambda t})$ and $\{B, \theta, \alpha, \sigma, \bar{A}, r, \psi, \lambda\} = \{0.8, 0.905, 5, 0.9, 0.9, 0.04, 0.02, 0.9\}$.

money responds to a change of the sunspot state from h to ℓ at Poisson rate ϖ^h. Equation (47) is the HJB equation for the value of money in the state where it is only partially accepted, $\chi < 1$. Finally, (48) is the condition for partial acceptability in the low state. Provided that the conditions for the existence of a monetary steady-state equilibrium hold, there is also a continuum of sunspot equilibria indexed by (ϖ^h, ϖ^ℓ). To see this, note that ω^ℓ is uniquely determined by (48) and coincides with the lowest steady state. Given ω^ℓ, ω^h is uniquely determined by (46). Finally, χ is determined by (47) and it is less than one provided ϖ^ℓ is not too large.

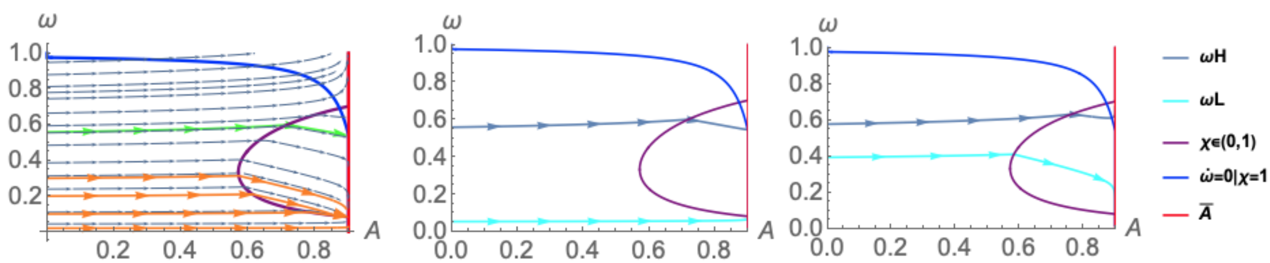

Figure 11: Equilibrium with Endogenous Acceptability. Left panel: All deterministic equilibria. Middle and right panels: Example of sunspot equilibria for $\varpi \in \{0.001, 0.01\}$.

From any stationary sunspot equilibrium, we can construct non-stationary sunspot equilibria starting from $A_0 = 0$. For the sake of illustration, we still restrict the set of sunspot states to $\{\ell, h\}$. The value of money is now a function of time, t, and the realization of the sunspot state, x, and it solves the following system of ODEs:

$$\begin{aligned} r\omega_t^x =& \alpha\sigma(1-A_t)\theta\chi_t^x(\omega_t^x)S(\omega_t^x) - \max\{-\psi + \alpha\sigma A_t(1-\theta)S(\omega_t^x), 0\} \\ &+ \varpi^x\left(\omega_t^{x'} - \omega_t^x\right) + \dot{\omega}_t^x, \text{ for } x, x' \in \{\ell, h\}. \end{aligned}$$

The middle and right panels of Figure 11 provide numerical examples of sunspot equilibria under the assumption that sunspot states are equally likely, $\varpi^h = \varpi^l = \varpi$. The middle panel plots equilibria for $\varpi = 0.001$ while the right panel assumes $\varpi = 0.01$. The equilibrium trajectory transitions between the dark green line, which corresponds to ω_t^h, and the cyan line, ω_t^ℓ, at Poisson rate ϖ. As ϖ increases, ω^h and ω^ℓ get closer to each other and the highest steady state rises.

4.4 Opportunity cost of mining

So far, we took the unit cost of mining as exogenous and constant. However, it seems reasonable to think that crypto mining, just like gold mining, has an endogenous opportunity cost: the inputs in the mining of crypto-currencies (e.g., labor, computer power, and electricity) can be devoted to the production of goods and services.[21] We capture this opportunity cost in a tractable way, as in Section 3, by assuming that agents are either producer or miner, $e \in \{0,1\}$. The opportunity cost of mining is then $C(1) = \alpha\sigma(1-\theta)A[u(q)-q]$, which corresponds to the flow expected surplus of a producer. This assumption breaks the dichotomy between ω and m and it makes the measure of miners relevant for allocations. We will also show in Section 4.5 that under this formulation the path for the money growth rate that implements price stability is reminiscent to the one of Bitcoin with some differences. For now we keep the mining technology as in (3). The money growth rate is $\pi(A) = \lambda[(\bar{A}-A)/A]\mathbb{I}_{(A\leq\bar{A})}$, which approximates the path of the supply of Bitcoins (see footnote 12).

By the same reasoning as above, the value of money and the measure of miners solve:

$$\dot{\omega} = \left[r + \frac{\pi(A)A}{m}\right]\omega - \alpha\sigma(1-A-m)\theta S(\omega), \tag{49}$$

$$m = \min\left\{\frac{\pi(A)\omega}{\alpha\sigma(1-\theta)S(\omega)}, 1-A\right\}. \tag{50}$$

At a steady state, $A^s = \bar{A}$ and

$$\frac{\omega^s}{S(\omega^s)} = \frac{\alpha\sigma\left(\theta-\bar{A}\right)}{r}. \tag{51}$$

A monetary steady state exists if $r < \alpha\sigma(\theta-\bar{A})/(1-\theta)$. Figure 12 shows the phase diagram and price trajectories.

Proposition 7 ***(Crypto mining with endogenous opportunity cost.)*** *Suppose $A_0 = 0$ and $r < \alpha\sigma(\theta-\bar{A})/(1-\theta)$.*

1. *There exists a unique monetary equilibrium such that (A_t, ω_t) converges asymptotically to (A^s, ω^s). The value of money, ω_t, rises over time from $\omega_0 = \hat{\omega}_0 > 0$ to ω^s if $\lambda \geq \bar{\lambda} \equiv (1-\theta)r\bar{A}/[\theta(\theta-\bar{A})]$; otherwise, ω_t rises and then falls before converging to ω^s.*

[21] CoinDesk reported that the number of blockchain jobs posted in the U.S. rised by 207% in 2017 and 631% since November 2015. Upwork, a large freelancing website, ranked blockchain as the top fastest-growing skill in the first quarter of 2018. This rapid growth is consistent with the rise in the number of crypto-currencies — according to investing.com, there were less than 1600 crypto-currencies in February 2018 and there are 2520 of them in February 2019.

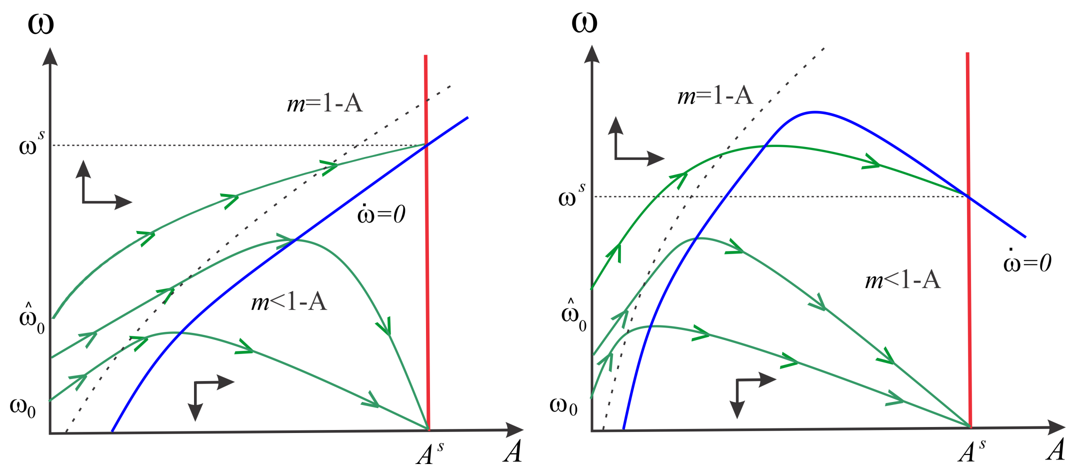

Figure 12: Phase diagram for crypto mining under occupation choice.

2. *There is a continuum of monetary equilibria indexed by* $\omega_0 \in (0, \hat{\omega}_0)$ *such that* $\lim_{t\to+\infty} \omega_t = 0$*. For each* ω_0*, there exist* $0 < T_0 < +\infty$ *such that* $\dot{\omega}_t \geq 0$ *for* $t \leq T_0$ *and* $\dot{\omega}_t < 0$ *for all* $t > T_0$*.*

The value of money rises along the equilibrium path leading to the unique steady state when the mining intensity λ is large (left panel of Figure 12). However, for smaller λ, the time-path for ω_t is hump-shaped (right panel of Figure 12). In order to understand why ω_t drops along the trajectory, recall that under the gold mining technology the individual mining speed, $\lambda(\bar{A} - A)$, falls over time. Thus ω_t must rise so that agents are willing to participate in mining. This logic fails for the crypto mining technology as the individual mining speed, $\lambda(\bar{A} - A)/m$, falls in m. The mining rate can rise over time if agents expect the value of money to fall and, as a result, fewer agents choose to mine.

In addition to the equilibrium leading to (A^s, ω^s), there is a continuum of equilibria where ω_t rises initially and then falls to 0. The velocity of money is 0 early on, in which case ω_t grows faster than the rate of time preference. When A_t is close to $\bar{A}$, the velocity of money is positive as producers and miners coexist.

4.5 A mining rule for price stability

In Proposition 7 we characterized price dynamics assuming a money growth rate analogous to that of Bitcoin. Suppose instead that we design the money growth rate, π_t, to implement price stability. How does this money growth rate compare to that of Bitcoin? Consider the model in Section 4.4 and take the potential money supply, $\bar{A}$, as given. The next proposition derives the path for π_t such that ω_t is constant over time.

Proposition 8 ***(Stabilizing the value of money)*** *In any equilibrium where the value of money is con-*

stant, $\omega = \omega^s(\bar{A})$, the rate of money creation evolves according to:

$$\pi_t^* = \pi^*(A_t) \equiv \lambda \left(\frac{\bar{A} - A_t}{\bar{A}} \right) \quad \textit{where } \lambda \equiv \frac{(1-\theta)\,\bar{A}}{\theta\left(\theta - \bar{A}\right)} r. \tag{52}$$

The money growth rate that achieves price stability in (52) is proportional to the fraction of the money supply that is left to mine. Interestingly, it resembles the growth rate of Bitcoin which is proportional to $\left(\bar{A} - A_t\right)/A_t$ instead of $\left(\bar{A} - A_t\right)/\bar{A}$. The coefficient of proportionality, λ, (which is equal to $\bar{\lambda}$ in Proposition 7) rises with $\bar{A}$, r, but falls with θ.

5 Conclusion

This paper was motivated by the recent multiplication of privately-produced monies and the questions raised about the dynamics of their prices and their use as medium of exchange. To address these questions, we studied the dynamics of a random-matching economy where money is produced according to a time-consuming mining technology. We distinguish different mining technologies for tangible and cryptomonies, both in steady states and out of steady states. We showed that our model can generate trajectories for prices and money velocity that are qualitatively similar to those observed: the price of a new money can increase initially as its supply increases even though it does not circulate as a medium of exchange; the currency price can feature booms and busts triggered by self-fulfilling beliefs; a new money can be valued and produced even when it is anticipated that its value will vanish in the long run.

Many interesting questions remain open, e.g., the determination of the relative prices of competing private monies, the competition among currency designers, and the role of reputation and public monitoring for the private provision of monies. In Choi and Rocheteau (2019a), we address some of these questions by extending our model to include two divisible monies and we study their joint price dynamics and the conditions under which they can coexist in the long run. But much more can be done on these topics.

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Appendix: Omitted Proofs

Proof of Proposition 1. The steady-state money supply is the lowest root of (20) at equality, i.e.,

$$A^2 - \left[\frac{\left(\theta + \bar{A}\right)\lambda + r(1-\theta)}{\lambda}\right] A + \bar{A}\theta = 0. \tag{53}$$

The lowest root of (53) is (23). If $\theta > 0$, then $A^s > 0$. From (19), $q^s > 0$ is a solution to

$$\Gamma(q) \equiv \{\alpha\sigma(\theta - A^s) - r(1-\theta)\}\,[u(q) - q] - rq = 0. \tag{54}$$

If $\alpha\sigma(\theta - A^s) - r(1-\theta) > 0$ then $\Gamma(q)$ is strictly concave with $\Gamma'(0) = +\infty$ and $\lim_{q\to\infty}\Gamma(q) = -\infty$. Hence, there exists a unique $q^s > 0$ solution to $\Gamma(q^s) = 0$. If $\alpha\sigma(\theta - A^s) - r(1-\theta) \leq 0$ then $\Gamma(q)$ is decreasing with $\Gamma(0) = 0$. Hence, there is no $q^s > 0$ solution to $\Gamma(q^s) = 0$. Comparative statics are straightforward. As an example, consider the effects of changes in λ. Differentiating (53) with respect to A and λ we obtain:

$$\frac{\partial A^s}{\partial \lambda} = \frac{\left(\bar{A} - A^s\right)(A^s - \theta)}{2\lambda A^s - \left[\left(\theta + \bar{A}\right)\lambda + r(1-\theta)\right]}.$$

Using that A^s is the lowest root of (53), it follows that the denominator is negative (graphically, the slope of the parabola is negative when it intersects the horizontal axis at A^s). Using that $A^s < \min\{\bar{A}, \theta\}$, the numerator is negative and $\partial A^s/\partial\lambda > 0$. From (54) $\Gamma(q)$ decreases with A^s for all q such that $u(q) - q > 0$. Hence, $\partial q^s/\partial\lambda > 0$. As another example, consider a change in r. From (53),

$$\frac{\partial A^s}{\partial r} = \frac{(1-\theta)A^s}{2\lambda A^s - \left[\left(\theta + \bar{A}\right)\lambda + r(1-\theta)\right]} < 0.$$

Since agents are indifferent between mining or producing at A^s, namely $\Delta(q^s, A^s) = 0$, by (15) we have

$$\frac{u(q^s) - q^s}{\omega(q^s)} = \left\{1 - \theta + \frac{q^s}{u(q^s) - q^s}\right\}^{-1} = \frac{\lambda\left(\bar{A} - A^s\right)}{\alpha\sigma A^s(1-\theta)}.$$

The left side falls in q^s and the right side falls in A^s. Thus q^s and A^s comove as r varies and $\partial q^s/\partial r < 0$. ■

Proof of Proposition 2. The first two claims and Part 1: We first provide a condition that determines which regime, $m = 1 - A$ or $m \in (0, 1 - A)$, is relevant along the equilibrium trajectory. We use the condition to create two ODEs that fully characterize the dynamics of q_t and A_t. Then we will use a standard result for systems of ODEs to prove the existence and uniqueness of q_t and A_t.

Suppose the economy is at (A', q') where $A' < A^s$ and consider the trajectory as we move backward in time. If $\mu(q') > A'$, then the equilibrium path cannot follow (17) because the solution path is continuous.

To see this, note that the path of A_t is continuous by (14) since A_t cannot jump and $\left|\dot{A}\right| \leq \lambda\bar{A}$. The value of money, $\omega(q_t) \equiv V_{1,t} - V_{0,t}$, is continuous over time because the continuation values $V_{1,t}$ and $V_{0,t}$ are integrals of payoffs that arrive randomly according to Poisson processes. As a result $m = 1 - A'$ when $\mu(q') > A'$. In this case it is optimal for agents to mine because $\Delta(q', A')$ in (15) is strictly positive when $\mu(q') > A'$.

Next suppose $\mu(q') = A'$. The equilibrium is in the regime with $m = 1 - A$ if and only if

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} \leq \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)} \tag{55}$$

where the first derivative is defined by (29) and the second is obtained by differentiating (17) with respect to A, namely $\partial q/\partial A = 1/\mu'(q)$. If (55) is binding, then both regimes imply $m = 1 - A$ and thus they imply the same trajectory. If (55) holds strictly, then the trajectory defined by (27) and (28) converges to (A', q') from the left of the line $\mu(q) = A$, and thus it is optimal for all agents without money to mine, $m = 1 - A$. The trajectory $\mu(q) = A$ is not an equilibrium near (A', q') because by (24) and (25)

$$\frac{\partial q}{\partial A} = \frac{m^{-1}\{\left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - \alpha\sigma\left(1 - A\right)\theta\left[u(q) - q\right]\} + \alpha\sigma\theta\left[u(q) - q\right]}{\lambda(\bar{A} - A)\omega'(q)}. \tag{56}$$

For any (A, q) such $A = \mu(q)$, the right side rises in m because the expression in the braces is negative by (17) and (20). Using that the left side of (55) coincides with (56) when $m = 1 - A$, it follows that if (55) holds strictly at (q', A'), then the measure of miners m implied by the trajectory $A = \mu(q)$ must strictly exceed $1 - A'$, and thus it cannot be an equilibrium path.

If (55) does not hold, then the trajectory defined by (27) and (28) converges to (q', A') from below $A = \mu(q)$ and, hence, no agent has an incentive to mine. In this case, the equilibrium path is in the regime where $m \in (0, 1 - A)$ and the measure of miners, m, implied by $A = \mu(q)$ satisfies $m < 1 - A$ by (56).

We are now ready to characterize the equilibrium by a system of ODEs in backward time. Let $y \equiv A^s - A$. Then we can define q as a function of y along the equilibrium path. From (17) and (29), the two sides of the inequality (55) can be expressed as:

$$\left.\frac{\partial q}{\partial y}\right|_{m=1-A} = g_1(y, q) \equiv -\frac{\omega(q)}{\omega'(q)}\frac{\left[r + \lambda\left(\bar{A} - A^s + y\right)\right]}{(1 - A^s + y)\lambda\left(\bar{A} - A^s + y\right)} \tag{57}$$

$$\left.\frac{\partial q}{\partial y}\right|_{m\in(0,1-A)} = g_2(y, q) \equiv -\frac{1}{\mu'(q)}. \tag{58}$$

By the discussion above, the equilibrium path $q(y)$ solves the following ODE:

$$\frac{\partial q}{\partial y} = f(y, q) \equiv \max\{g_1(y, q), g_2(y, q)\}\mathbf{1}_{\{\mu(q)\leq A^s-y\}} + g_1(y, q)\mathbf{1}_{\{\mu(q)>A^s-y\}} \tag{59}$$

with the initial condition $q(0) = q^s$, where $\mathbf{1}_{\{\cdot\}}$ is an indicator function. It is easy to check that $f(y, q)$ is bounded and continuous for $y \in [0, A^s]$ and $q \in (0, \bar{q}]$ where $\bar{q} > 0$ is the solution to $u(\bar{q}) - \bar{q} = 0$.

Next we show that the equilibrium eventually enters the regime with $m = 1 - A$ as y increases. As q tends to 0, then $\omega(q)\mu'(q)/\omega'(q) \to 0$ and, thus, there exists $\underline{q} > 0$ such that (57) exceeds (58) for all $q < \underline{q}$. Therefore, the equilibrium stays in the regime $m = 1 - A$ for all $q < \underline{q}$, as shown in Figure 13. Let $\underline{y} \equiv A^s - \mu(\underline{q})$ so that the equilibrium has $m = 1 - A$ for all $y \geq \underline{y}$.

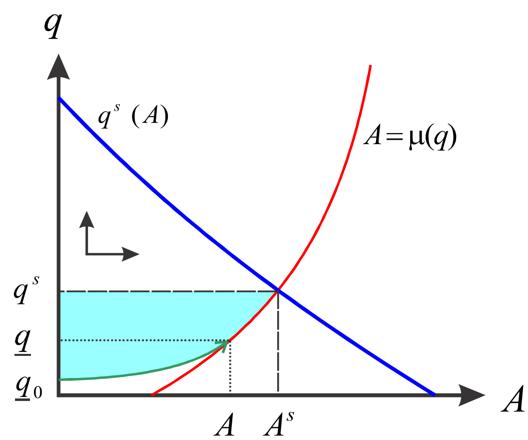

Figure 13: The equilibrium path is contained in the shaded area.

Consider the existence and uniqueness of equilibrium for $q \in [\underline{q}, q^s]$ and $y \in [0, \underline{y}]$. By Theorem 58.5 in Tennenbaum and Pollard (1985), there is a unique solution for $q(y)$ in (59) provided that $f(y, q)$ is Lipschitz continuous, namely there is a real constant $K > 0$ such that

$$|f(y, q') - f(y, q'')| \leq K|q' - q''|$$

for every $y \in [0, \underline{y}]$ and $q', q'' \in [\underline{q}, q^s]$. For any $y \in [0, \underline{y}]$, one can check that the slope of f with respect to q is bounded provided that u'' is bounded. Therefore, f is Lipschitz continuous.

Next, we express m as a function of y. By using the solution of $q(y)$ and (56),

$$m(y) = \frac{\alpha\sigma\left(1 - A^s + y\right)\theta\{u[q(y)] - q(y)\} - \left[r + \lambda\left(\bar{A} - A^s + y\right)\right]\omega[q(y)]}{\lambda(\bar{A} - A^s + y)\omega'(q)\partial q/\partial y + \alpha\sigma\theta\{u[q(y)] - q(y)\}}. \tag{60}$$

By (25) the ODE that determines y_t is

$$\dot{y}_t = -\lambda m(y_t)(\bar{A} - A^s + y_t).$$

One can check that the slope of the right side with respected to y is bounded for all $y \leq \underline{y}$. Therefore, the right side is Lipschitz continuous and the solution for y_t is unique for any given initial condition. Since $q(y)$ and y_t exist and are unique, q_t and A_t exist and also are unique.

For $A < A^s - \underline{y}$, the equilibrium is in the regime with $m = 1 - A$. The ODE for A, (28), is a Riccati equation, which has a closed form solution (see Section 2.15 in Ince (1956)):

$$A_t = \frac{\bar{A} - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}, \tag{61}$$

where $A_0 = 0$ is the initial condition. By (27) and (61) we can solve for the value of money in closed form,

$$\omega(q_t) = \omega_0 e^{rt}\left\{1 + \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{(1-\bar{A})}\right\}, \tag{62}$$

where ω_0 is the initial value of $\omega(q_t)$ at $t = 0$. We can solve for ω_0 by first solving t_0 in $A_{t_0} = A^s - \underline{y}$ where A_t is given by (61). Then we derive ω_0 by solving $\omega(q_{t_0}) = \omega(\underline{q})$ where $\omega(q_{t_0})$ is derived by evaluating (62) at $t = t_0$. Since $\underline{q} > 0$ as discussed above, $\omega_0 > 0$ by (62).

Part 2: By (29) the inequality (55) is equivalent to

$$\frac{\omega(q)}{\omega'(q)}\frac{r + \lambda\left(\bar{A} - A\right)}{(1-A)\lambda\left(\bar{A} - A\right)} \leq \frac{1}{\mu'(q)}. \tag{63}$$

Suppose $A = A^s$ and $q = q^s$. Since $A^s = \mu(q^s)$ and A^s solves the equality (20), the inequality above is equivalent to

$$\frac{\mu'(q^s)}{\mu(q^s)}\frac{\omega(q^s)}{\omega'(q^s)} \leq \frac{1-\theta}{\theta}.$$

This proves the claim concerning (33) in the proposition. Finally, if $m = 1 - A$ near the neighborhood of the steady state, then $\dot{A}$ in (14) is strictly positive because $\bar{A} - A_t > 0$ and $1 - A_t > 0$ near the steady state. Therefore A_t converges to A^s in finite time near the steady state. When $m < 1 - A$, $m \to 0$ as $(q, A) \to (q^s, A^s)$ because the denominator in (60) is positive by (17) and (20), and it vanishes as $(q, A) \to (q^s, A^s)$. Moreover, since $\partial q/\partial y = -1/\mu'[q(y)]$ when $m < 1 - A$, the numerator in (60) can be written as

$$-\lambda(\bar{A} - A^s + y)\frac{\omega'[q(y)]}{\mu'[q(y)]} + \alpha\sigma\theta\{u[q(y)] - q(y)\} = (1-\theta)\alpha\sigma\{u[q(y)] - q(y)\}\left[\frac{\theta}{1-\theta} - \frac{\omega'[q(y)]/\omega[q(y)]}{\mu'[q(y)]/\mu[q(y)]}\right]. \tag{64}$$

The equation is true because when $m < 1 - A$ we have $A = \mu(q)$, and thus by (17) and $A = A^s - y$

$$\lambda\left(\bar{A} - A^s + y\right) = \frac{(1-\theta)\alpha\sigma\mu[q(y)]\left\{u[q(y)] - q(y)\right\}}{\omega(q)}. \tag{65}$$

As $q(y) \to q^s$, the right side of (64) converges to a strictly positive value by (33) and, therefore, m in (60) vanishes as $(q, A) \to (q^s, A^s)$. It follows that $\dot{A}$ vanishes by (25) and thus A_t converges to A^s only asymptotically. ■

Proof of Lemma 1. **Part 1:** By Proposition 2, mining and trades coexist near the steady state if

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

By differentiating (17),

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} = \frac{\left(\bar{A} - A^s\right)[1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]}. \tag{66}$$

Hence, (33) can be rewritten as:

$$\frac{\left(\bar{A} - A^s\right)[1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]} > \frac{1-\theta}{\theta}. \tag{67}$$

The left side rises in q^s by $\epsilon'(q) \leq 0$ and the concavity of $u(q)$. As λ rises, A^s rises and q^s falls by Proposition 1, and thus the left side of (67) falls. As $\lambda \to \infty$, $A \to \bar{A}$ and the right side of (67) vanishes, so λ^* is finite.

Next, by Proposition 1, as $\sigma\alpha$ increases, A^s remains constant but q^s rises. Hence, the left side of (67) rises. When $\alpha\sigma$ is sufficiently small, q is arbitrarily close to 0. As $q \to 0$, the left side of (67) vanishes by L'Hospital's Rule, and thus the inequality fails. From (19), as $\alpha\sigma \to \infty$, $q^s/u(q^s) \to 1$, and the left side of (67) goes to infinity. This proves that (33) holds if $\alpha\sigma$ is sufficiently large and thus $\kappa^* \in (0, \infty)$.

Now we prove the last claim of Part 1. The transition from the regime with $m = 1 - A$ to one with $m \in (0, 1 - A)$ must occur on the indifference condition $A = \mu(q)$ and the slope of the two trajectories must be the same, namely

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} = \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)}.$$

The equality holds because if the right side is strictly larger, then the measure of miners implied by the trajectory $A = \mu(q)$ strictly exceeds $1 - A$ and thus it cannot be an equilibrium. If the left side is strictly larger, then the trajectory in the $m=1-A$ regime will cut the line $A=\mu(q)$ from below, but in this case no agents has incentive to mine and, thus it is impossible to have $m=1-A$. Hence, the slope of the trajectories must be the same at the transition point. From (17) and (29), the displayed equation above is the same as

$$\frac{\omega(q)}{\omega'(q)} \frac{r + \lambda\left(\bar{A} - A\right)}{(1 - A)\lambda\left(\bar{A} - A\right)} = \frac{1}{\mu'(q)}.$$

By the definition of μ in (17) and $A = \mu(q)$, the equation can be rewritten as

$$\left[1 - \frac{\omega(q)\{\alpha\sigma(1-\theta)\left[u'(q)-1\right] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)\}}\right] \frac{A[r+\lambda\left(\bar{A}-A\right)]}{(1-A)\lambda\left(\bar{A}-A\right)} = 1$$

$$\iff \frac{\alpha\sigma(1-\theta)[1-\epsilon(q)]}{\omega'(q)} \left[\frac{u(q)}{\alpha\sigma(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)}\right] \frac{A[r+\lambda\left(\bar{A}-A\right)]}{(1-A)\lambda\left(\bar{A}-A\right)} = 1.$$

Now we argue that the left side rises monotonically as we move along the line $A = \mu(q)$ in the (A, q) space. As we move along $A = \mu(q)$, both q and A increase. The fraction $1 - \epsilon(q)$ rises in q provided that u has decreasing elasticity. The fraction $1/\omega'(q)$ rises in q by the concavity of u. The fraction in the large bracket rises in q when $\alpha\sigma(1-\theta) > \lambda\theta$. The last fraction in the left side rises in A. Altogether, the left side increases as the trajectory moves along $A = \mu(q)$. It follows that there can be at most one transition from the $m = 1 - A$ regime to the one with $m \in (0, 1 - A)$.

Finally, once the equilibrium enters the regime with $m \in (0, 1-A)$, it cannot switch regime again. For suppose it does, then the equilibrium must stay in the regime with $m = 1-A$ as explained in the above paragraph. But then the equilibrium trajectory cannot converge to (A^s, q^s) as it cannot intersect the locus $A = \mu(q)$.

Part 2: We show $m \in (0, 1-A)$ is impossible when $\theta < 1/2$. As discussed before (64), the denominator in the right side of (60) is strictly negative for all $q < q^s$ and $A < A^s$. Suppose $m \in (0, 1-A)$, then the numerator can be written as (64) and it is negative if and only if

$$\frac{\omega(q)}{\omega'(q)} \frac{\mu'(q)}{\mu(q)} > \frac{1-\theta}{\theta} \iff 1 - \frac{\omega(q)\{\alpha\sigma(1-\theta)\left[u'(q)-1\right] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)\}} > \frac{1-\theta}{\theta}.$$

The second inequality uses the definition of μ in (17). If λ is sufficiently large, then $q^s < q^*$ by Proposition 1. In this case $u'(q) - 1 > 0$ for all $q \le q^s$ and thus the left side is less than 1. If $\theta < 1/2$, then $(1-\theta)/\theta > 1$ and, therefore, this condition always fails. As a result, $m < 0$ and thus it is impossible for the equilibrium path to be in the regime with $m \in (0, 1-A)$. ■

Proof of Proposition 3. Part 1: From the proof of Proposition 2, for any (A', q') where $A' < A^s$ and $A' \le \mu(q')$, there is a unique q_0 such that $(A_t, q_t) = (A', q')$ for some $t > 0$ from the initial condition $A_0 = 0$. As a result if two equilibrium trajectories have different initial values for q, then they will not intersect in the (A, q) space. It follows that any equilibrium trajectory with $q_0 < \hat{q}_0$ is located below the trajectory with $q_0 = \hat{q}_0$ that converges to (A^s, q^s) as illustrated in the left panel of Figure 14. Since $A_0 = 0 \le \mu(q_0)$, the trajectory is located in the regime with $m_t = 1 - A_t$ if t is not too large. From (31), in this regime the trajectory solves:

$$\begin{aligned} A_t &= \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \\ \omega(q_t) &= e^{rt}\omega(q_0)\left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}}\right]. \end{aligned}$$

Since the trajectory cannot intersect the one that converges to steady state, it must cross the locus $A = \mu(q)$ in the (A, q) space at some $A < A^s$. Let T_0 be the first time the trajectory satisfies $A_t = \mu(q_t)$. Since $A_{T_0} < A^s$ and $m = 1 - A_t$ for all $t < T_0$, the value of A_t reaches A_{T_0} in finite time, thus $T_0 < +\infty$. This proves Part 1(a).

Next, we argue that the trajectory only crosses the locus $A = \mu(q)$ once. Suppose $A_t > \mu(q_t)$, then $m_t = 0$ because no agent wants to mine. Therefore, A_t remains constant. The value of money, $\omega_t = \omega(q_t)$, solves (11) with $m = 0$, i.e.,

$$\dot{\omega} = r\omega - \alpha\sigma\left(\theta - A_{T_1}\right)\left[u(q) - q\right]. \tag{68}$$

Using that $\dot{\omega} = 0$ when $(A_t, q_t) = (A^s, q^s)$ and $\dot{\omega}/\omega$ increases in q, it follows that $\dot{\omega} < 0$. See right panel of Figure 14. It follows that the trajectory (A_t, q_t) falls vertically whenever (A_t, q_t) lies below the locus $A = \mu(q)$ and $A_t < A^s$. Since the trajectory must and can only cross $A = \mu(q)$ once, there is $T_1(q_0) \in (T_0(q_0), +\infty)$ such that $A_{T_1} = \mu(q_{T_1})$ and $A_t > \mu(q_t)$ for all $t > T_1$. For all $t < T_1$, $m_t > 0$ and $\dot{q}_t > 0$ since otherwise the equilibrium trajectory would fall permanently below the locus $A = \mu(q)$. These properties are also illustrated in the left panel of the Figure 14.

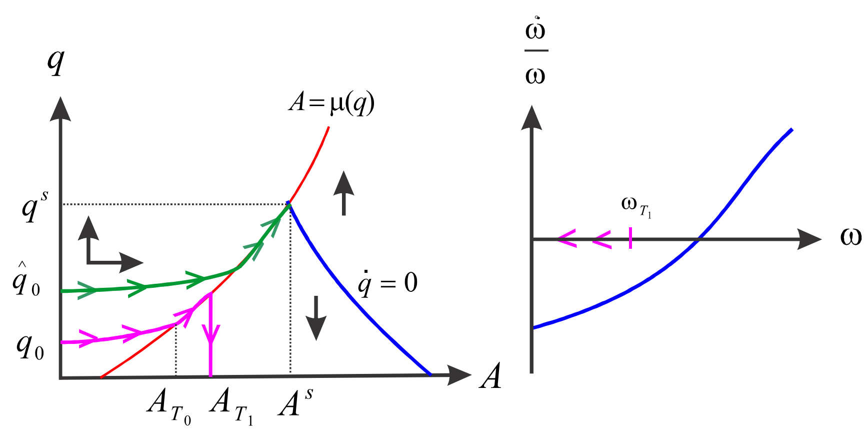

Figure 14: Failing currency equilibria.

For all $t > T_1$, $m_t = 0$, $A_t = A_{T_1}$, and $\dot{\omega} < 0$ as discussed above. See right panel of Figure 14. By (68) and L'Hospital's Rule

$$\lim_{q \searrow 0} \frac{\dot{\omega}}{\omega} = r - \frac{\alpha\sigma\left(\theta - A_{T_1}\right)}{u'(0)/[u'(0)-1] - \theta}.$$

Since the right side is constant for $t \geq T_1$, ω_t falls at a constant rate when $q \approx 0$. Therefore, ω_t converges to 0 asymptotically. This proves Part 1(b).

Part 2: If (33) holds, then there is a $\underline{T} < +\infty$ such that for all $t \geq \underline{T}$, $A_t = \mu(q_t)$ along the unique equilibrium, $\left(\hat{A}_t, \hat{q}_t\right)$, leading to (A^s, q^s). For all $T \geq \underline{T}$, we can construct an equilibrium such that $(A_t, q_t) = \left(\hat{A}_t, \hat{q}_t\right)$ for all $t \leq T$ and $m_t = 0$ for all $t \geq T$. The trajectory up to T is the solution to the system of ODEs in backward time characterized in the proof of Proposition 2. Since $A_T = \mu(q_T)$, at time T agents are indifferent between mining or not. We select $m_T = 0$. As a result, $\dot{A}_T = 0$ and

$$\dot{q}_T = \frac{r\omega(q_T) - \alpha\sigma\left(\theta - A_T\right)\left[u(q_T) - q_T\right]}{\omega'(q_T)} < 0.$$

Since $\dot{q}_T < 0$, the trajectory falls below the locus $A = \mu(q)$. As a result, for all $t > T$, $m_t = \dot{A}_t = 0$, and $\dot{q}_t < 0$. The rest of the argument is similar to the proof of Part 1 of Proposition 3. ■

Lemma 2 *There exists a pair of* (m_t, A_t) *that solves the planner's problem (69)-(71), provided that* $q_t = q^*$.

Proof. The planner's problem is given by:

$$\max_{q_t, m_t, A_t} \int_0^{+\infty} e^{-rt} \alpha\sigma A_t (1 - A_t - m_t)[u(q_t) - q_t] dt \tag{69}$$

$$\text{s.t. } \dot{A} = m_t \lambda(\bar{A} - A_t), \tag{70}$$

$$m_t \leq 1 - A_t \text{ and } A(0) = A_0. \tag{71}$$

The objective is the discounted sum of all trade match surpluses where the aggregate measure of trade matches between a money holder and a producer is $\alpha\sigma A\,(1 - A - m)$. The state variable is the money supply which increases with the measure of miners who successfully dig money from the ground. There are two control variables, the measure of miners and output in a match. The measure of miners has an upper bound given the measure of agents without money. If the planner can dictate the output traded in each match, she will choose $q_t = q^*$ for all t. One can rewrite (70) as

$$\dot{A} = \min\{m_t, 1 - A_t\}\lambda(\bar{A} - A_t). \tag{72}$$

This reformulation is useful because it ensures the planner never chooses $m_t > 1 - A_t$ even when it is feasible. As a result we could drop the constraint $m_t \leq 1 - A_t$ and only impose $m_t \in [0, 1]$. Next we apply a standard result to show the existence of solution for an infinite horizon optimization problem. By Theorem 15 in Seierstad and Sydsaeter (1986) there exists a (A_t, m_t) that solves the new planner's problem if

1. The right side of (72) and the integrand in (69) are continuous in m_t and A_t.
2. There exists a function $\phi(t)$ such that $\phi(t) \geq |e^{-rt}\alpha\sigma A_t(1 - A_t - m_t)[u(q^*) - q^*]|$ for all admissible (m_t, A_t) and $\int_0^\infty \phi(t)dt < \infty$.
3. There exists non-negative functions $a(t)$ and $b(t)$ such that

$$\min\{m_t, 1 - A_t\}\lambda(\bar{A} - A_t) \leq a(t)A_t + b(t)$$

   for all $A_t \in [0, \bar{A}]$ and $m_t \in [0, 1]$.

4. The set

$$N(A, t) = \{(e^{-rt}\alpha\sigma A_t(1 - A_t - m_t)[u(q^*) - q^*] + \gamma, \min\{m_t, 1 - A_t\}\lambda(\bar{A} - A_t))|m_t \in [0, 1], \gamma \leq 0\}$$

   is convex for all A_t and t.

It is easy to see condition (1) is satisfied. Condition (2) is satisfied by assuming $\phi(t) = e^{-rt}\alpha\sigma\bar{A}[u(q^*) - q^*]$. Condition (3) is satisfied because the right side of (72) is bounded above by $\lambda\bar{A}$ for all A_t and m_t. The last condition is satisfied because the first component of $N(A, t)$ is linear in m_t and γ and the second component is concave in m_t and constant in γ. It follows that there is a pair, (A_t, m_t), that solves the planner's problem provided that $q_t = q^*$. ■

Proof of Proposition 4. Part 1: It is obvious that the optimal output is $q_t = q^*$ provided that trade happens. By Lemma 2, there is a solution to the planner's provided that $q_t = q^*$. Now we characterize this solution and argue it is unique. The current value Hamiltonian corresponding to (69)-(71) is:

$$\mathcal{H}(A, m, \xi, \nu) = \alpha\sigma A(1 - A - m)[u(q^*) - q^*] + \xi m\lambda\left(\bar{A} - A\right) + \nu(1 - A - m),$$

where ξ is the co-state variable associated with A, and ν is the Lagrange multiplier associated with $m \leq 1 - A$. The FOC with respect to m is:

$$m \begin{array}{c} = 0 \\ \in [0, 1 - A] \end{array} \quad \text{if } -\alpha\sigma A[u(q^*) - q^*] + \xi\lambda\left(\bar{A} - A\right) - \nu \begin{array}{c} < 0 \\ = 0 \end{array}, \tag{73}$$

together with the complementary slackness condition, $\nu(1-A-m)=0$. The co-state variable satisfies the following ODE:

$$r\xi = \alpha\sigma\left(1-2A-m\right)\left[u\left(q^{*}\right)-q^{*}\right]-\xi m\lambda-\nu+\dot{\xi}. \tag{74}$$

The stationary solutions to (70) and (74), $\dot{A}=\dot{\xi}=0$, are such that $m=0$ and

$$r\xi = \alpha\sigma\left(1-2A\right)\left[u\left(q^{*}\right)-q^{*}\right] \tag{75}$$

$$\xi\lambda\left(\bar{A}-A\right) \leq \alpha\sigma A\left[u\left(q^{*}\right)-q^{*}\right]. \tag{76}$$

We denote A^{*} the lowest value of A that satisfies (76) where ξ is given by (75). It is the lowest root of the following quadratic equation,

$$2A^{2}-\left(1+2\bar{A}+\frac{r}{\lambda}\right)A+\bar{A}=0.$$

In closed form:

$$A^{*}=\frac{\left(1+2\bar{A}+r/\lambda\right)-\sqrt{\left(1+2\bar{A}+r/\lambda\right)^{2}-8\bar{A}}}{4}.$$

It is easy to check that $A^{*}<\min\{1/2,\bar{A}\}$. We denote

$$\xi^{*}=\alpha\sigma\left(1-2A^{*}\right)\left[u\left(q^{*}\right)-q^{*}\right]/r.$$

Now we argue that A_t converges to A^{*}. Since A_t is continuous, non-decreasing and bounded above by $\bar{A}$, eventually it converges and m vanishes. The process A_t cannot converge to any $A'<A^{*}$. Suppose it does. Since A^{*} is the smallest solution to (75) and (76), for all $A'<A^{*}$ we have

$$-\alpha\sigma A'\left[u\left(q^{*}\right)-q^{*}\right]+\xi\lambda\left(\bar{A}-A'\right)>0.$$

This implies $\nu>0$ by (73) and thus $m=1-A$ by the complementary slackness condition. Since $A'<1$, $m=1-A'>0$ and thus A_t cannot converge to A'. The process A_t also cannot converge to any $A>A^{*}$ because when A_t goes above A^{*} the inequality (76) holds strictly and thus $m=0$ by (73). It follows that the optimal solution can only converge to A^{*}.

We conjecture and then verify that the solution to the planner's problem is such that for all $A<A^{*}$, $m=1-A$. Then, the ODEs (70) and (74) can be rewritten as

$$\dot{A} = \lambda(1-A)\left(\bar{A}-A\right) \tag{77}$$

$$\dot{\xi} = \left[r+(1+\bar{A}-2A)\lambda\right]\xi. \tag{78}$$

The ODE for A, (77), is a Riccati equation that can be solved in closed form. See Section 2.15 in Ince (1956) for details. The solutions are

$$A_t = \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \quad \text{and} \quad \xi_t = \xi_0 e^{[r+\lambda(1-\bar{A})]t}\left(\frac{1-\bar{A}e^{-\lambda(1-\bar{A})t}}{1-\bar{A}^2}\right)^2,$$

where we used that $A_0 = 0$. Hence, there is a unique solution to (77)-(78). By the formula for A^* and A_t, one can solve for the time T^*. We denote the path defined by (77)-(78) by $\xi = \xi^p(A)$. From (77)-(78) the slope of $\xi = \xi^p(A)$ is

$$\xi^{p\prime}(A) = \frac{\dot{\xi}}{\dot{A}} = \frac{\left[r + (1 + \bar{A} - 2A)\lambda\right]\xi}{\lambda(1-A)\left(\bar{A} - A\right)}.$$

From (73) $m = 1 - A$ is optimal only if

$$\xi^p(A) \geq \Omega(A) \equiv \frac{\alpha\sigma A\left[u\left(q^*\right) - q^*\right]}{\lambda\left(\bar{A} - A\right)} \quad \text{for all } A < A^*.$$

We now show that whenever $\xi^p(A) = \Omega(A)$ then $0 < \xi^{p\prime}(A) < \Omega'(A)$. To see this, we evaluate $\xi^{p\prime}(A)$ at $\xi = \Omega(A)$:

$$\begin{aligned} \xi^{p\prime}(A)\big|_{\xi=\Omega(A)} &= \frac{A\left[r + (1+\bar{A}-2A)\lambda\right]}{\lambda(1-A)\bar{A}}\,\frac{\alpha\sigma\bar{A}\left[u\left(q^*\right) - q^*\right]}{\lambda\left(\bar{A}-A\right)^2} \\ &< \Omega'(A) = \frac{\alpha\sigma\bar{A}\left[u\left(q^*\right)-q^*\right]}{\lambda\left(\bar{A}-A\right)^2} \quad \text{for all } A < A^*. \end{aligned}$$

Given that $\xi^p(A^*) = \Omega(A^*)$, there is no other solution $A < A^*$ to $\xi^p(A) = \Omega(A)$, and thus $\xi^p(A) \geq \Omega(A)$ for all $A < A^*$.

Finally we argue $m \in (0, 1-A)$ cannot be optimal. Suppose $m \in (0, 1-\tilde{A})$ at certain $(\tilde{A}, \tilde{\xi})$ where $\tilde{A} < A^*$. Then $\tilde{\xi} = \Omega(\tilde{A})$ by (73). By the ODE (70) and (74),

$$\xi^{p\prime}(\tilde{A}) = \frac{\dot{\xi}}{\dot{A}} = \frac{\tilde{\xi}}{\lambda(\bar{A}-\tilde{A})}\left[\frac{1}{m}\left(r + \frac{\lambda(\bar{A}-\tilde{A})(2\tilde{A}-1)}{\tilde{A}}\right) + \lambda + \frac{\alpha\sigma[u(q^*)-q^*]}{\tilde{\xi}}\right]. \tag{79}$$

Since $rA + \lambda(\bar{A} - A)(2A-1) < 0$ for all $A < A^*$ by (75) and (76), the right side of (79) strictly increases in m. As discussed above, if $m = 1 - \tilde{A}$, then $\xi^{p\prime}(\tilde{A}) < \Omega'(\tilde{A})$ and thus $\xi^p(A)$ cuts $\Omega(A)$ from above at $(\tilde{A}, \tilde{\xi})$. By a similar argument, $\xi^p(A)$ must be lower than $\Omega(A)$ for all $A \in (\tilde{A}, A^*)$. But then it is impossible for $\xi(A)$ to reach ξ^* as $A \to A^*$ because $m = 0$ when $\xi^p(A) < \Omega(A)$ by (73). Therefore $m \in (0, 1-A)$ is sub optimal.

Part 2: In order to guarantee that $m_t = 1 - A_t$ for all $t < T^*$ we set $\theta_t = 1$ for all $t < T^*$ so that producers receive no gains from trade. From (19) the buyer's bargaining power that implements q^* when T^*

has been reached is θ^* defined in (38). Incentive feasibility means $\theta^* \in [0,1]$, which holds if and only if (36) holds. From (20) agents stop mining when A^* is reached if

$$r \leq \frac{(1-A^*)\left[u(q^*)-q^*\right]}{q^*}\left[\alpha\sigma A^* - \lambda\left(\bar{A}-A^*\right)\right] - \lambda\left(\bar{A}-A^*\right).$$

This inequality can be rearranged to give (37). ■

Proof of Proposition 5. From (40) and the assumption that $\pi(A_t)=0$ for all $A_t \geq \bar{A}$, any steady-state monetary equilibrium is such that $A \geq \bar{A}$. We focus on $A^s = \bar{A}$ since it is the value that is reached from $A_0 = 0$. From (41), the value of money at a steady state solves:

$$r\omega^s = \alpha\sigma\left(\theta - \bar{A}\right)S(\omega^s).$$

Using that $S'(0) = 1/(1-\theta)$ and the fact that $S(0)=0$ and $S(\omega)$ is concave, there exists $\omega^s > 0$ solution to the equation above if and only if (42) holds. The system of ODEs, (40) and (41), in the neighborhood of the steady state can be approximated by:

$$\begin{pmatrix} \dot{A} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \pi'(\bar{A})\bar{A} & 0 \\ \alpha\sigma S(\omega^s) & r - \alpha\sigma\left(\theta-\bar{A}\right)S'(\omega^s) \end{pmatrix} \begin{pmatrix} A - \bar{A} \\ \omega - \omega^s \end{pmatrix}.$$

Using that $r > \alpha\sigma\left(\theta - \bar{A}\right)S'(\omega^s)$ and $\pi'(\bar{A}) < 0$, then the determinant of the Jacobian matrix is negative and hence there is a unique saddle path leading to the steady state. Starting from an initial condition on that saddle path near the steady state, the system (40) and (41) in backward time generates a unique solution. This solution is located below the ω-isocline and is such that $\dot{\omega} < 0$. See phase diagram in Figure 12. It can also be checked from the phase diagram that any $\omega_0 \in (0, \hat{\omega}_0)$ is such that $\dot{\omega} < 0$ and $\lim_{t\to\infty}\omega_t = 0$. ■

Proof of Proposition 6. **Part 1:** At a steady-state monetary equilibrium with $\chi = 1$, $\bar{\omega}^s$ solves

$$r\bar{\omega}^s = \alpha\sigma\left(\theta - \bar{A}\right)S(\bar{\omega}^s) + \psi. \tag{80}$$

The left side is linear and unbounded while the right side is concave in $\bar{\omega}^s$, bounded, and it is strictly positive when $\bar{\omega}^s = 0$. Hence, there is a unique $\bar{\omega}^s > 0$ solution to (80). Moreover, $\bar{\omega}^s$ increases with ψ and, under the assumption (42), $\lim_{\psi\to 0}\bar{\omega}^s = \underline{\omega}^s > 0$. The solution is consistent with $\chi = 1$ if $\psi \leq \alpha\sigma\bar{A}(1-\theta)S(\bar{\omega}^s)$. This inequality holds at $\psi = 0$ and, by continuity, it holds over some interval $\left[0, \bar{\psi}\right]$ with $\bar{\psi} > 0$.

In the neighborhood of the steady state $(\bar{A}, \bar{\omega}^s)$, the system of ODEs for (A_t, ω_t) is approximated by:

$$\begin{pmatrix} \dot{\omega} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} r - \alpha\sigma\left(\theta-\bar{A}\right)S'(\bar{\omega}^s) & \alpha\sigma S(\bar{\omega}^s) \\ 0 & \pi'(\bar{A})\bar{A} \end{pmatrix} \begin{pmatrix} \omega - \bar{\omega}^s \\ A - \bar{A} \end{pmatrix}.$$

Using that $r > \alpha\sigma\left(\theta - \bar{A}\right) S'(\omega^s)$ and $\pi'(\bar{A}) < 0$, the determinant of the Jacobian matrix is negative, i.e., the steady state is a saddle point. Hence, there is a unique saddle path in the neighborhood of the steady state leading to it. In order to compute the equilibrium path from $A_0 = 0$, we take a point on the saddle path arbitrarily close to the steady state and move backward in time according to the system of ODEs, (40) and (45). As shown in the phase diagram in Figure 10, as long as the system remains in the region where $\chi_t = 1$, when A_t is sufficiently close to $\bar{A}$, i.e., t is larger than some threshold $\bar{t}$, then $\dot{\omega}_t < 0$. When the system enters the region where $\chi_t = 0$ then, from (45), $\dot{\omega}/\omega = r > 0$. Let $\underline{t} > 0$ be the time such that $A_t = \psi/\alpha\sigma(1-\theta)\left[u(q^*) - q^*\right]$. From (44), for all $t < \underline{t}$, $\chi_t = 0$, and $\underline{t} \leq \bar{t}$ since the opposite would contradict the existence of a steady state with $\chi = 1$.

Part 2: By (44) and (45) a monetary steady-state with partial acceptability is a pair $(\underline{\omega}^s, \chi^s)$ solving

$$\psi = \alpha\sigma\bar{A}(1-\theta)S(\underline{\omega}^s), \tag{81}$$

$$\chi^s = \frac{r\underline{\omega}^s}{\alpha\sigma(1-\bar{A})\theta S(\underline{\omega}^s)}. \tag{82}$$

This system is solved recursively: (81) gives $\underline{\omega}^s$ while (82) gives χ^s. It is easy to check that (81) admits two solutions, as $S(\cdot)$ is hump-shaped. From Part 1, since $\chi^s = 1$ in the steady state $\bar{\omega}^s$,

$$\psi \leq \alpha\sigma\bar{A}(1-\theta)S(\bar{\omega}^s),$$

the lowest candidate solution for $\underline{\omega}^s$ is smaller than $\bar{\omega}^s$ while the largest candidate solution is greater than $\bar{\omega}^s$. Given that from (82) χ^s is increasing with $\underline{\omega}^s$ and $\chi^s \geq 1$ when $\underline{\omega}^s = \bar{\omega}^s$, it follows that the unique steady state with partial acceptability corresponds to the lowest solution to (81). From the phase diagram, one can check that pairs (A_t, ω_t) in the positive quadrant located underneath the trajectory leading to the steady state with full acceptability and such that $A_t \leq \bar{A}$ form a basin of attraction for the steady state with partial acceptability. Hence, all equilibria such that $\omega_0 \in (0, \hat{\omega}_0)$ lead to $(\underline{\omega}^s, \bar{A})$. ■

Proof of Proposition 7. Part 1: The determinant of the Jacobian matrix of the system of ODEs given by (40) and (49) in the neighborhood of (A^s, ω^s) is negative. Hence, by the argument in the proof of Proposition 6, there exists a unique equilibrium path converging to (A^s, ω^s). To characterize the equilibrium path, we focus first on the state space where $m = 1 - A$, which corresponds to the black dashed line in Figure 12. Since the first term in the min operator of (50) rises in q, the measure of miners is $m = 1 - A$ if and only if (A_t, q_t) lies on or above the line characterized by

$$\frac{\omega(q)}{u(q)-q} = \frac{\alpha\sigma(1-\theta)(1-A)}{\pi(A)} \tag{83}$$

in the (A, q) space, which defines a positive relationship between q and A.

Next, we characterize the state space where $\dot{\omega} > 0$. When $m = 1 - A$, clearly $\dot{\omega} > 0$ by (49). When $m \in (0, 1-A)$, substituting the value of m from (50) into (49) and assuming $\dot{\omega} = 0$, we have

$$\frac{\omega(q)}{u(q)-q} = \frac{\alpha\sigma(1-\theta)(1-A/\theta)}{\frac{1-\theta}{\theta}r + \pi(A)}. \tag{84}$$

Hence $\dot{\omega} > 0$ if and only if (A_t, q_t) lies above the line characterized by (84). But (84) characterizes a line that is strictly below (83) in the (A, q) space. Hence, if (A_t, q_t) lies on or above (84), then either $m = 1 - A$ or $m < 1 - A$, but in both cases $\dot{\omega} \geq 0$. If (A_t, q_t) lies below (84), then $m < 1 - A$ and $\dot{\omega} < 0$. Since $\omega(q)$ increases in q by (8), Equation (83) and (84) also define a locus in the (A, ω) space, we plot them in Figure 12 (dashed black and solid blue line respectively).

Call the line characterized by (84) the q *locus*. Since $\dot{q} > 0$ iff (A_t, q_t) lies above the q locus, the trajectory (A_t, q_t) can change direction (from up to down or from down to up) at most the same number of times as the q locus does. We now show the q locus is either increasing or hump-shaped. Let $\nu(A)$ be the right side of (84). Since the left side of (84) rises in q, the q locus rises in the (A, q) space if and only if $\nu'(A) > 0$. So it suffices to show that $\nu(A)$ is either increasing in A or first increases and then decreases in A for $A \leq \bar{A}$. By (84) the slope of $\nu(A)$ is proportional to

$$\frac{d\nu(A)}{dA} \propto \zeta(A) \equiv [\lambda\theta - (1-\theta)r]A^2 - 2\theta\lambda\bar{A}A + \theta^2\lambda\bar{A},$$

where $\zeta(0) = \theta^2\lambda\bar{A} > 0$, and hence $\nu(A)$ is initially rising. If $\lambda\theta - (1-\theta)r \leq 0$, then the right side falls in A. In this case $\nu(A)$ is either increasing or first rises and then falls. Next, assume $\lambda\theta - (1-\theta)r > 0$ so that $\zeta(A)$ is U-shaped in A. The turning point of $\zeta(A)$ is $\tilde{A} \equiv \theta\lambda\bar{A}/[\lambda\theta - (1-\theta)r]$ and $\zeta(\tilde{A}) > 0$ iff $\lambda\frac{(\theta-\bar{A})}{1-\theta} > r$. Hence, $\nu(A)$ rises in A when $\lambda\frac{(\theta-\bar{A})}{1-\theta} > r$. If $\lambda\frac{(\theta-\bar{A})}{1-\theta} \leq r$, then $\tilde{A} \geq \theta > \bar{A}$, in this case $\nu(A)$ is either increasing or first rises and then falls. Altogether, the q locus is either rising or hump-shaped. It follows that the trajectory (A_t, q_t) must either be increasing, decreasing, or first increasing and then decreasing.

But since the q locus is strictly to the right of the y-axis when $q = 0$ by (84) (the blue line in Figure 12), the trajectory (A_t, q_t) must lie above the q locus when $A \approx 0$, and hence is initially rising. Thus, (A_t, q_t) is either increasing or hump-shaped. It is increasing if and only if (A^s, q^s) lies in the increasing part of the

q locus. Therefore if $\nu'(\bar{A}) \propto \zeta(\bar{A}) \geq 0$ then (A^s, q^s) is increasing and if $\zeta(\bar{A}) < 0$ then it is hump-shaped. Since $\zeta(\bar{A}) = \bar{A}[-(1-\theta)r\bar{A} + \theta\lambda(\theta - \bar{A})]$, $\zeta(\bar{A}) \geq 0$ iff $\lambda > \bar{\lambda}$.

Part 2: Let $\hat{q}_0$ be the initial value of q in the equilibrium described in Part 1. By the proof logic of Proposition 3, different equilibrium trajectories cannot cross each other. Hence, if $q_0 < \hat{q}_0$, then the trajectory must stay strictly below the trajectory leading to $(\bar{A}, q^s)$ at all $t < \infty$. By (40) and (49), the slope of any trajectory is

$$\frac{dq}{dA} = \frac{\dot{\omega}}{\omega'(q)\dot{A}} = \frac{r\omega(q) - \alpha\sigma[\theta(1-m) - A]\,[u(q) - q]}{\omega'(q)\pi(A)A}.$$

At any point $(\bar{A}, q')$ where $q' \in (0, q^s)$, the slope of the trajectory is $dq/dA = -\infty$, hence no trajectory can converge to $(\bar{A}, q')$. Also there cannot be any trajectory converging to any $(A', 0)$ for $A' < \bar{A}$. Suppose there exists one. Since $\dot{q}$ becomes vanishing small as $q \to 0$, the trajectory converges to the x-axis asymptotically. But since $\dot{A} = \pi(A)A > 0$ whenever $q > 0$, the money supply eventually reaches $\bar{A}$ and thus cannot converge to A'. It follows that any trajectory must eventually converge to either $(\bar{A}, 0)$ or $(\bar{A}, q^s)$. But there is a unique trajectory converging to $(\bar{A}, q^s)$ by Part 1, hence all other trajectories must converge to $(\bar{A}, 0)$. The rest of the results follow directly from the phase diagram in Figure 12. ■

Proof of Proposition 8.

A necessary condition for $\dot{\omega} = 0$ is $m < 1 - A$. Substituting m by its expression from (50) into (49) and setting $\dot{\omega} = 0$ yields (52). ■

Online Appendix for "Money Mining and Price Dynamics"

May 2020

A Granger test

In this section we test whether the prices of gold and Bitcoin affect their production or mining intensity.

Gold: We use the historical mine production index and purchasing power of gold from Jastram (2009). This is an annual data covering 1870-1970, see Figure 1. Consider the two-variable VAR

$$\begin{bmatrix} \text{production}_t \\ \text{price}_t \end{bmatrix} = \mathbf{b_0} + \mathbf{B_1} \begin{bmatrix} \text{production}_{t-1} \\ \text{price}_{t-1} \end{bmatrix} + \cdots + \mathbf{B_k} \begin{bmatrix} \text{production}_{t-k} \\ \text{price}_{t-k} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

where $\mathbf{b_0}$ is a vector of intercept terms and each of $\mathbf{B_1}$ to $\mathbf{B_k}$ is a matrix of coefficients. The lag length $k = 3$ is recommended by the likelihood ratio test, final prediction error and Akaike's information criterion.

We use the Granger test to test the null hypothesis that all coefficients on lags of the price in the production equation are equal to zero, against the alternative that at least one is not non-zero. The p-value is 0.02 and thus we conclude that the real price of gold Granger-causes the production at the 5% level.

Bitcoin: We use the monthly data on mining difficulty and Bitcoin price from the web site Bitcoinity, covering the period Aug 2010 to Oct 2018. We consider the following VAR model

$$\begin{bmatrix} \text{growth of diff level}_t \\ \text{growth of price}_t \end{bmatrix} = \mathbf{b_0} + \mathbf{B_1} \begin{bmatrix} \text{growth of diff level}_{t-1} \\ \text{growth of price}_{t-1} \end{bmatrix} + \cdots + \mathbf{B_k} \begin{bmatrix} \text{growth of diff level}_{t-k} \\ \text{growth of price}_{t-k} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}.$$

The recommended lag is $k = 2$ and the p-value of the causality test is 0.0004. Hence we conclude that the growth rate of prices Granger-causes the growth rate of the difficulty level at the 1% level.

B Search while mining

Now we allow agents to produce and mine at the same time, specifically we let miners produce with probability $\eta \in [0,1]$ when in contact with a buyer. The opportunity cost of mining becomes $C(1) = \alpha\sigma(1-\eta)A(-q + V_1 - V_0)$ and the probability that a random non-money holder can produce is then

$$\chi_t = \frac{1 - A - m(1-\eta)}{1-A}.$$

The gold mining model in Section 3 is the special case $\eta = 0$. Another polar case is $\eta = 1$ where agents can engage in mining without forgiving any trading opportunity. We will show that this possibility can change the dynamics of prices depending on the efficiency of the mining technology.

Proposition 9 ***(Search while mining)*** *There exists a steady-state monetary equilibrium iff*

$$r < \frac{\alpha\sigma}{1-\theta}\left[\frac{\alpha\sigma\theta(1-\eta) + \lambda\left(\theta - \bar{A}\right)}{\alpha\sigma(1-\eta)+\lambda}\right]. \tag{85}$$

The steady-state money supply, A^s, increases with η while the value of money, q^s, decreases with η.

Suppose $\eta = 1$. There exists a monetary equilibrium if $r < \alpha\sigma\left(\theta - \bar{A}\right)/(1-\theta)$ and it is such that A^s tends to $\bar{A} < \theta$. For all $A_0 < \bar{A}$ the unique equilibrium leading to the steady state is such that: A increases over time until it reaches $\bar{A}$; q increases over time if $\lambda > r/\left(\theta - \bar{A}\right)$, decreases if $\lambda < r/\left(\theta - \bar{A}\right)$, and remains constant if $\lambda = r/\left(\theta - \bar{A}\right)$.

Proof of Proposition 9. Agents' value functions solve:

$$rV_1 = \alpha\sigma\left[1 - A - m(1-\rho)\right]\theta\left[u(q) - q\right] + \dot{V}_1 \tag{86}$$

$$rV_0 = \alpha\sigma A\rho(1-\theta)\left[u(q)-q\right] + \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A}-A\right)\omega(q)\right\} + \dot{V}_0. \tag{87}$$

The key novelty in (87) is that the opportunity cost of mining has been multiplied by $1-\rho$. In particular, if $\rho = 1$ there is no opportunity cost of mining and all agents without money mine. Subtracting (87) from (86) the value of money solves:

$$\begin{aligned} r\omega(q) = & \left\{1 - \left[1 + \rho\left(\frac{1-\theta}{\theta}\right)\right]A - m(1-\rho)\right\}\theta\alpha\sigma\left[u(q)-q\right] \\ & - \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A}-A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned} \tag{88}$$

The law of motion for A is:

$$\dot{A} = m\lambda\left(\bar{A}-A\right). \tag{89}$$

The locus of pairs (A, q) such that agents are indifferent between mining or not is given by:

$$A = \mu(q) \equiv \frac{\lambda \bar{A} \omega(q)}{\alpha\sigma(1-\rho)(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)}$$

The μ-locus shifts to the right as ρ increases and it becomes vertical at $A = \bar{A}$ when $\rho = 1$.

By the same reasoning as in Section 3.1, q^s solves (18),

$$r\omega(q) = (\theta - A)\,\alpha\sigma\left[u(q) - q\right],$$

and A^s is the smallest root to

$$\lambda\left(\bar{A} - A\right)(\theta - A) - A(1-\eta)(1-\theta)r = 0. \tag{90}$$

It is easy to check that A^s increases with ρ while q^s decreases with η. Moreover, as η approaches to 1, A^s approaches to $\min\{\theta, \bar{A}\}$. By the same reasoning as in the proof of Proposition 1 there exists a steady-state monetary equilibrium iff

$$\lim_{q\to 0}\left\{r\omega(q) - \left[\theta - \mu(q)\right]\alpha\sigma\left[u(q) - q\right]\right\} < 0.$$

Dividing by $\omega(q) > 0$ this condition can be rewritten as:

$$\lim_{q\to 0}\left\{r - \alpha\sigma\frac{\left[\theta - \mu(q)\right]\left[u(q) - q\right]}{\omega(q)}\right\} < 0.$$

Using that $\lim_{q\to 0}\left\{\left[u(q) - q\right]/\omega(q)\right\} = 1/(1-\theta)$ and $\lim_{q\to 0}\mu(q) = \lambda\bar{A}/\left[\alpha\sigma(1-\eta) + \lambda\right]$ the condition above can be rewritten as (85). In particular, when $\eta = 1$,

$$r < \frac{\alpha\sigma}{1-\theta}\left(\theta - \bar{A}\right).$$

In that case a necessary condition for a steady-state monetary equilibrium is $\bar{A} < \theta$. Hence, $A^s = \theta < \bar{A}$.

The condition $\alpha\sigma(\theta - \bar{A}) > r(1-\theta)$ guarantees the existence of a steady-state monetary equilibrium when $\eta = 1$. The system of ODEs, (88) and (89), becomes:

$$\begin{aligned} \omega'(q)\dot{q} &= \left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - (\theta - A)\,\alpha\sigma\left[u(q) - q\right] \\ \dot{A} &= \lambda(1 - A)\left(\bar{A} - A\right) \end{aligned}$$

Linearizing the system around the steady state we obtain:

$$\begin{pmatrix} \dot{q} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} \frac{r\omega'(q^s) - \left(\theta - \bar{A}\right)\alpha\sigma\left[u'(q^s) - 1\right]}{\omega'(q^s)} & \frac{-\lambda\omega(q^s) + \alpha\sigma\left[u(q^s) - q^s\right]}{\omega'(q^s)} \\ 0 & -\lambda(1 - \bar{A}) \end{pmatrix}\begin{pmatrix} q - q^s \\ A - A^s \end{pmatrix}.$$

If $\left(\theta-\bar{A}\right)\alpha\sigma > r(1-\theta)$ then $r\omega'(q^s) > \left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]$. It follows that the determinant of the Jacobian matrix is negative, i.e., the steady state is a saddle point. The negative eigenvalue is $e_1 = -\lambda(1-\bar{A})$ and the associated eigenvector is

$$\overrightarrow{v}_1 = \begin{pmatrix} \frac{\left[\lambda - r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]} \\ 1 \end{pmatrix}$$

where we used that $r\omega(q^s) = \left(\theta-\bar{A}\right)\alpha\sigma\left[u(q^s)-q^s\right]$. The first component of $\overrightarrow{v}_1$ is of the same sign as $\lambda - r/\left(\theta-\bar{A}\right)$. The solution to the linearized system is

$$\begin{pmatrix} q-q^s \\ A-A^s \end{pmatrix} = Ce^{-\lambda(1-\bar{A})t}\overrightarrow{v}_1,$$

where C is some constant. Hence, in the neighborhood of the steady state,

$$\frac{\partial q}{\partial A} = \frac{\left[\lambda - r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]},$$

which is of the same sign as $\lambda - r/\left(\theta-\bar{A}\right)$. If $\lambda > r/\left(\theta-\bar{A}\right)$, then the saddle path in the neighborhood of the steady state is upward sloping, i.e., q and A increase over time. We can show that this result holds globally since the equation of the q-isocline is:

$$\frac{\omega(q)}{u(q)-q} = \frac{(\theta-A)\,\alpha\sigma}{r+\lambda\left(\bar{A}-A\right)}.$$

The q-isocline is upward sloping when $\lambda > r/\left(\theta-\bar{A}\right)$. See left panel of Figure 15. By the same reasoning, if $\lambda < r/\left(\theta-\bar{A}\right)$, then the saddle path is downward sloping and along the equilibrium path, q decreases while A increases. See middle panel of Figure 15. Finally, if $\lambda = r/\left(\theta-\bar{A}\right)$, then the q-isocline is horizontal. In that case q is constant over time. See right panel of Figure 15. ■

According to (85) the set of parameter values for which a steady-state monetary equilibrium exists shrinks as η increases. If agents can meet trading partners more frequently while mining, then the opportunity cost of mining is lower and the incentives to mine are greater, which leads to a higher supply of money. But for a monetary equilibrium to exist, the money supply cannot be too large. A higher η also reduces the value of money. In the limiting case where $\eta = 1$, there is no opportunity cost to engage in mining and all agents without money mine, $m = 1 - A$. At the steady state the money supply is equal to the maximum stock of money that could be mined, $\bar{A}$. We now turn to the transition dynamics for this special case.

Proposition 9 shows that when there is no opportunity cost of mining, the correlation between the value of money and the money stock along the transitional path depends on the efficiency of the mining technology.[22]

[22] While Proposition 9 focuses on the unique equilibrium leading to the steady state, there is also a continuum of equilibria where the value of money vanishes asymptotically. In the left panel of Figure 15, when λ is high, the value of money increases

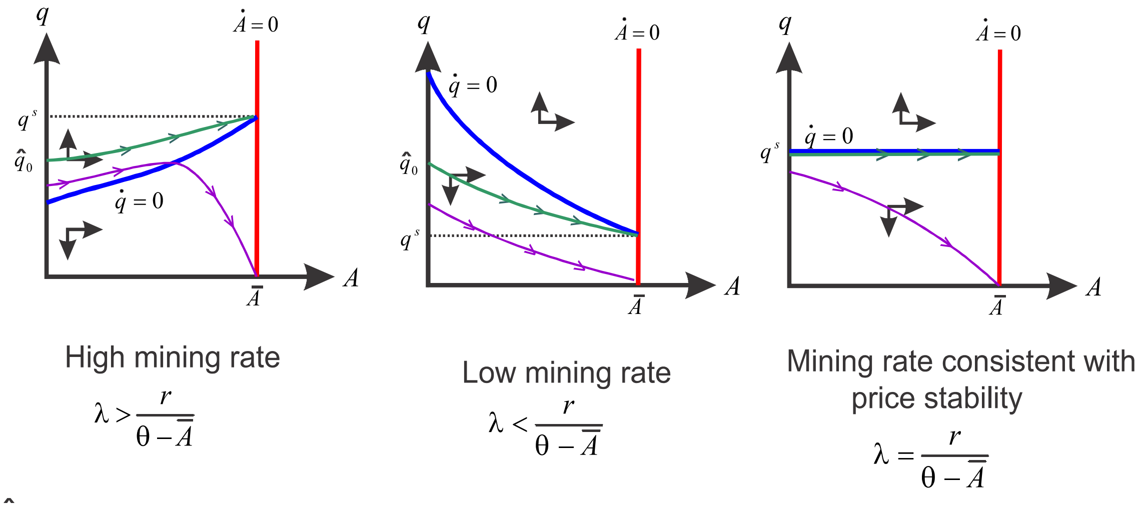

Figure 15: Phase diagrams when agents can mine while searching for trading partners ($\eta = 1$).

If the mining intensity is high, the value of money increases with the money supply. If the mining intensity is low, then the opposite correlation prevails and the value of money decreases as the money supply increases. Finally, there is a mining rate such that the price level is constant, the value of money is independent of the money stock.

first and then decreases. In the middle and right panels, when λ is low, the value of money is monotone decreasing in time.

C General matching function

Consider the gold mining model in Section 3. But suppose now that only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function. The matching probability of a buyer is $\alpha(\tau)$ where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. As is standard, we assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. A matching function that satisfies these properties is the Cobb-Douglas matching function.

The HJB equations of agents with and without money are:

$$rV_1 = \alpha(\tau)\sigma\theta\left[u(q) - q\right] + \dot{V}_1 \tag{91}$$

$$rV_0 = \max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \dot{V}_0. \tag{92}$$

The novelty is that the matching rate of a buyer is $\alpha(\tau)$ while the matching rate of a seller is $\alpha(\tau)/\tau$. Using that $\lim_{\tau\to 0}\alpha(\tau)/\tau = +\infty$, it follows that $\tau > 0$ in equilibrium, i.e., $m < 1 - A$. The goods market is always active and

$$\max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} = \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right]. \tag{93}$$

Subtracting (92) from (91) the value of money solves:

$$r\omega(q) = \left[\alpha(\tau)\sigma\theta - \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\right]\left[u(q) - q\right] + \omega'(q)\dot{q}. \tag{94}$$

From (93) market tightness in the goods market solves:

$$\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right] \geq \lambda\left(\bar{A} - A\right)\omega(q), \quad \text{“} = \text{"} \text{ if } \tau < \frac{1-A}{A}.$$

Solving for τ we obtain:

$$\tau(\omega, A) = \min\left\{g^{-1}\left[\frac{\lambda\left(\bar{A} - A\right)\omega}{\sigma(1-\theta)S(\omega)}\right], \frac{1-A}{A}\right\}. \tag{95}$$

where $S(\omega) \equiv u\left[q(\omega)\right] - q(\omega)$ and $g(\tau) \equiv \alpha(\tau)/\tau$. For all (ω, A) such that $\frac{\lambda(\bar{A}-A)\omega}{\sigma(1-\theta)S(\omega)} \geq g\left(\frac{1-A}{A}\right)$, $m > 0$ and $\tau(\omega, A)$ is decreasing in ω and increasing in A. Moreover, $\tau(+\infty, A) = 0$ and $\tau(0, A) > 0$. The money supply evolves according to

$$\dot{A} = \left[1 - A\left(1 + \tau\right)\right]\lambda\left(\bar{A} - A\right), \tag{96}$$

where we used that $1 - A\left(1 + \tau\right) = m$.

We summarize the equilibrium by a system of two ODEs in ω and A:

$$\dot{\omega} = r\omega - \{\alpha [\tau (\omega, A)] \sigma\theta - g [\tau (\omega, A)] \sigma(1-\theta)\} S(\omega) \tag{97}$$

$$\dot{A} = \{1 - A [1 + \tau (\omega, A)]\} \lambda \left(\bar{A} - A\right). \tag{98}$$

The locus of the points such that $\dot{A} = 0$ corresponds to all pairs (ω, A) such that $\tau(\omega, A) = (1-A)/A$. From (95) it is given by:

$$\frac{\lambda \left(\bar{A} - A\right) \omega}{\sigma(1-\theta)S(\omega)} \leq g\left(\frac{1-A}{A}\right). \tag{99}$$

Condition (99) at equality gives a positive relationship between ω and A. As ω approaches 0, A tends to the solution to $\lambda \left(\bar{A} - A\right) = \sigma g \left(\frac{1-A}{A}\right)$. As ω tends to $+\infty$, A tends to $\bar{A}$. This locus is represented by a red upward-sloping curve in Figure 16.

The locus of the points such that $\dot{\omega} = 0$ and $\dot{A} > 0$ is such that

$$r\frac{\omega}{S(\omega)} = \{\alpha [\tau (\omega, A)] \sigma\theta - g [\tau (\omega, A)] \sigma(1-\theta)\}. \tag{100}$$

The left side is increasing in ω while the right side is decreasing in ω but increasing in A. For given A there is a unique ω solution to (100) provided that

$$r(1-\theta) < \{\alpha [\tau (0, A)] \sigma\theta - g [\tau (0, A)] \sigma(1-\theta)\},$$

where $\tau(0, A)$ is the solution to $g(\tau) = \lambda \left(\bar{A} - A\right)/\sigma$. If this condition holds for $A = 0$, then it holds for all A. Hence, we assume

$$r(1-\theta) < [\alpha (\tau_0) \sigma\theta - g (\tau_0) \sigma(1-\theta)] \text{ where } \tau_0 = g^{-1}\left[\lambda \left(\bar{A} - A\right)/\sigma\right]. \tag{101}$$

Assuming this condition is satisfied, the ω-isocline is upward sloping as illustrated in Figure 16. As A goes to zero, ω tends to a positive value.

There is a unique steady state such that agents are indifferent between mining or not and it solves

$$g(\tau) = g\left(\frac{1-A}{A}\right) = \frac{\lambda \left(\bar{A} - A\right) \omega}{\sigma(1-\theta)S(\omega)} \tag{102}$$

$$r\frac{\omega}{S(\omega)} = [\alpha (\tau) \sigma\theta - g (\tau) \sigma(1-\theta)]. \tag{103}$$

Equation (102) specifies the market tightness such that agents are indifferent between mining or participating in the goods market. Equation (103) gives the value of money given market tightness. Combining (102) and

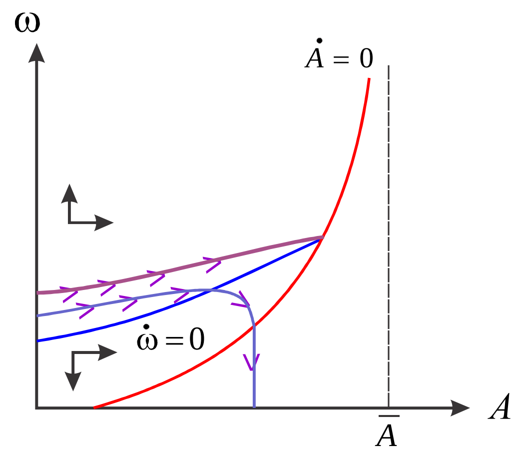

Figure 16: Phase diagram under matching function satisfying Inada conditions.

(103), steady-state market tightness solves:

$$\sigma(1-\theta)\left[\frac{r}{\lambda\left(\bar{A}-\frac{1}{1+\tau}\right)}+1\right]g(\tau)=\alpha(\tau)\,\sigma\theta. \tag{104}$$

It is easy to check that there is a unique $\tau^s \in \left(0, \frac{1-\bar{A}}{\bar{A}}\right)$ solution to this equation. The supply of money at the steady state is then $A^s = 1/(1+\tau^s)$. The equilibrium is monetary if (101) holds. The existence of a unique steady state guarantees that the A-isocline and ω-isocline only intersect once, i.e., the ω-isocline is located above the A-isocline as illustrated in Figure 16.

In Figure 16 we represent the phase diagram of the dynamic system (97)-(98) and its arrows of motion. It can be checked that the steady state is a saddle path and given the initial condition $A_0 = 0$ there is a unique path leading to it. Along that path the value of money rises over time. There is also a continuum of other equilibria where the value of money vanishes asymptotically.

In order to characterize the path for market tightness, we can rewrite (100) as

$$r\frac{\omega(\tau,A)}{S[\omega(\tau,A)]}=[\alpha(\tau)\,\sigma\theta-g(\tau)\,\sigma(1-\theta)],$$

where $\omega(\tau,A)$ is defined implicitly by $\tau=\tau(\omega,A)$. Assuming $m>0$, ω is a decreasing function of τ and an increasing function of A. Hence, the τ-isocline is upward sloping. The A-isocline becomes $A=1/(1+\tau)$. By the same reasoning as above, the saddle path is upward sloping, which means that τ increases over time.

D Interest-bearing/commodity monies

Suppose money is a commodity that provides some direct utility, e.g., gold or silver, or a financial asset that pays dividends. We denote $d>0$ the dividend flow enjoyed by each money holder. The Bellman equation of a money holder is:

$$rV_1 = d + \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] + \dot{V}_1. \tag{105}$$

The only novelty is the first term on the right side representing the dividend flow. The Bellman equation for an agent without money is unchanged. It follows that the dynamic equation for the value of money is:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] \\ &\quad - \max\left\{\alpha\sigma A(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned} \tag{106}$$

A steady-state equilibrium, (q^s, A^s), solves:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma\left(\theta - A\right)\left[u(q) - q\right] \\ A &= \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1-\theta)\left[u(q) - q\right] + \lambda\omega(q)} \end{aligned} \tag{107}$$

The first equation gives a negative relationship between q and A while the second equation gives a positive relationship between A and q. So there is a unique steady state and $\partial q^s/\partial d > 0$ and $\partial A^s/\partial d > 0$.

Out of steady state, if $m<1-A$, then the trajectory is $A=\mu(q)$ as in the baseline. If $m=1-A$, then:

$$\dot{q} = \frac{\left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - d}{\omega'(q)} \tag{108}$$

$$\dot{A} = (1-A)\lambda\left(\bar{A} - A\right). \tag{109}$$

The slope $\partial q/\partial A = \dot{q}/\dot{A}$ falls in d for any given (A, q), but one can show that $\dot{q}>0$ at all time. If $\dot{q}=0$ at some time t, then $\dot{q}<0$ after t by (108). The trajectory cannot change regime after t as a regime switch requires both trajectories to have the same slope but the locus $A=\mu(q)$ is always upward sloping. In the regime $m=1-A$, $\dot{q}=\alpha\sigma(1-A)\theta\left[u(q)-q\right]/\omega'(q)>0$ when $q\approx q^s$ by (106) and (107). Hence $\dot{q}\neq 0$ at all t.

By the proof of Proposition 2, mining and production co-exist near the steady state if only if

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} > \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)} \iff \frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

As d increases there are two opposing effects. Since $\partial q/\partial A|_{m=1-A}$ falls in d for any given (A, q), it is more likely that $m = 1 - A$ near the steady state when d is large. On the other hand q^s and A^s increase in d and therefore agents have less incentive to mine around the steady state. The net effect is ambiguous in general.

E Divisible assets

We now study price dynamics when all assets are perfectly divisible and individual asset holdings are unrestricted, $a \in \mathbb{R}_+$. This model, which is a continuous-time version of the New-Monetarist model of Lagos and Wright (2005), will be useful to check the robustness of our earlier results. Choi and Rocheteau (2019b) provide a detailed description of the New Monetarist model in continuous time and its solution methods.

Consider the gold mining model in Section 3. We add a centralized market (CM) where price-taking agents can trade continuously a good, distinct from the one traded in pairwise meetings, for money. The purpose of these CMs is to allow agents to readjust their money holdings to some targeted level in-between pairwise meetings, so as to keep the distribution of money holdings degenerate. In reality, the CMs could correspond to the several exchanges where individuals trade crypto-currencies for different government-supplied currencies using credit or debit cards (e.g., Coinbase, Coinmama, Luno...). In the following we take the CM good as the numéraire. Agents have the technology to produce h units of the numéraire good at a linear cost h ($h < 0$ is interpreted as consumption). Hence, agents' discounted lifetime utility in-between pairwise meetings is $-\int_0^{+\infty} e^{-rt} dH(t)$ where $H(t)$ is a measure of the cumulative production of the numéraire good (net of its consumption) up to t. This formulation allows agents to produce or consume the numéraire good in flows (in which case $H(t)$ admits a density $h(t)$) or in discrete amounts (in which case $H(t^+) - H(t^-) \neq 0$). Preferences during pairwise meetings are as before. Money is a Lucas tree that pays a dividend flow $d \geq 0$. The case $d = 0$ corresponds to fiat money. The CM price of the asset is denoted ϕ_t.

Let $V(a)$ be the value function of an agent with a units of assets expressed in terms of the numéraire. At any point in time between pairwise meetings, an agent can readjust her asset holdings by consuming or producing the numéraire good. Formally,

$$V(a) = \max_h \{-h + V(a+h)\} = a + \max_{a^* \geq 0} \{-a^* + V(a^*)\},$$

where h is the production of the numéraire and a^* is the agent's targeted asset holdings (expressed in terms of the numéraire). The value function, $V(a)$, is linear in a.

We now consider the bargaining problem in a pairwise meeting between a buyer holding a^b units of assets and a seller holding a^s units of asset. The outcome of the negotiation is a pair $(q, p) \in \mathbb{R}_+ \times [-a^s, a^b]$ where q is the amount of goods produced by the seller for the buyer and p is the transfer of assets from the buyer to the seller. Feasibility requires that $-a^s \leq p \leq a^b$. By the linearity of $V(a)$ the buyer's surplus is

$u(q) + V(a^b - p) - V(a^b) = u(q) - p$ and the seller's surplus is $-q + V(a^s + p) - V(a^s) = -q + p$. According to the Kalai proportional solution, the buyer's surplus is equal to a fraction θ of the total surplus of the match, i.e., $u(q) - p = \theta\,[u(q) - q]$. Moreover, the solution is pairwise Pareto efficient, which implies that $q \leq q^*$ with an equality if $p \leq a^b$ does not bind. Using the notation $\omega(q)$ from (8), the buyer's consumption as a function of her asset holdings, $q(a^b)$, is such that $q(a^b) = q^*$ if $a^b \geq \omega(q^*)$ and $\omega(q) = a^b$ otherwise.

Consider the lifetime expected utility of the agent holding her targeted asset holdings, $V(a^*)$. Choi and Rocheteau (2019b) show that it solves the following HJB equation that is similar to (9) and (10) combined:

$$rV(a^*) = \rho a^* + \alpha(1-m)\sigma\theta\left\{u\left[q(a^*)\right] - q(a^*)\right\}$$
$$+\max\left\{\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}, \lambda(\bar{A} - A)\phi\right\} + \dot{V}(a^*), \tag{110}$$

where the rate of return of assets is

$$\rho = \frac{d + \dot{\phi}}{\phi}. \tag{111}$$

The first term on the right side of (110) is the flow return of the asset. The second term is analogous to the first term on the right side of the HJB equation for V_1, (9). The agent receives an opportunity to consume at Poisson arrival rate $\alpha\sigma$. The partner can produce if she is not a miner, with probability $1 - m$. The third term on the right side of (110) is analogous to the right side of the HJB equation for V_0 in (10). It corresponds to the occupational choice according to which agents can choose to be producers and enjoy the flow payoff $\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}$ or miners and enjoy $\lambda(\bar{A} - A)\phi$. The term $\bar{a}$ represents asset holdings of other agents in the economy. The expected gain from mining describes the assumption that at Poisson arrival rate $\lambda(\bar{A} - A)$ the miner digs a unit of money which is worth ϕ units of numéraire. The last term is the change in the value function for a given asset position, $\dot{V}(a) = \partial V_t(a)/\partial t$.

The envelope condition associated with (110) together with $V'(a^*) = 1$ gives

$$r - \rho = \alpha(1-m)\sigma\theta\left\{\frac{u'\left[q(a^*)\right] - 1}{(1-\theta)u'\left[q(a^*)\right] + \theta}\right\}, \tag{112}$$

where we used $q'(a) = 1/\omega'(q)$ if $a < \omega(q^*)$ and $\partial^2 V(a)/\partial a \partial t = 0$ as $V'(a^*) = 1$ at all t. The opportunity cost of holding the asset on the left side is the difference between the rate of time preference and the real rate of return of the asset. The right side is the marginal value of an asset if a consumption opportunity arises.

Since now agents can carry money and mine at the same time, the measure of miners solves

$$m \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \text{ if } \lambda(\bar{A} - A)\phi \begin{cases} > \\ = \\ < \end{cases} \alpha\sigma(1-\theta)\left[u\left(q\right) - q\right]. \tag{113}$$

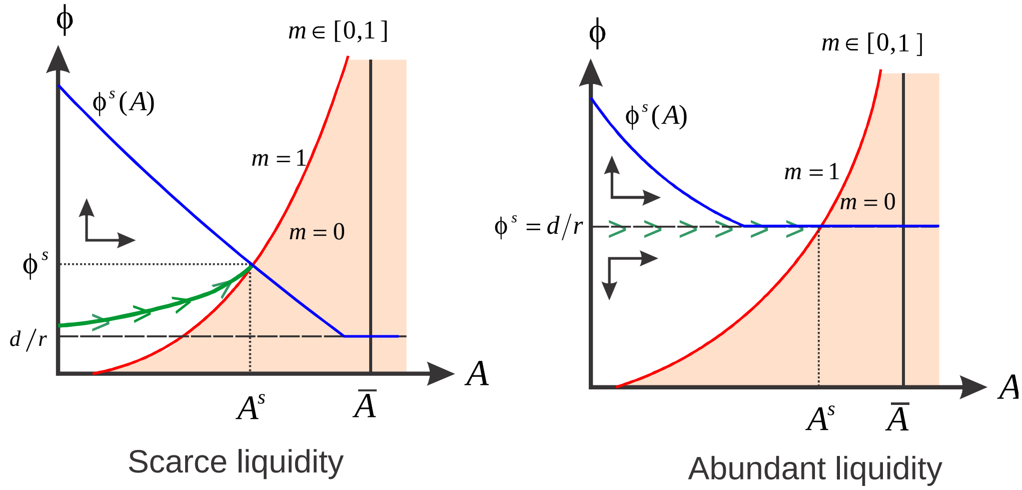

Figure 17: Phase diagram with divisible assets.

By market clearing:

$$a^* = \phi A. \tag{114}$$

The supply of assets evolves according to:

$$\dot{A} = \lambda m(\bar{A} - A). \tag{115}$$

An equilibrium is a list, $\langle a_t^*, m_t, \phi_t, A_t \rangle$, that solves (112), (113), (114), and (115).

We represent the phase diagram and equilibrium trajectory in Figure 17. There are two main insights relative to the model with indivisible money and no centralized exchanges. First, there is a regime where the asset supply at the steady state is abundant enough to satiate agents' liquidity needs and to allow agents to trade q^* in all matches. In such equilibria, the asset is priced at its fundamental value at all dates, see the right panel of Figure 17. A necessary (but not sufficient) condition is that the potential asset supply when valued at its fundamental price, $\bar{A}d/r$, is larger than agents' liquidity needs, $\omega(q^*)$. It is the standard condition in the literature for abundant liquidity since Geromichalos et al. (2007), except that it applies to the potential asset supply, $\bar{A}$, and not the actual asset supply, A, which is endogenous. The second insight is that there is a regime with scarce liquidity that is qualitatively similar to the equilibria of the model with indivisible money. The price of the asset is above its fundamental value at all dates and it keeps increasing over time until it reaches a steady state as shown in the left panel of Figure 17.