Characterization and Implementation of the Shapley Value for Fair Reordering Problems

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Outline

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   - Fair Reordering Problems
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2. The Model
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Take a number and wait

▲ People queue to receive a dose of Covid-19 vaccine at a healthcare centre in Delhi on Saturday. Photograph: Prakash Singh/AFP/Getty Images

Fair allocation problems

- Toy allocation
- Airport problem (cost sharing problem)
- Bankruptcy problem
- Estate division problem
- Profit division problem
- Queueing problem
- Reordering problem: queueing problem with an initial queue

Research approach

- Axiomatic approach:
  applying the fairness properties to explore and compare the solutions. A social planner is required in this approach.

- Strategic approach:
  design a non-cooperative game to implement a solution.

A motivating example

A variety of priorities can be observed in reality, such as COVID-19 vaccination. Governments in various countries usually list medical staff as a priority group, followed by the elderly, high-risk professionals sequentially.

Senior citizens stand in a long queue for getting the COVID-19 vaccine at a vaccination center, in Mumbai on Tuesday. (ANI Photo)

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Motivations

- In the literature, the fair queuing problem assumes that the players have equal right to the object, that is, no initial allocation.
- In reality, people pick ticket numbers and an initial queue is predetermined.
- When everyone's waiting costs are different, is a first-come, first-served system a better system?

Fair reordering

- different unit waiting costs of the customers
- each customer needs one unit service
- The server serves one customer at a time
- each customer will pick a service order number when entering the system, and the number will not be repeated
- a social planner rearranges the order of services to improve the overall efficiency
- if customers are required to postpone service, they must be compensated for the positions they gave up

Research objective

- establishes a new queuing model with an initial priority
- individually rational
- characterization of the solution concept
- design the market rules for remuneration indicated by the concept of continuous solution for players to continue to play

Queueing problems

- Agents stand to receive a public service and each agent has equal opportunity to be the first-served.
- No two agents can be served simultaneously.
- Each agent demands one unit time of service.
- The agents differ in their unit waiting costs.
- Efficiency requires that agents be served in the order of their waiting costs.
- An efficient queue has to be organized, and monetary compensations should to set up for those who have to wait.

Queueing problems with an initial queue

- Agents stand to receive a public /private service according to their waiting numbers.
- No two agents can be served simultaneously.
- Each agent demands one unit time of service.
- The agents differ in their unit waiting costs.
- Efficiency requires that agents be served in the order of their waiting costs.
- An efficient queue has to be organized, and monetary compensations should to set up for participation the queue.

Relative Literature

- Maniquet(2003). A characterization of the Shapley value in queueing problems, JET.
- Chun(2006). A pessimistic approach to the queueing problem, MSS.
- Ergin(2002). Efficient resource allocation on the basis of priorities, ECTA.
- Chun, Mitra, and Mutuswami(2017). Reordering an existing queue, SCW.

Our results

- We propose a new model which extends the fair queueing problems in literature.
- We show that the Shapley value is the unique solution satisfying the individual rationality, budget balanced, efficiency, independence of irrelevant alternatives and an equal cost sharing property.
- We propose a non-cooperative game to justify the Shapley value, and the social planner does not require any private information.

Reordering problem: setting I

- $N \in \mathbb{N}$: a finite set of agents, $N = \{1, 2, ..., n\}, n \geq 2$, $\mathbb{N}$ denotes the set of natural numbers.
- $\Sigma(N)$: the set of all possible queues, which no two agents are assigned the same position.
- Let $\sigma : N \to \{1, 2, ..., n\}$ be an onto function, which is called a queue, for all $\sigma \in \Sigma(N)$. $\sigma \prec$ denotes the initial priority queue.
- Agent $i \in N$ consumes a position $\sigma_i \in N$.
- $\sigma_i^{\prec}$: an initial priority position for agent i, which is the number that the agent picks in advance.
- $t_i \in \mathbb{R}$: cash transfer of agent i, $\mathbb{R}$: the set of reals. $t \in \mathbb{R}^n$: the profile of cash transfer for agents in N. Note that the cash transfer may be positive, negative or zero.

Reordering problem: setting II

- $\theta_i \geq 0$: agent i's waiting cost per unit of time, at the beginning every agent i incurs the waiting cost $(\sigma_i \prec -1)\theta_i \geq 0$, $i \in N$.
- $\theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n_+$: the profile of waiting costs for N; $\theta_S = (\theta_i)_{i \in S}, S \subseteq N$ is a vector of waiting costs of agents in coalition S.
- Preferences are quasi-linear in position and cash transfer, $u_i(\sigma_i, t_i, \theta_i) = -(\sigma_i - 1)\theta_i + t_i$.
- The net utility gain $U_i(\sigma_i, t_i, ; \theta_i) = (\sigma_i^{\prec} - \sigma_i)\theta_i = u_i(\sigma_i, t_i; \theta_i) + (\sigma_i^{\prec} - 1)\theta_i$, which denotes the net gain when the agent moving from the initial priority position to the other queueing position.

Reordering problem: setting III

- A queueing problem with an initial priority is defined by a list $q = (N, \theta, \sigma^{\prec}) \in 2^{\mathbb{N}} \times \mathbb{R}_+^{|N|} \times \mathbb{N}_+^{|N|} \equiv \mathcal{Q}$.
- An allocation for $q \in \mathcal{Q}$ is a list $z = (\sigma, t) \in \mathbb{N}^n \times \mathbb{R}^n$, which assigns a position σ_i and cash transfer t_i to each agent $i \in N$.
- Let $\Sigma(q)$ be the set of all allocations with $q \in \Sigma(N)$.
- Let φ be be an allocation rule which associates to each problem $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$ a non-empty subset $\varphi(q)$ of allocations.

Fair and IR properties I

The first property requires that the allocation not generate a surplus or deficit at every profile.

Definition 1

An allocation $z = (\sigma, t)$ *is budget balance (BB) if for each profile* θ, $\sum_{i \in N} t_i = 0$.

A weaker concenpt with respect to the initial priority: feasibility requires that the net transfers not exceed the saving in waiting cost achieved by the allocation.

Definition 2

An allocation $z = (\sigma, t)$ *is feasible if for each profile* θ, $\sum_{i \in N} t_i \leq \sum_{i \in N} [\sigma_i^{\prec} - \sigma_i]\theta_i$, *for all* $\sigma \in \Sigma(N)$.

Fair and IR properties II

The next property requires that at all profiles, the reordered queue should minimize the total waiting cost. A queue is efficient for each profile θ if the queue $\sigma = \arg\min_{\sigma' \in \Sigma(N)} \Sigma_{i \in N}(\sigma'_i - 1)\theta_i$. Let $E(\theta)$ be the collection of all efficient queues.

Definition 3

An allocation $z = (\sigma, t)$ is efficient for $q = (N, \theta; \sigma^{\prec})$ if $\sigma \in E(\theta)$.

Note that an efficient queue of a queueing problem with initial priority only depends on the profile of waiting costs θ and is independent of the cash transfer and initial priority. And the efficient queue is unique if the agents have no identical impatience.

Fair and IR properties III

Finally, individual rationality requires that each agent's utility in the new queue is at least as large as the utility she would receive if the jobs were processed according to the initial queue and no transfers are given. If a new queue does not satisfy this property, then agents may not agree to join the reordered system.

Definition 4

An allocation $z = (\sigma, t)$ is individually rational (IR) if for each profile θ, $u_i(\sigma_i, t_i, \theta_i) \geq -(\sigma_i^{\prec} - 1)\theta_i$ for all $i \in N$.

Queueing games I

To solve the queueing problem with an initial priority $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$, define the worth of a coalition and it is a cooperative game deal.
Firstly, for each coalition $S \subseteq N$, the worth of moving from the priority queue to another queue is defined by

$$v_q^{\prec}(S) = \sum_{i \in S} (\sigma_{i|S}^{\prec} - \sigma_{i|S})\theta_i, \forall q \in \mathcal{Q}, \forall i \in S,$$

which is the sum of its members' net gain whenever they move from initial priority position to the new position assigned by σ, assuming that the coalition has the power to be served first, which is suggested in Maniquet(2003).

Queueing games II

Secondly, as suggested in Chun(2006), if the coalition is being the last to be served, then the coalition value of $S \subseteq N$ is defined by

$$v_q^{\prec}(S) = \sum_{i \in S} (|N| - |S| + \sigma_{i|S}^{\prec} - \sigma_{i|S})\theta_i, \forall q \in \mathcal{Q}, \forall i \in S,$$

To attain the efficient queue, let $\sigma^* \in E(\theta)$ then we have

$$v_q^{\prec}(S) = \sum_{i \in S} (\sigma_{i|S}^{\prec} - \sigma_{i|S}^{*})\theta_i, \forall q \in \mathcal{Q}, \forall i \in S, \qquad (1)$$

and

$$v_q^{\prec}(S) = \sum_{i \in S} (|N| - |S| + \sigma_{i|S}^{\prec} - \sigma_{i|S}^{*})\theta_i, \forall q \in \mathcal{Q}, \forall i \in S, \qquad (2)$$

Queueing games III

respectively.

The marginal contribution of an agent $i \in N$ to a coalition S in $v_q^{\prec}, i \notin S$, is a sum of the costs associated to each member of S. In other words, the marginal contribution is composed of the net contribution of the agent's moving, and the cost changes on those agents who follow certain agent in the new efficient queue. For each problem and a queue $\sigma \in \Sigma(N)$, let $F_i(\sigma)$ denote the set of agents who are the followers of agent $i \in N$, that is,
$F_i(\sigma) = \{j \in N | \sigma_j > \sigma_i\}$;
let $P_i(\sigma)$ denote the set of agents who are the predecessors of agent $i \in N$, that is, $P_i(\sigma) = \{j \in N | \sigma_j < \sigma_i\}$.

Queueing games IV

Then the marginal contribution for a queueing problem with an initial priority is defined by
For all $q \in \mathcal{Q}, S \subseteq N, i \in N \setminus S$,

$$v_q^{\prec}(S \cup \{i\}) - v_q^{\prec}(S) = \sum_{j \in S \cup \{i\}} (\sigma_{j|S \cup \{i\}}^{\prec} - \sigma_{j|S \cup \{i\}}^{*})\theta_j - \sum_{j \in S} (\sigma_{j|S}^{\prec} - \sigma_{j|S}^{*})\theta_j$$

$$= (\sigma_i^{\prec} - \sigma_i^{*})\theta_i + \sum_{j \in F_i(\sigma^{\prec}) \setminus F_i(\sigma^{*})} \theta_j + \sum_{j \in F_i(\sigma^{*}) \setminus F_i(\sigma^{\prec})} \theta_j$$

The Shapley value $SV(v_q^{\prec})$

For each problems $(N, \theta; \sigma^{\prec}) \in \mathcal{Q}$ and $i \in N$, the payoff assigned to agent i is given by

$$SV_i(v_q^{\prec}) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}[v_q^{\prec}(S \cup \{i\}) - v_q^{\prec}(S)] \quad (3)$$

Monetary compensation

The following lemma suggests the cash transfer according to the definition of Shapley value.

Lemma 1

For each problem $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$, let $z = (\sigma, t) \in \Sigma(q)$ give the agents utilities correspinding to the Shapley value of $v_q^{\prec}$. Then for $\sigma \in E(\theta)$ and for all $i \in N$,

$$t_i(\sigma_i, \theta_i, \sigma^{\prec}) = \sum_{j \in P_i(\sigma)} \frac{\theta_i}{2} - \sum_{j \in P_i(\sigma^{\prec})} \frac{\theta_i}{2} - \sum_{j \in F_i(\sigma)} \frac{\theta_j}{2} + \sum_{j \in F_i(\sigma^{\prec})} \frac{\theta_j}{2} \tag{4}$$

The Shapley value

The Shapley value is a set of solution defined as follows:

Shapley value, Sh**:** For each $N \in \mathcal{N}$ and each problem $(\theta, \sigma^{\prec}) \in \mathcal{S}^N$,

$$Sh(\theta, \sigma^{\prec}) = \left\{ (\sigma, t) \in \mathcal{E}(\theta, \sigma^{\prec}) \,\middle|\, \begin{array}{c} \sigma \in E(\theta), \text{ and for each } i \in N, \\ t_i \text{ is defined by (4)} \end{array} \right\}.$$

The axioms I

1. Efficiency (E)
   An allocation rule φ satisfies *Efficiency* if and only if for all $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$, and all $z = (\sigma, t) \in \varphi(q)$, $\sigma \in E(\theta)$.
2. Budget Balance (BB)
   An allocation rule φ satisfies *Budget Balance* if and only if for all $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$, and all $z = (\sigma, t) \in \varphi(q)$, $\sum_{i \in N} t_i = 0$.

The axioms II

3. Individual Rationality (IR)
   An allocation rule φ satisfies *Individual Rationality* if and only if for all $q = (N, \theta; \sigma^{\prec}) \in \mathcal{Q}$, and all $z = (\sigma, t) \in \varphi(q)$, $u_i(\sigma_i, t_i, \theta_i) \geq -(\sigma_i^{\prec} - 1)\theta_i$.

4. Pareto Indifference (PI)
   A rule φ satisfies *Pareto indifference* if and only if for each $N \in \mathcal{N}$, each $(\theta, \sigma^{\prec}) \in \mathcal{S}^N$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \mathcal{E}(\theta, \sigma^{\prec})$ such that for each $i \in N$, $U_i(\sigma', t'; \theta, \sigma^{\prec}) = U_i(\sigma, t; \theta, \sigma^{\prec})$, $(\sigma', t') \in \varphi(\theta, \sigma^{\prec})$.

The axioms III

5. Independence of local position change(ILPC)
A rule φ satisfies *independence of local position change* if and only if for each $N \in \mathcal{N}$, each pair of problems $(\theta, \sigma^{\prec}), (\theta', \sigma^{\prec'}) \in \mathcal{S}^N$ in which $\theta = \theta'$ and there are $i, j \in N$ such that $(\sigma_i^{\prec} - \sigma_i^{\prec'})(\sigma_j^{\prec} - \sigma_j^{\prec'}) = -1$ and for each $k \in N \setminus \{i, j\}$, $\sigma_k^{\prec} = \sigma_k^{\prec'}$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \varphi(\theta', \sigma^{\prec'})$, $U_k(\sigma, t; \theta, \sigma^{\prec}) = U_k(\sigma', t'; \theta', \sigma^{\prec'})$ for all $k \in N \setminus \{i, j\}$.

The axioms IV

6. Equal sharing of local (dis)improvement(ESLI)
   A rule φ satisfies *equal sharing of local (dis)improvement* if and only if for each $N \in \mathcal{N}$, each pair of problems $(\theta, \sigma^{\prec}), (\theta', \sigma^{\prec'}) \in \mathcal{S}^N$ in which $\theta = \theta'$ and there are $i, j \in N$ such that $(\sigma_i^{\prec} - \sigma_i^{\prec'})(\sigma_j^{\prec} - \sigma_j^{\prec'}) = -1$ and for each $k \in N \setminus \{i, j\}$, $\sigma_k^{\prec} = \sigma_k^{\prec'}$, each $(\sigma, t) \in \varphi(\theta, \sigma^{\prec})$, and each $(\sigma', t') \in \varphi(\theta', \sigma^{\prec'})$,

$$\begin{aligned} & U_i(\sigma, t; \theta, \sigma^{\prec}) - U_i(\sigma', t'; \theta', \sigma^{\prec'}) \\ = \; & U_j(\sigma, t; \theta, \sigma^{\prec}) - U_j(\sigma', t'; \theta', \sigma^{\prec'}) \\ = \; & \frac{1}{2}(\sigma_i^{\prec} - \sigma_i^{\prec'})(\theta_i - \theta_j). \end{aligned}$$

The Characterization of the Shapley value

Theorem 1

A reordering rule φ satisfies efficiency, Pareto indifference, individual rationality, independence of local position change and equal sharing of local (dis)improvement if and only if it is the Shapley value.

Game procedure I

Let Γ be a finite-round position-negotiation game, in which each round of this game includes finite number of bilateral bargaining procedures Ω. We first define the bilateral bargaining procedure as follows. For each $N \in \mathcal{N}$ with $|N| = 2$ and each problem $(\theta, \sigma^{\prec}) \in \mathcal{S}^N$, the bilateral bargaining procedure $\Omega(N, \theta, \sigma^{\prec})$ contains the following two stages:

1. *Nature* picks an agent $i \in N$ with equal probability, and then agent i propose either a money payment $p_{ij} \in R$ for position exchange to the other agent $j \in N \setminus \{j\}$ or $p_{ij} = \kappa$ for keeping her position. In the latter case, the procedure ends up with the outcome $(U_i, U_j) = (0, 0)$, which means that the queue doesn't change and there is no money transfer between the agents. In the former case, the procedure goes to Stage 2.

Game procedure II

2. Given agent i's proposal $p_{ij} \in R$, agent j decides to either accept $(q_j = A)$ or reject $(q_j = R)$ it. In the case of rejection, the procedure ends up with the outcome $(U_i, U_j) = (0, 0)$, which means that the queue doesn't change and there is no money transfer between the agents. In the case of acceptance, the procedure ends up with the outcome $(U_i, U_j) = (-(\sigma_i - \sigma_i^{\prec}) - p_{ij}, -(\sigma_j - \sigma_j^{\prec}) + p_{ij})$, which means that the agents exchange their position in the queue and agent i pays p_{ij} to agent j, where $\sigma_i = \sigma_j^{\prec}$ and $\sigma_j = \sigma_i^{\prec}$.

Define the ordering for the game Γ:

Game procedure III

- In **Round 1**, there are n_1 bilateral bargaining procedure. The first bilateral bargaining procedure is $\Omega(N^{11}, \theta_{N^{11}}, \sigma^{11}_{N^{11}})$ played by the agents in N^{11}, where $N^{11} \equiv \{i^{11}_1, i^{11}_2\}$ and i^{11}_1, $i^{11}_2 \in N$ such that $\sigma^{11}_{i^{11}_1} = 1 = \sigma^{11}_{i^{11}_2} - 1$. Let $t_{i^{11}_1 i^{11}_2}$ be the money payment from agent i^{11}_1 to agent i^{11}_2 in the first bargaining procedure, and let σ^{12} be the queue obtained in the end of the first bargaining procedure, then the second bilateral bargaining procedure is $\Omega(N^{12}, \theta_{N^{12}}, \sigma^{12}_{N^{12}})$ played by the agents in N^{12}, where $N^{12} \equiv \{i^{12}_2, i^{12}_3\}$ and i^{12}_2, $i^{22}_3 \in N$ such that $\sigma^{12}_{i^{12}_2} = 2 = \sigma^{12}_{i^{12}_3} - 1$. Let $t_{i^{12}_2 i^{12}_3}$ be the money payment from agent i^{12}_2 to agent i^{12}_3 in the second bargaining procedure, and let σ^{13} be the queue obtained in the end of the second bargaining procedure. For each $m \in \{3, \cdots, n_1\}$,

Game procedure IV

let σ^{1m} be the queue obtained in the end of the $(m-1)$th bargaining procedure, then the mth bilateral bargaining procedure is $\Omega(N^{1m}, \theta_{N^{1m}}, \sigma^{1m}_{N^{1m}})$ played by the agents in N^{1m}, where $N^{1m} \equiv \{i^{1m}_m, i^{1m}_{m+1}\}$ and $i^{1m}_m, i^{1m}_{m+1} \in N$ such that $\sigma^{1m}_{i^{1m}_m} = m = \sigma^{1m}_{i^{1m}_{m+1}} - 1$. Let $t_{i^{1m}_m i^{1m}_{m+1}}$ be the money payment from agent i^{1m}_m to agent i^{1m}_{m+1} in the mth bargaining procedure.

- Let σ^{21} be the queue obtained in the end of Round 1. If all pairs $\{\{i,j\} | \sigma^{21}_i = m, \sigma^{21}_j = m+1, m \in \{1, \cdots, n-1\}\}$ have met to bargain positions before, then the game ends up in Round 1. Otherwise, the game proceeds to the next round.

Game procedure V

In the former case, $\sigma = \sigma^{21}$ is the final queue, and the outcome of the game is $U = (U_i)_{i\in N}$, where for each $i \in N$,

$$U_i = -(\sigma_i - \sigma_i^{\prec})\theta_i - \sum_{\{m\in\{1,\cdots,n_1\}|i_m^{1m}=i\}} t_{i_m^{1m} i_{m+1}^{1m}}$$

$$+ \sum_{\{m\in\{1,\cdots,n_1\}|i_{m+1}^{1m}=i\}} t_{i_m^{1m} i_{m+1}^{1m}}.$$

Game procedure VI

- In **Round 2**, there are $n_2 \in N$ bilateral bargaining procedure with $n_2 \leq n-2$. The first bilateral bargaining procedure is $\Omega(N^{21}, \theta_{N^{21}}, \sigma^{21}_{N^{21}})$ played by the agents in N^{21}, where $N^{21} \equiv \{i_1^{21}, i_2^{21}\}$ and i_1^{21}, $i_2^{21} \in N$ satisfying that $\sigma^{21}_{i_1^{21}} \in \{1, \cdots, n-1\}$ is the smallest number such that agents i_1^{21} and i_2^{21} never meet to bargain positions until now with $\sigma^{21}_{i_2^{21}} \equiv \sigma^{21}_{i_1^{21}} + 1$. Let $t_{i_1^{21} i_2^{21}}$ be the money payment from agent i_1^{21} to agent i_2^{21} in this bargaining procedure, and let σ^{22} be the queue obtained in the end of the first bargaining procedure.

Game procedure VII

- Next, the second bilateral bargaining procedure is $\Omega(N^{22}, \theta_{N^{22}}, \sigma^{22}_{N^{22}})$ played by the agents in N^{22}, where $N^{22} \equiv \{i^{22}_2, i^{22}_3\}$ and i^{22}_2, $i^{22}_3 \in N$ satisfying that $\sigma^{22}_{i^{22}_2} \in \{\sigma^{21}_{i^{22}_2}, \cdots, n-1\}$ is the smallest number such that agents i^{22}_2 and i^{22}_3 never meet to bargain positions until now with $\sigma^{22}_{i^{22}_3} \equiv \sigma^{22}_{i^{22}_2} + 1$. Let $t_{i^{22}_2 i^{22}_3}$ be the money payment from agent i^{22}_2 to agent i^{22}_3 in this bargaining procedure. For each $m \in \{3, \cdots, n_2\}$, let σ^{2m} be the queue obtained in the end of the $(m-1)$th bargaining procedure, then the mth bilateral bargaining procedure is $\Omega(N^{2m}, \theta_{N^{2m}}, \sigma^{2m}_{N^{2m}})$ played by the agents in N^{2m}, where $N^{2m} \equiv \{i^{2m}_m, i^{2m}_{m+1}\}$ and i^{2m}_m, $i^{2m}_{m+1} \in N$ satisfying that $\sigma^{2m}_{i^{2m}_m} \in \{\sigma^{2(m-1)}_{i^{2m}_m}, \cdots, n-1\}$ is the smallest number such that agents i^{2m}_m and i^{2m}_{m+1} never meet to

Game procedure VIII

bargain positions until now with $\sigma^{2m}_{i^{2m}_{m+1}} \equiv \sigma^{2m}_{i^{2m}_{m}} + 1$. Let $t_{i^{2m}_m i^{2m}_{m+1}}$ be the money payment from agent i^{2m}_m to agent i^{2m}_{m+1} in the mth bargaining procedure.

- Let σ^{31} be the queue obtained in the end of Round 2. If all pairs $\{\{i,j\} | \sigma^{31}_i = m, \sigma^{31}_j = m+1, m \in \{1, \cdots, n-1\}\}$ have met to bargain positions before, then the game ends up in Round 2. Otherwise, the game proceeds to the next round. In the former case, $\sigma = \sigma^{31}$ is the final queue, and the outcome of the game is $U = (U_i)_{i \in N}$, where for each $i \in N$,

Game procedure IX

$$U_i = -(\sigma_i - \sigma_i^{\prec})\theta_i - \sum_{h=1}^{2} [\sum_{\{m \in \{1, \cdots, n_h\} | i_m^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}}$$

$$+ \sum_{\{m \in \{1, \cdots, n_h\} | i_{m+1}^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}}].$$

Game procedure X

- Suppose that Rounds 1 to $(l-1)$ of the game are defined with $l \geq 3$, let σ^{l1} be the queue obtained in the end of Round $(l-1)$. If all pairs $\{\{i,j\} | \sigma_i^{l1} = m, \sigma_j^{l1} = m+1, m \in \{1, \cdots, n-1\}\}$ have met to bargain positions before, then the game ends up in Round $(l-1)$. Otherwise, the game proceeds to the next round. In the former case, $\sigma = \sigma^{l1}$ is the final queue, and the outcome of the game $U = (U_i)_{i \in N}$ is defined by

$$U_i = -(\sigma_i - \sigma_i^{\prec})\theta_i$$

$$-\sum_{h=1}^{l-1}\left[\sum_{\{m \in \{1,\cdots,n_h\} | i_m^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}} + \sum_{\{m \in \{1,\cdots,n_h\} | i_{m+1}^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}}\right].$$

Game procedure XI

- In **Round** l, there are $n_l \in N$ bilateral bargaining procedure with $n_l \leq n - l$. The first bilateral bargaining procedure is $\Omega(N^{l1}, \theta_{N^{l1}}, \sigma^{l1}_{N^{l1}})$ played by the agents in N^{l1}, where $N^{l1} \equiv \{i^{l1}_1, i^{l1}_2\}$, and i^{l1}_1, $i^{l1}_2 \in N$ satisfying that $\sigma^{l1}_{i^{l1}_1} \in \{1, \cdots, n-1\}$ is the smallest number such that agents i^{l1}_1 and i^{l1}_2 never meet to bargain positions until now with $\sigma^{l1}_{i^{l1}_2} \equiv \sigma^{l1}_{i^{l1}_1} + 1$. Let $t_{i^{l1}_1 i^{l1}_2}$ be the money payment from agent i^{l1}_1 to agent i^{l1}_2 in this bargaining procedure, and let σ^{l2} be the queue obtained in the end of the first bargaining procedure. Next, the second bilateral bargaining procedure is $\Omega(N^{l2}, \theta_{N^{l2}}, \sigma^{l2}_{N^{l2}})$ played by the agents in N^{l2}, where $N^{l2} \equiv \{i^{l2}_2, i^{l2}_3\}$ and i^{l2}_2, $i^{l2}_3 \in N$ satisfying that $\sigma^{l2}_{i^{l2}_2} \in \{\sigma^{l1}_{i^{l1}_2}, \cdots, n-1\}$ is the smallest number such that

Game procedure XII

agents i_2^{l2} and i_3^{l2} never meet to bargain positions until now, and $\sigma^{l2}_{i_3^{l2}} \equiv \sigma^{l2}_{i_2^{l2}} + 1$. Let $t_{i_2^{l2} i_3^{l2}}$ be the money payment from agent i_2^{l2} to agent i_3^{l2} in this bargaining procedure.

- For each $m \in \{3, \cdots, n_2\}$, let σ^{lm} be the queue obtained in the end of the $(m-1)$th bargaining procedure, then the mth bilateral bargaining procedure is $\Omega(N^{lm}, \theta_{N^{lm}}, \sigma^{lm}_{N^{lm}})$ played by the agents in N^{lm}, where $N^{lm} \equiv \{i_m^{lm}, i_{m+1}^{lm}\}$ and i_m^{lm}, $i_{m+1}^{lm} \in N$ satisfying that $\sigma^{lm}_{i_m^{lm}} \in \{\sigma^{l(m-1)}_{i_m^{l(m-1)}}, \cdots, n-1\}$ is the smallest number such that agents i_m^{lm} and i_{m+1}^{lm} never meet to bargain positions until now with $\sigma^{lm}_{i_{m+1}^{lm}} \equiv \sigma^{lm}_{i_m^{lm}} + 1$. Let $t_{i_m^{lm} i_{m+1}^{lm}}$ be the money payment from agent i_m^{lm} to agent i_m^{lm} in this bargaining procedure.

Game procedure XIII

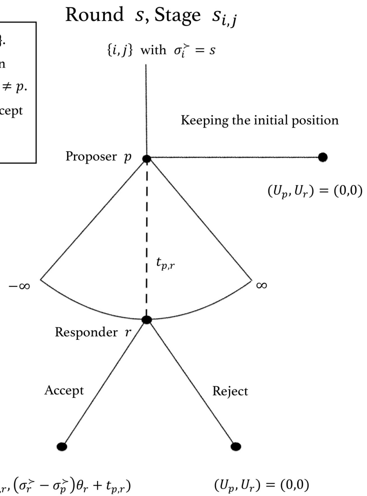
- Let $\sigma^{(l+1)1}$ be the queue obtained in the end of Round l. If all pairs $\{\{i,j\} | \sigma_i^{(l+1)1} = m, \sigma_j^{(l+1)1} = m+1, m \in \{1, \cdots, n-1\}\}$ have met to bargain positions before, then the game ends up in Round l. Otherwise, the game proceeds to the next round. In the former case, $\sigma = \sigma^{(l+1)1}$ is the final queue, and the outcome of the game $U = (U_i)_{i \in N}$ is defined by

$$U_i = -(\sigma_i - \sigma_i^{\prec})\theta_i$$

$$-\sum_{h=1}^{l} \left[\sum_{\{m \in \{1,\cdots,n_h\} | i_m^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}} + \sum_{\{m \in \{1,\cdots,n_h\} | i_{m+1}^{hm} = i\}} t_{i_m^{hm} i_{m+1}^{hm}} \right].$$

Non-cooperative justification

Step 1. Nature picks any agent $p \in \{i,j\}$.
Step 2. The chosen agent p proposes an exchange proposal $t_{p,r}$ to the other $r \neq p$.
Step 3. The other agent r decides to accept or reject the proposal.



Existence and uniqueness of a SPNE

Theorem 2

For each $N \in \mathcal{N}$ *and each problem* $(\theta, \sigma^{\prec}) \in \mathcal{S}^N$, $\Omega(N, \theta, \sigma^{\prec})$ *has a unique SPNE outcome* $U^{Sh} = (U_i^{Sh})_{i \in N}$*, and in particular, for each* $i \in N$,

$$U_i^{Sh} \equiv U_i(\sigma^{Sh}, t^{Sh}; \theta, \sigma^{\prec}),$$

where $(\sigma^{Sh}, t^{Sh}) \in Sh(\theta, \sigma^{\prec})$.

Discussion

Further research:

- the model can be extended in multiple identical or non-identical facilities
- Extend by dropping the identical service time assumption

Thank you! Any comments or suggestion are welcome.